Indian Institute of Technology Kanpur
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A First Course in Linear Algebra (MTH 201A)
Exercise Set #7

Notation: In what follows, we let $\mathbb{F}$ be $\mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$.

(1) A *Quadric* is a generalization of conics, i.e., it is defined as the locus of a quadratic equation with real coefficients in n variables as follows:

$$\left\{ \mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{1 \le i,j \le n} a_{ij} x_i x_j + \sum_{r=1}^{n} b_r x_r + c = \mathbf{x}^t A \mathbf{x} + B\mathbf{x} + c = 0 \right\}, \qquad (*\ 1)$$

where $A = [a_{ij}]_{1 \le i,j \le n} \in M_n(\mathbb{R})$ is symmetric and $B = (b_1, \ldots, b_n)$.

(a) Show that, by an orthogonal change in variables, the above equation can be brought to the following form:

$$\lambda_1 y_1^2 + \cdots + \lambda_n y_n^2 + b_1 y_1 + \cdots + b_n y_n + c = 0 \qquad (*\ 2)$$

for some $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$.

(b) Show that, by a translation and permutation of variables if necessary, ($*$ 2) can be brought to

$$\lambda_1 z_1^2 + \cdots + \lambda_r z_r^2 + b_{r+1} z_{r+1} + \cdots + b_n z_n + c' = 0\,, \qquad (*\ 3)$$

for some $r \le n$ and $c' \in \mathbb{R}$.

(c) Suppose that some $b_i \neq 0$ in ($*$ 3). Then show that, by an appropriate translation, ($*$ 3) can be brought to

$$\lambda_1 \eta_1^2 + \cdots + \lambda_r \eta_r^2 + b_{r+1} \eta_{r+1} + \cdots + b_n \eta_n = 0\,. \qquad (*\ 4)$$

(d) Now show that ($*$ 4) can be further reduced to the following by an orthogonal change in variables:

$$\lambda_1 \xi_1^2 + \cdots + \lambda_r \xi_r^2 + d \xi_{r+1} = 0\,,$$

for some $d \in \mathbb{R}$.

(2) Let V be a finite dimensional real vector space and f is a bilinear form on V, i.e., $f : V \times V \longrightarrow \mathbb{R}$ which is linear in both the variables. For any ordered basis $\mathcal{B} = \{e_1, \ldots, e_n\}$, the *matrix of f in the ordered basis* $\mathcal{B}$, denoted by $[f]_{\mathcal{B}}$ is $[f(e_i, e_j)]_{1 \le i,j \le n}$.

(a) Show that, for any $v, w \in V$, $f(v, w) = [v]_{\mathcal{B}}^t [f]_{\mathcal{B}} [w]_{\mathcal{B}}$.

(b) If $\mathcal{B}$ and $\mathcal{B}'$ are two ordered bases of V then show that there exists an invertible $n \times n$ real matrix P such that $[f]_{\mathcal{B}'} = P^t [f]_{\mathcal{B}} P$.

(c) Show that the rank of $[f]_{\mathcal{B}}$ does not depend on the choice of $\mathcal{B}$. We define this as the *rank* of the bilinear form f.

(3) Let V be a finite dimensional real vector space and f is a bilinear form on V. We say that f is *symmetric* if $f(v, w) = f(w, v)$ holds for all $v, w \in V$.

(a) Show that f is symmetric if and only if $[f]_{\mathcal{B}}$ is symmetric for any ordered basis $\mathcal{B}$ of V.

(b) Show that if f is symmetric then there exists an ordered basis $\mathcal{B} = \{e_1, \ldots, e_n\}$ of V such that $[f]_{\mathcal{B}}$ is is the following diagonal matrix:

$$\begin{bmatrix} I_k & & \\ & -I_\ell & \\ & & \mathbf{0} \end{bmatrix}, \qquad (*\ 5)$$

where k and ℓ are nonnegative integers.

(c) Let $V^+ := L(\{e_1, \ldots, e_k\}), V^- := L(\{e_{k+1}, \ldots, e_{k+\ell}\})$ and $V^\perp := L(\{e_{k+\ell+1}, \ldots, e_n\})$. Suppose that $W \le V$ is such that $f(w, w) > 0$, for all $0 \neq w \in W$. Then the sum of W, V^{-1} and $V^\perp$ is direct and consequently, $\dim W \le \dim V^+$.

(d) Show that in ($*$ 5), k and ℓ are unique. The number $k-\ell$ is called the *signature* of the bilinear form f.

(4)* Let $A=[a_{ij}]_{1\le i,j\le n}\in M_n(\mathbb{C})$ and $M>0$ be such that $|a_{ij}|\le M$, for all $i,j=1,\dots,n$. For any $m\in\mathbb{N}$ and $i,j=1,\dots,n$, we denote the (i,j)-th entry of A^m by $a_{ij}^{(m)}$.

(a) Show that, for any $m\in\mathbb{N}$ and $i,j=1,\dots,n$, one has $|a_{ij}^{(m)}|\le(nM)^m$.

(b) Show that the following series is convergent

$$I+A+\frac{A^2}{2!}+\cdots+\frac{A^m}{m!}+\cdots; \qquad (*\ 6)$$

and the convergence is uniform on every $B_R:=\{C\in[c_{ij}]_{1\le i,j\le n}\in M_n(\mathbb{C}):|c_{ij}|\le R\}$, where $R>0$. The limit of ($*$ 6) is called the *exponential* of A and denoted by e^A.

(c) Let $A\in M_n(\mathbb{C})$. Show that, for any $P\in GL_n(\mathbb{C})$, $e^{PAP^{-1}}=Pe^AP^{-1}$.

