Indian Institute of Technology Kanpur
Department of Mathematics and Statistics

A First Course in Linear Algebra (MTH 201A)
Exercise Set #6

Notation: In what follows, we let $\mathbb{F}$ be $\mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$.

(1) Let $n \in \mathbb{N}$ and $A \in M_n(\mathbb{F})$. The minimal polynomial of A is defined to the minimal polynomial of the linear operator $\mathbf{x} \mapsto A\mathbf{x}$, for all $\mathbf{x} \in \mathbb{F}^n$, on $\mathbb{F}^n$. Show that, the minimal polynomial of a square matrix over $\mathbb{F}$ remains the same if it is regarded as a complex matrix as well.

(2) Let $n \in \mathbb{N}$ and $A \in M_n(\mathbb{F})$. Assume that the minimal polynomial of A splits over $\mathbb{F}$. Show that A and A^t are similar.

(3) Let $m, n \in \mathbb{N}$. Suppose that $A \in M_m(\mathbb{F})$, $D \in M_n(\mathbb{F})$, $B \in M_{m\times n}(\mathbb{F})$ and $C \in M_{n\times m}(\mathbb{F})$. If A is invertible then show the following:

$$\det\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = \det(A)\det(D - CA^{-1}B).$$

Deduce from this that, if $n = m$ and A commutes with C then one has $\det\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = \det(AD - BC)$.

(4) Let $n \in \mathbb{N}$. For $\mathbf{x} = (x_1 \ldots, x_n) \in \mathbb{C}^n$, we define $||\mathbf{x}||_1 := |x_1| + \cdots + |x_n|$. Prove that, for any n vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in $\mathbb{C}^n$, one always has the following:

$$\det([\mathbf{v}_1 \cdots \mathbf{v}_n]) \leq \prod_{i=1}^{n} ||\mathbf{v}_i||_1. \qquad (*\ 1)$$

Find necessary and sufficient condition(s) so that equality occurs in ($*$ 1).

(5) Let V be an n dimensional vector space over $\mathbb{F}$ and $T : V \longrightarrow V$ be linear. We say that T *stabilizes* a maximal flag $\{V_i\}_{i=0}^n$ in V if $T(V_i) \subseteq V_i$, for all $i = 1, \ldots, n$. Show that, T stabilizes a maximal flag in V if and only if V can be expressed as the direct sum of its generalised eigenspaces.

(6) Let V be a finite dimensional vector space over $\mathbb{F}$ and $T : V \longrightarrow V$ be nilpotent. Assume that k is the smallest positive integer that $T^k = 0$. Show that any linearly independent $S \subseteq V$ with the property that $L(S) \cap \ker T^{k-1} = \{0\}$ can be extended to a Jordon basis for T.

(7) Let V be a finite dimensional vector space over $\mathbb{F}$ and $T : V \longrightarrow V$ be linear. Assume that m_T splits over $\mathbb{F}$. Show the following for any eigenvalue λ of A:
   (a) The number of Jordon blocks corresponding to λ in the Jordon canonical form of A is $\dim V_\lambda$.
   (b) The size of the largest Jordon block corresponding to λ in the Jordon canonical form of A is the multiplicity of the root λ in m_T.
   (c) The sum of the sizes of the Jordon blocks corresponding to λ in the Jordon canonical form of A is the multiplicity of the root λ in the characteristic polynomial of T.