Indian Institute of Technology Kanpur

Department of Mathematics and Statistics

End-Sem-Exam.: 2020-21-I, Part II

Date: December 17, 2020

Time: 2.5 hours | A First Course in Linear Algebra (MTH 201A) | Total marks: 40

Throughout we let $\mathbb{F} = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$.

1. Let V be a finite dimensional vector space over $\mathbb{F}$. Show that, given any finite $\emptyset \neq \mathcal{S} \subseteq V \setminus \{0\}$, there exist nonzero functionals f and g on V such that, $\forall x \in \mathcal{S}, f(x) \neq g(x)$. [4]

2. Let V be a finite dimensional vector space over $\mathbb{F}$. Show that, for every linear operator $T : V \longrightarrow V$, there exists $r \in \mathbb{N}$ such that $V = \ker T^r \oplus \operatorname{Im} T^r$. [4]

3. Let the set up be as in Question 2 and $T : V \longrightarrow V$ be linear. Suppose that K and L are T invariant subspaces of V such that the following hold:

   (a) $V = K \oplus L$, and

   (b) the restriction of T to K and L are nilpotent and invertible respectively.

   Then show that, for some $m \in \mathbb{N}$, one has $K = \ker T^m$ and $L = \operatorname{Im} T^m$. [4]

4. Prove or disprove the following:
   Every real square matrix is similar to its transpose, i.e., $\forall n \in \mathbb{N}$ and $A \in M_n(\mathbb{R})$, $P^{-1}AP = A^t$, for some $P \in GL_n(\mathbb{R})$. [5]

5. Let $n \in \mathbb{N}$. Find all $n \times n$ matrices over $\mathbb{R}$ which are row equivalent to some orthogonal matrix. [5]

6. Let $n \in \mathbb{N}$ and $B \in M_n(\mathbb{R})$ has determinant 1. Show that there exist $n \times n$ matrices K, A and N with the following four properties:

   (a) $B = KAN$,

   (b) K is an orthogonal matrix with $\det(K) = 1$,

   (c) N is an upper triangular matrix with all diagonal entries 1; and

   (d) A is a diagonal matrix whose all diagonal entries are positive and $\det(A) = 1$.

   [3]

7. Show that, in Question 6, the matrices K, A and N with the properties (a)-(d) are unique. [5]

8. Recall that, a *Quadratic form* Q on $\mathbb{R}^n$, where $n \in \mathbb{N}$, is a function of the following form:

$$Q(\mathbf{x}) = \mathbf{x}^t A \mathbf{x}, \forall \mathbf{x} \in \mathbb{R}^n,$$

   where A is an $n \times n$ real symmetric matrix.

We say a quadratic form Q on $\mathbb{R}^n$ is *nondegenrate* if $Q(\mathbf{y}) \neq 0$ for all $\mathbf{y} \neq \mathbf{0}$. Two quadratic forms Q_1 and Q_2 are said to be *equivalent* if there exists $P \in GL_n(\mathbb{R})$ such that one has the following:

$$Q_1(\mathbf{x}) = Q_2(P\mathbf{x}), \forall \mathbf{x} \in \mathbb{R}^n.$$

Note that, this defines an equivalence relation on the set of all quadratic forms on $\mathbb{R}^n$ and furthermore, Q_1 is nondegenrate if and only if Q_2 is nondegenerate whenever Q_1 and Q_2 are equivalent. Show that, for $n = 3$, every nondegenrate quadratic form is equivalent to exactly one of the following:

(a) $Q_0(x_1, x_2, x_3) := x_1^2 + x_2^2 + x_3^2$

(b) $Q_1 := -Q_0$. where Q_0 is defined above in (a),

(c) $Q_3(x_1, x_2, x_3) := 2x_1x_3 - x_2^2$, and

(d) $Q_4 := -Q_3$, where Q_3 is defined above in (c).

[5]

9. Let V be a finite dimensional inner product space over $\mathbb{C}$. Show that, for any linear operator $A : V \longrightarrow V$, one has the following:

$$A \text{ is normal if and only if } \exists f(x) \in \mathbb{C}[x] \text{ such that } A^* = f(A).$$

You may use the following if necessary:

$$\det\left(\begin{bmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdot & \cdot & \cdot & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \cdot & \cdot & \cdot & \alpha_2^{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \alpha_n & \alpha_n^2 & \cdot & \cdot & \cdot & \alpha_n^{n-1} \end{bmatrix}\right) = \prod_{1 \leq i < j \leq n} (\alpha_i - \alpha_j),$$

where $n \geq 2$ *and* $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$. [5]