Indian Institute of Technology Kanpur

Department of Mathematics and Statistics

A First Course in Linear Algebra (MTH 201A)

Exercise Set #5

<u>Notation</u>: In what follows, we let $\mathbb{F} = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$.

(1) Let $n \in \mathbb{N}$ and $\emptyset \neq U \subseteq \mathbb{R}^n$ be open. For any $p \in U$, by $\{dx_1|_p, \ldots, dx_n|_p\}$ denote the basis of $T_p(U)^*$ dual to $\{\frac{\partial}{\partial x_1}|_p, \ldots, \frac{\partial}{\partial x_n}|_p\}$. Suppose $f : U \longrightarrow \mathbb{R}$ such that $\frac{\partial f}{\partial x_i}$ is continuos for all $i = 1, \ldots, n$. Show the following for any $p \in U$:

$$df_p = \frac{\partial f}{\partial x_1}(p)\, dx_1|_p + \cdots + \frac{\partial f}{\partial x_n}(p)\, dx_n|_p\,.$$

(2) Let $n \in \mathbb{N}$.

(a) Find all linear functionals $f : M_n(\mathbb{F}) \longrightarrow \mathbb{F}$ with the following property:

$$AB - BA \in \ker f \ \text{ for all } A, B \in M_n(\mathbb{F}). \qquad (*\,1)$$

(b) Let f be a nonzero linear functional on $M_n(\mathbb{F})$ with property $(*\,1)$. Show that $\ker f$ is spanned by the matrices of the form $AB - BA$, where $A, B \in M_n(\mathbb{F})$.

(3) Let $n \in \mathbb{F}$ and $B \in M_n(\mathbb{F})$. Define $f_B : M_n(\mathbb{F}) \longrightarrow \mathbb{F}$ by $f_B(A) = tr(B^t A)$, $\forall A \in M_n(\mathbb{F})$.

(a) Show that $f_B \in M_n(\mathbb{F})^*$.

(b) Show that every linear functional on $M_n(\mathbb{F})$ is of the form f_B for some $B \in M_n(\mathbb{F})$.

(c) Conclude that $B \mapsto f_B$, for all $B \in M_n(\mathbb{F})$, is an ismomorphism from $M_n(\mathbb{F})$ to $M_n(\mathbb{F})^*$.

(4) Let V and W be two finite dimensional vector spaces over $\mathbb{F}$ and $T : V \longrightarrow W$ be a linear map. We define a map $T^* : W^* \longrightarrow V^*$ by $f \mapsto f \circ T$, $\forall f \in W^*$.

(a) Show that T^* defined above is linear.

(b) Assume that $\mathcal{B} := \{v_1, \ldots, v_n\}$ and $\mathcal{B}_1 := \{w_1, \ldots, w_m\}$, where $m, n \in \mathbb{N}$, are ordered bases of V and W. Denote the bases dual to $\mathcal{B}$ and $\mathcal{B}_1$ by $\mathcal{B}^*$ and $\mathcal{B}_1^*$ respectively. Find the relation between $[T]_{\mathcal{B}_1}^{\mathcal{B}}$ and $[T^*]_{\mathcal{B}^*}^{\mathcal{B}_1^*}$.

(c) Show that $\operatorname{Im} T^* = (\ker T)^0$.

(5) Let V be a finite dimensional vector space over $\mathbb{F}$. Show that the following is an ismomorphism:

$$L(V, V) \longrightarrow L(V^*, V^*), T \mapsto T^*, \ \forall T \in L(V, V).$$

(6) Let V be a finite dimensional vector space over $\mathbb{F}$. Show that, for any $\lambda \in \mathbb{F}$, one always has the following:

$$\ker(T - \lambda I) \neq \{0\} \iff \ker(T^* - \lambda I) \neq \{0\}.$$