# Indian Institute of Technology Kanpur
## Department of Mathematics and Statistics
### A First Course in Linear Algebra (MTH 201A)
### Exercise Set #4

Notation: In what follows, we let $\mathbb{F} = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$.

(1) Let $X \neq \emptyset$ be a set. Show the following:

(a) There exists a vector space $V$ over $\mathbb{F}$ and a map $\iota : X \longrightarrow V$ with the following property:

**Property 1.** *whenever one has a vector space $W$ over $\mathbb{F}$ and $\varphi : X \longrightarrow W$, then there exists a unique linear map $\tilde{\varphi} : V \longrightarrow W$ such that $\tilde{\varphi} \circ \iota = \varphi$, i.e., making the following diagram commutative:*

$$\begin{array}{ccc} X & \xrightarrow{\iota} & V \\ & \underset{\varphi}{\searrow} & \downarrow{\scriptstyle \tilde{\varphi}} \\ & & W \end{array}.$$

(b) Show that, for any pair $(V', \iota')$, where $V'$ is a vector space over $\mathbb{R}$ and $\iota' : X \longrightarrow V'$, satisfying the Property 1 in (1a), there exists a unique isomorphism $\psi : V \longrightarrow V'$ such that $\psi \circ \iota = \iota'$, i.e., making the following diagram commutative:

$$\begin{array}{ccc} X & \xrightarrow{\iota} & V \\ & \underset{\iota'}{\searrow} & \downarrow{\scriptstyle \psi} \\ & & V' \end{array}$$

(2) Let $V$ and $W$ be two finite dimensional vector spaces over $\mathbb{F}$ and $T : V \longrightarrow W$ be a linear map. Is it possible to have $[T]_{\mathcal{B}'}^{\mathcal{B}}$ a row echelon matrix by choosing appropriate ordered bases $\mathcal{B}$ and $\mathcal{B}'$ of $V$ and $W$ respectively?

(3) Let $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in M_2(\mathbb{R})$ and consider the following linear map

$$\mathbb{R}^2 \longrightarrow \mathbb{R}^2, \mathbf{x} \mapsto A\mathbf{x}, \forall \mathbf{x} \in \mathbb{R}^2. \qquad (*\ 1)$$

Recall that $\mathbb{C}$ can be identified with $\mathbb{R}^2$ in view of the correspondence $x + iy \longleftrightarrow (x, y)$. Find a necessary and sufficient condition on $a, b, c$ and $d$ so that the map given by ($*$ 1) becomes a linear operator on the complex vector space $\mathbb{C}$.

(4) Let $n \in \mathbb{N}$ and $\emptyset \neq U \subseteq \mathbb{R}^n$ be open. Recall that, for any $p \in U$, one has the following isomorphism:

$$T_p(U) \longrightarrow \mathbb{R}^n, \ [\gamma] \mapsto \gamma'(0), \forall [\gamma] \in T_p(U).$$

We denote preimages of the standard basis vectors $e_1, \ldots, e_n$ of $\mathbb{R}^n$ under this isomorphism by $\frac{\partial}{\partial x_1}|_p, \ldots, \frac{\partial}{\partial x_n}|_p$ respectively. Suppose that $m \in \mathbb{N}$ and $V \subseteq \mathbb{R}^m$ is open and $\mathbf{f} : U \longrightarrow V$. We denote the $i$-th coordinate of the function $\mathbf{f}$ by $f_i$, for all $i = 1, \ldots, m$. Assume that $\frac{\partial f_i}{\partial x_j}$ is continuous for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

(a) Show that, $\forall p \in U$ the following is a linear map:

$$d\mathbf{f}_p : T_p(U) \longrightarrow T_{\mathbf{f}(p)}(V), \ [\gamma] \mapsto [\mathbf{f} \circ \gamma], \ \forall [\gamma] \in T_p(U). \qquad (*\ 2)$$

(b) For any $p \in U$, find the matrix of $d\mathbf{f}_p$ (defined above in ($*$ 2)) relative to the ordered bases $\left\{ \frac{\partial}{\partial x_1}|_p, \ldots, \frac{\partial}{\partial x_n}|_p \right\}$ and $\left\{ \frac{\partial}{\partial x_1}|_{\mathbf{f}(p)}, \ldots, \frac{\partial}{\partial x_m}|_{\mathbf{f}(p)} \right\}$.

(5) Let $V$ be a finite dimensional vector space over $\mathbb{F}$ and $T$ be a linear operator on $V$ such that $T^2 = T$. Show the following:

(a) $V = \ker T \oplus \ker(T - I)$; and

(b) By choosing an appropriate ordered basis, the matrix of $T$ can be brought to the following form:

$$\begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \text{ where } r \geq 0.$$

(6) Suppose that $V$ is an $n$ dimensional vector space over $\mathbb{F}$, where $n \in \mathbb{N}$, and $T$ is a linear operator on $V$ such that $T^n = 0$ but $T^{n-1} \neq 0$.

(a) Show that there exists an ordered basis $\mathcal{B} = \{v_1, \ldots, v_n\}$ such that

$$Tv_1 = v_2, \cdots, Tv_{n-1} = v_n, Tv_n = 0.$$

(b) Let $\mathcal{B}$ be as in (6a). Find $[T]_{\mathcal{B}}$.

(c) Deduce that, if $A, B \in M_n(\mathbb{F})$ such that $A^n = B^n = 0$ and $A^{n-1} \neq 0 \neq B^{n-1}$, then $A$ and $B$ are similar.

(7) Let $V$ be a finite dimensional vector space over $\mathbb{F}$ and $A, B : V \longrightarrow V$ be linear. Show that the following are equivalent:

(a) There exists ordered bases $\mathcal{B}$ and $\mathcal{B}'$ of $V$ such that $[A]_{\mathcal{B}} = [B]_{\mathcal{B}'}$; and

(b) There exists an invertible linear operator $P$ on $V$ such that $A = PBP^{-1}$.