Indian Institute of Technology Kanpur
Department of Mathematics and Statistics

Date: October 17, 2020

Mid-Sem-Exam.: 2020-21-I

Time: 2 hours | A First Course in Linear Algebra (MTH 201A) | Total marks: 35

Throughout we let $\mathbb{F} = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$.

1. Fix $m \in \mathbb{N}$. For each $n \in \mathbb{N}$, let A_n be an $m \times m$ matrix and we denote the (i, j)-th entry of A_n by $a_{ij}^{(n)}$, for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$. We say that *the sequence of matrices* $\{A_n\}_{n=1}^{\infty}$ *converges to a matrix* $A = [a_{ij}]_{1 \le i \le m;\, 1 \le j \le m} \in M_m(\mathbb{R})$ if, for all $(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, m\}$, one has $a_{ij}^{(n)} \xrightarrow[n\to\infty]{} a_{ij}$. For example,

$$\begin{bmatrix} \frac{1}{n} & 2 + \frac{3}{n} \\ 5 & \frac{1}{2^n} \end{bmatrix} \xrightarrow[n\to\infty]{} \begin{bmatrix} 0 & 2 \\ 5 & 0 \end{bmatrix}.$$

   Show that, for any $A \in M_m(\mathbb{R})$, there exists a sequence $\{A_n\}_{n=1}^{\infty}$ of <u>invertible</u> $m \times m$ real matrices such that $A_n \xrightarrow[n\to\infty]{} A$. [5]

2. Let $n \in \mathbb{N}$ and $1 \le r \le n$. Denote the collection of all r dimensional subspaces of $\mathbb{F}^n$ by $\mathfrak{S}_r$. Show that, for every $W \in \mathfrak{S}_r$, there exists $\mathscr{U} \subseteq \mathfrak{S}_r$ containing W such that $\mathscr{U}$ can be made a vector space over $\mathbb{F}$ (in a natural way) of dimension $r(n - r)$. [6]

3. Recall that, a line in a vector space V over $\mathbb{F}$ is a subset of the following form:

$$\ell(p; d) := \{p + td : t \in \mathbb{F}\}, \text{ where } p \in V \text{ and } d \in V \setminus \{0\}.$$

   Show that, if V is a finite dimensional real or complex vector space with $\dim V > 1$, then V cannot be written as the union of countably many lines. [4]

4. Let $m, n \in \mathbb{F}$ and $A = [a_{ij}]_{1 \le i \le m, 1 \le j \le n} \in M_{m \times n}(\mathbb{F})$. Assume that rank $A = k$. Let $r_1 < \cdots < r_k$ and $c_1 < \cdots < c_k$ be such that $r_1, \ldots, r_k$ rows of A and $c_1, \ldots, c_k$ columns of A are linear linearly independent in $\mathbb{F}^n$ and $\mathbb{F}^m$ respectively. Show that the following submatrix of A is invertible:

$$\begin{bmatrix} a_{r_1 c_1} & \cdots & a_{r_1 c_k} \\ \vdots & \ddots & \vdots \\ a_{r_k c_1} & \cdots & a_{r_k c_k} \end{bmatrix}.$$

   [5]

5. Let V and W be finite dimensional vector spaces over $\mathbb{F}$ and $T : V \longrightarrow W$ be linear. Show that, there exists a unique nonnegative integer r such that, by choosing appropriate ordered bases $\mathcal{B}$ and $\mathcal{B}'$ of V and W respectively, $[T]_{\mathcal{B}'}^{\mathcal{B}}$ can be brought to the form $\begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$. [4]

6. Suppose that, we are given two pairs of linear maps $V \xrightarrow{T} W \xrightarrow{S} U$ and $V' \xrightarrow{T'} W' \xrightarrow{S'} U'$, where V, V', W, W', U and U' are vector spaces over $\mathbb{F}$. Assume that, T and T' are injective, S and S' are surjective and $\text{Im } T = \ker S$ and $\text{Im } T' = \ker S'$. Let A, B and C are the linear maps such that the following diagram commutes:

$$\begin{array}{ccccc} V & \xrightarrow{T} & W & \xrightarrow{S} & U \\ {\scriptstyle A}\downarrow & & {\scriptstyle B}\downarrow & & {\scriptstyle C}\downarrow \\ V' & \xrightarrow{T'} & W' & \xrightarrow{S'} & U' \end{array} \text{, i.e.,}$$

one has $B \circ T = T' \circ A$ and $C \circ S = S' \circ B$. Show the following:

(a) A and C are injective $\Longrightarrow B$ is injective; and

(b) A and C are surjective $\Longrightarrow B$ is surjective.

[3+3=6]

7. Let V and W be finite dimensional vector spaces over $\mathbb{F}$ and $T : V \longrightarrow W$ be a linear map. Show that, there exists a linear operator $P : V \longrightarrow V$ with the following three properties:

(a) $P^2 = P$,

(b) the restriction of T to $\text{Im } P$, denoted by T', is an isomorphism from $\text{Im } P$ to $\text{Im } T$; and

(c) $T = T' \circ P$.

Is the above P unique? [4+1=5]
