

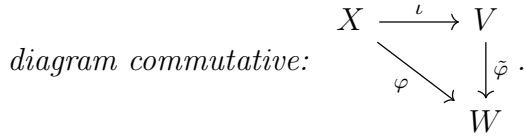
Indian Institute of Technology Kanpur
Department of Mathematics and Statistics
A First Course in Linear Algebra (MTH 201A)
Exercise Set #4

Notation: In what follows, we let $\mathbb{F} = \mathbb{Q}, \mathbb{R}$ or \mathbb{C} .

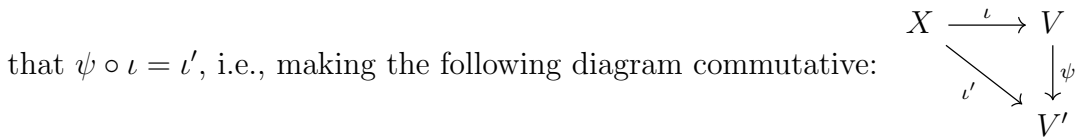
(1) Let $X \neq \emptyset$ be a set. Show the following:

(a) There exists a vector space V over \mathbb{F} and a map $\iota : X \rightarrow V$ with the following property:

Property 1. whenever one has a vector space W over \mathbb{F} and $\varphi : X \rightarrow W$, then there exists a unique linear map $\tilde{\varphi} : V \rightarrow W$ such that $\tilde{\varphi} \circ \iota = \varphi$, i.e., making the following



(b) Show that, for any pair (V', ι') , where V' is a vector space over \mathbb{R} and $\iota' : X \rightarrow V'$, satisfying the Property 1 in (1a), there exists a unique isomorphism $\psi : V \rightarrow V'$ such



(2) Let V and W be two finite dimensional vector spaces over \mathbb{F} and $T : V \rightarrow W$ be a linear map. Is it possible to have $[T]_{\mathcal{B}'}^{\mathcal{B}}$ a row echelon matrix by choosing appropriate ordered bases \mathcal{B} and \mathcal{B}' of V and W respectively?

(3) Let $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in M_2(\mathbb{R})$ and consider the following linear map

$$\mathbb{R}^2 \rightarrow \mathbb{R}^2, \mathbf{x} \mapsto A\mathbf{x}, \forall \mathbf{x} \in \mathbb{R}^2. \quad (* 1)$$

Recall that \mathbb{C} can be identified with \mathbb{R}^2 in view of the correspondence $x + iy \leftrightarrow (x, y)$. Find a necessary and sufficient condition on a, b, c and d so that the map given by (* 1) becomes a linear operator on the complex vector space \mathbb{C} .

(4) Let $n \in \mathbb{N}$ and $\emptyset \neq U \subseteq \mathbb{R}^n$ be open. Recall that, for any $p \in U$, one has the following isomorphism:

$$T_p(U) \rightarrow \mathbb{R}^n, [\gamma] \mapsto \gamma'(0), \forall [\gamma] \in T_p(U).$$

We denote preimages of the standard basis vectors e_1, \dots, e_n of \mathbb{R}^n under this isomorphism by $\frac{\partial}{\partial x_1}|_p, \dots, \frac{\partial}{\partial x_n}|_p$ respectively. Suppose that $m \in \mathbb{N}$ and $V \subseteq \mathbb{R}^m$ is open and $\mathbf{f} : U \rightarrow V$. We denote the i -th coordinate of the function \mathbf{f} by f_i , for all $i = 1, \dots, m$. Assume that $\frac{\partial f_i}{\partial x_j}$ is continuous for all $i = 1, \dots, m$ and $j = 1, \dots, n$.

(a) Show that, $\forall p \in U$ the following is a linear map:

$$d\mathbf{f}_p : T_p(U) \rightarrow T_{\mathbf{f}(p)}(V), [\gamma] \mapsto [\mathbf{f} \circ \gamma], \forall [\gamma] \in T_p(U). \quad (* 2)$$

(b) For any $p \in U$, find the matrix of $d\mathbf{f}_p$ (defined above in (* 2)) relative to the ordered bases $\left\{ \frac{\partial}{\partial x_1}|_p, \dots, \frac{\partial}{\partial x_n}|_p \right\}$ and $\left\{ \frac{\partial}{\partial x_1}|_{\mathbf{f}(p)}, \dots, \frac{\partial}{\partial x_m}|_{\mathbf{f}(p)} \right\}$.

(5) Let V be a finite dimensional vector space over \mathbb{F} and T be a linear operator on V such that $T^2 = T$. Show the following:

(a) $V = \ker T \oplus \ker(T - I)$; and

- (b) By choosing an appropriate ordered basis, the matrix of T can be brought to the following form:

$$\begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \text{ where } r \geq 0.$$

- (6) Suppose that V is an n dimensional vector space over \mathbb{F} , where $n \in \mathbb{N}$, and T is a linear operator on V such that $T^n = 0$ but $T^{n-1} \neq 0$.

- (a) Show that there exists an ordered basis $\mathcal{B} = \{v_1, \dots, v_n\}$ such that

$$Tv_1 = v_2, \dots, Tv_{n-1} = v_n, Tv_n = 0.$$

- (b) Let \mathcal{B} be as in (6a). Find $[T]_{\mathcal{B}}$.

- (c) Deduce that, if $A, B \in M_n(\mathbb{F})$ such that $A^n = B^n = 0$ and $A^{n-1} \neq 0 \neq B^{n-1}$, then A and B are similar.

- (7) Let V be a finite dimensional vector space over \mathbb{F} and $A, B : V \rightarrow V$ be linear. Show that the following are equivalent:

- (a) There exists ordered bases \mathcal{B} and \mathcal{B}' of V such that $[A]_{\mathcal{B}} = [B]_{\mathcal{B}'}$; and
(b) There exists an invertible linear operator P on V such that $A = PBP^{-1}$.