Indian Institute of Technology Kanpur
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A First Course in Linear Algebra (MTH 201A)
Exercise Set #2

Notation: In what follows, we let $\mathbb{F} = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$. For any $n \in \mathbb{N}$, we denote the set of all invertible $n \times n$ matrices over $\mathbb{F}$ by $\mathrm{GL}_n(\mathbb{F})$.

(1) Let S be the set of all prime numbers. Which of the following statements is/are true (justify your answer):
  (a) S can be made a vector space over $\mathbb{Q}$,
  (b) S can be made a vector space over $\mathbb{R}$, and
  (c) S can be made a vector space over $\mathbb{C}$.

(2) Let X be a set and $\mathscr{C}(X, \mathbb{F})$ denote the space of all functions on X. For $x_0 \in X$, denote $\{f \in \mathscr{C}(X, \mathbb{F}) : f(x_0) = 0\}$ by W_{x_0}.
  (a) Show that W_{x_0} is a proper subspace of $\mathscr{C}(X, \mathbb{F})$.
  (b) Show that, for all $x_0 \in X$, W_{x_0} is a maximal subspace of $\mathscr{C}(X, \mathbb{F})$, i.e., there is no proper subspace of $\mathscr{C}(X, \mathbb{F})$ containing W_{x_0} properly.
  (c) Is the converse true, i.e., is every maximal subspace of $\mathscr{C}(X, \mathbb{F})$ of the form W_{x_0} for some $x_0 \in X$?
  (d) Let V be a vector space over $\mathbb{F}$ and for $x_0 \in X$, let $W_{x_0} := \{f \in \mathscr{C}(X, V) : f(x_0) = 0\}$. Are these subspaces maximal?

(3) Let V be a vector space over $\mathbb{F}$. Show that, for any $k \in \mathbb{N}$ and proper subspaces $W_1, \ldots, W_k$ of V, one always has $\displaystyle\bigcup_{i=1}^{k} W_i \subsetneqq V$.

(4) Consider the vector space $\mathbb{R}$ over $\mathbb{Q}$. Prove or disprove the following: there exists a countable $S \subseteq \mathbb{R}$ such that $L(S) = \mathbb{R}$.

(5) Let $n \in \mathbb{N}$. A subspace W of $\mathbb{F}^n$ is said to be *invariant* under $\mathrm{GL}_n(\mathbb{F})$ if $\forall \mathbf{x} \in W$ and $A \in \mathrm{GL}_n(\mathbb{F})$, one has $A\mathbf{x} \in W$. Find all subspaces of $\mathbb{F}^n$ which are invariant under $\mathrm{GL}_n(\mathbb{F})$.

(6) Let $m, n \in \mathbb{N}$. Let $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_m\}$ be bases of $\mathbb{F}^n$ and $\mathbb{F}^m$ respectively. Does $\{w_i v_j^t : i = 1, \ldots, m; j = 1, \ldots, n\}$ form a basis of $M_{m \times n}(\mathbb{F})$?

(7) For a vector space V, a finite collection of subspaces $\{V_i\}_{i=0}^{n}$ is said to be a *flag* if it satisfies the following:

$$\{0\} = V_0 \subsetneqq V_1 \subsetneqq \cdots \subsetneqq V_n = V.$$

The number n in the above, i.e., the number of nontrivial subspaces in the flag $\{V_i\}_{i=0}^{n}$ is called the *length* of the flag. We say a flag $\{W_i\}_{i=0}^{m}$ in V is a *refinement* of the flag $\{V_i\}_{i=0}^{n}$ if $\{V_i\}_{i=0}^{n} \subsetneqq \{W_i\}_{i=0}^{m}$. A flag in V is said to be *maximal* if it does not have a refinement. For V finite dimensional, show the following:
  (a) The length of any flag in V cannot exceed $\dim V$.
  (b) For every nonnegative integer n not exceeding $\dim V$, V has a flag of length n.
  (c) The length of a flag in V is equal to $\dim V$ if and only if the flag is maximal.
  (d) For every flag $\{V_i\}_{i=0}^{n}$ in V, there exists a maximal flag $\{W_i\}_{i=0}^{m}$ in V such that $\{V_i\}_{i=0}^{n} \subseteq \{W_i\}_{i=0}^{m}$.

(8) Let $n \in \mathbb{N}$. A matrix $A \in M_n(\mathbb{F})$ is said to *stabilize* a flag $\{V_i\}_{i=0}^{k}$ in $\mathbb{F}^n$ if one has the following:

$$A\mathbf{x} \in V_i, \forall i = 1, \ldots, k \text{ and } \mathbf{x} \in V_i.$$

(a) Find all $n \times n$ matrices which stabilize the flag $\{W_i\}_{i=0}^{n}$, where $W_0 = \{0\}, W_i := L(\{e_1, \ldots, e_i\}), \forall i = 1, \ldots, k$.

(b) Let $\{V_i\}_{i=0}^{n}$ be a maximal flag in $\mathbb{F}^n$. Describe all the matrices that stabilize this flag.

(c) Can you generalize (8b) to any flag in $\mathbb{F}^n$?