(d) Show that, if $\lambda_1,\dots,\lambda_n$ are the eigenvalues of A (counting multiplicities) then the eigenvalues of e^A are $e^{\lambda_1},\dots,e^{\lambda_n}$. Deduce that $\det(e^A)=e^{tr(A)}$.

(e) Show that, $e^{A^t}=(e^A)^t$ and $e^{\overline{A}}=\overline{e^A}$.

(5)* Let $A,B\in M_n(\mathbb{C})$ be two commuting matrices. Observe that, for any $m\in\mathbb{N}$,

$$\sum_{i=0}^{2m}\frac{(A+B)^i}{i!}=\left(\sum_{i=0}^{m}\frac{A^i}{i!}\right)\left(\sum_{i=0}^{m}\frac{B^i}{i!}\right)+R_m\,, \qquad (*\ 7)$$

where

$$R_m:=\sum_{\substack{(k,\ell),\,\max(k,\ell)>m\\ k+\ell\le 2m}}\frac{A^k}{k!}\frac{B^\ell}{\ell!}$$

(a) From ($*$ 7), show that $R_m\xrightarrow[m\to\infty]{}0$. Deduce that $e^{A+B}=e^Ae^B$.

(b) Show that, $e^{-A}=(e^A)^{-1}$.

(6)* Let $A\in M_n(\mathbb{C})$. Show the following:

(a) A is hermitian $\Longrightarrow$ so is e^A.

(b) A is positive definite hermitian $\Longrightarrow$ so is e^A.

(c) For every positive definite hermitian matrix $B\in M_n(\mathbb{C})$, there exists a unique hermitian matrix $A\in M_n(\mathbb{C})$ such that $e^A=B$.

(7)** Let $\{A_n\}_{n\ge1}$ be a sequence of $d\times d$ complex matrices, where $d\in\mathbb{N}$, and $A\in M_d(\mathbb{C})$. Assume that $A_n\xrightarrow[n\to\infty]{}A$. Denote the eigenvalues of A by $\lambda_1,\dots,\lambda_d$. We aim to prove that for each $n\in\mathbb{N}$, there exists an ordering $\lambda_{n,1},\dots,\lambda_{n,d}$ of the eigenvalues of A_n such that $\lambda_{n,j}\xrightarrow[n\to\infty]{}\lambda_j$, for all $j=1,\dots,d$. This follows as an immediate consequence of the following theorem:

Theorem 1. *Fix $d\in\mathbb{N}$. For every $n\in\mathbb{N}$, let $P_n(z)$ denote the complex polynomial*

$$a_{n,d}z^d+a_{n,d-1}z^{d-1}+\dots a_{n,1}z+a_{n,0}$$

of degree d. Suppose that $P(z)=a_dz^d+a_{d-1}z^{d-1}+\dots a_1z+a_0\in\mathbb{C}[z]$ is such that $\deg P(z)=d$ *and P_n* converges *to P* coefficientwise *as $n\to\infty$, i.e., for every $j=0,\dots,d$, $a_{n,j}\xrightarrow[n\to\infty]{}a_j$. Denote the roots of P by $\lambda_1,\dots,\lambda_d$. Then for each $n\in\mathbb{N}$, there exists an ordering $\lambda_{n,1},\dots,\lambda_{n,d}$ of the roots of $P_n(z)$ such that $\lambda_{n,j}\xrightarrow[n\to\infty]{}\lambda_j$, for all $j=1,\dots,d$.*

Without any loss in generality, we assume from now that all $P_n(z)$'s and $P(z)$ are monic. The above theorem will be proved by induction and using the two lemmas:

Lemma 1. *Let the set up be as above in the Theorem 1 and λ be a root of $P(z)$. Suppose that, for each $n \in \mathbb{N}$, λ_n is a root of $P_n(z)$ and $\lambda_n \xrightarrow[n\to\infty]{} \lambda$. Define $Q_n(z) \in \mathbb{C}[z]$, for each $n \in \mathbb{N}$, and $Q(z) \in \mathbb{C}[z]$ by the following:*

$$P_n(z) = (z - \lambda_n)Q_n(z) \text{ and } P(z) = (z - \lambda)Q(z).$$

Then $Q_n \xrightarrow[n\to\infty]{} Q$ coefficientwise.

Proof. Left as an exercise. □

Lemma 2. *Let the set up be as above. Then for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\forall n \geq N$, one has $|\lambda_n - \lambda| < \varepsilon$ for some root λ_n of $P_n(z)$.*

Proof. Left as an exercise. □

We now give a brief sketch of the Proof of the Theorem 1, the task of working out the details is left as an exercise:

The statement if obvious when $d = 1$. So assume $d > 1$ and the conclusion of the theorem holds for $d - 1$. Put $\lambda = \lambda_1$ and now by Lemma 2, one can find a strictly increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ of positive integers such that, for all $k \in \mathbb{N}$ and all $j \geq n_k$, there is a root λ_{j,n_k} of $P_j(z)$ satisfying $|\lambda_{j,n_k} - \lambda| < \frac{1}{k}$. Pick a root λ_i of $P_i(z)$ for all $i = 1, \ldots, n_1 - 1$.

Next show that, the following sequence of roots converges to λ:

$$\lambda_1, \ldots, \lambda_{n_1-1}, \lambda_{n_1,n_1}, \lambda_{n_1+1,n_1}, \ldots, \lambda_{n_2-1,n_1}, \lambda_{n_2,n_2}, \lambda_{n_2+1,n_2}, \ldots, \lambda_{n_3-1,n_2}, \lambda_{n_3,n_3}, \ldots$$

Finish the proof using Lemma 1 and induction hypothesis.