Indian Institute of Technology Kanpur
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A First Course in Linear Algebra (MTH 201A)
Exercise Set #3

Notation: In what follows, we let $\mathbb{F} = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$.

(1) Let $m, n \in \mathbb{N}$. Consider the system of linear equations $A\mathbf{x} = B$, where $A \in M_{m\times n}(\mathbb{F})$ and $B \in M_{m\times 1}(\mathbb{F})$. Show that the above system has a solution if and only if the ranks of the matrices A and $[A|B]$ are same.

(2) Let $n \in \mathbb{N}$. We say Γ *is a discrete subgroup of* $\mathbb{R}^n$ if there exists a linearly independent $\{v_1, \ldots, v_k\} \subseteq \mathbb{R}^n$ such that $\Gamma = \{n_1v_1 + \cdots + n_kv_k : n_1, \ldots, n_k \in \mathbb{Z}\}$. In this case, we say that $\{v_1, \ldots, v_k\}$ *is a basis of* Γ.

(a) Let $1 \leq k \leq n$ and $\Gamma_k := \{(n_1, \ldots, n_k, 0, \ldots, 0) : n_1, \ldots, n_k \in \mathbb{Z}\}$. Find all bases of Γ_k.
(b) Show that, for any discrete subgroup Γ of $\mathbb{R}^n$, any two bases of Γ always have same number of elements. We call this number the *rank* of Γ.
(c) Show that, if $A \in \mathrm{GL}_n(\mathbb{R})$ and Γ is a discrete subgroup of $\mathbb{R}^n$ then $A(\Gamma) := \{A\mathbf{x} : \mathbf{x} \in \Gamma\}$ is also a discrete subgroup of $\mathbb{R}^n$ having the same rank with Γ.
(d) If Γ and Γ' are discrete subgroups of $\mathbb{R}^n$ having the same rank then show that $\Gamma' = A(\Gamma)$, for some $A \in \mathrm{GL}_n(\mathbb{R})$.

(3) Let $n \in \mathbb{N}$ and $W \leq \mathbb{F}^n$. Show that there is a homogeneous system of linear equations whose solution space is W.

(4) Let $n \in \mathbb{N}$ and $1 \leq r \leq n$. Denote the collection of all r dimensional subspaces of $\mathbb{F}^n$ by $\mathfrak{S}_r$. Show that, for every $W \in \mathfrak{S}_r$, there exists $\mathfrak{U} \subseteq \mathfrak{S}_r$ containing W such that $\mathfrak{U}$ can be made a vector space over $\mathbb{F}$ (in a natural way) of dimension $r(n-r)$.

(5) Let $n \in \mathbb{N}$ and $A \in M_{m\times n}(\mathbb{F})$. For nonzero $f(t) = \sum_{i=0}^{n} a_i t^i \in \mathbb{F}[t]$, where $n \in \mathbb{N} \cup \{0\}$, we define $f(A) := a_nA^n + \cdots + a_1A + a_0I$.

(a) Show that, there exists a nonzero polynomial $p(t) \in \mathbb{F}[t]$ such that $p(A) = 0$.
(b) Show that there exists a unique monic polynomial $p_A(t) \in \mathbb{F}[t]$ such that $p_A(A) = 0$ and $p_A(t)$ divides every nonzero $p(t) \in \mathbb{F}[t]$ such that $p(A) = 0$.
(c) Let $p_A(t)$ be as above in (5b). If $f(t) \in \mathbb{F}[t]$ and $\gcd(f(t), p_A(t)) = 1$ then show that $f(A)$ is invertible.