

On extensions of characters of affine pro- p Iwahori–Hecke algebra.

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Abstract

Let K be a non-discrete non-Archimedean local field with residue characteristic p . Let G be the group of K rational points of an algebraic connected reductive group defined over K . In this article we compute the extensions between characters of affine pro- p Iwahori–Hecke algebra \mathcal{H}^{aff} over an algebraically closed field R of characteristic p .

1 Introduction

Let K be a non-discrete non-Archimedean local field and \mathbf{G} be a connected reductive group scheme over K with an irreducible root system. In this article we are interested in computing extensions of supersingular characters of affine pro- p Iwahori–Hecke algebra, denoted by \mathcal{H}^{aff} (see section 2). In the context of mod- p local Langlands correspondence the Iwahori–Hecke algebra, denoted by \mathcal{H} , plays a very important role. For instance, when $\mathbf{G} = \text{GL}_n/K$ a numerical correspondence between absolutely simple supersingular \mathcal{H} -modules of dimension n and n -dimensional absolutely irreducible mod- p representations of the absolute Galois group of \mathbb{Q}_p was conjectured by Vigneras (see [Vig05]) and is proved by Ollivier for $\text{GL}_n(\mathbb{Q}_p)$ (see [Oll10a, Theorem 1.1]). This numerical correspondence is extended as an exact functor by the work of Grosse-Klönne (see [GK16, Theorem 8.8]). This article is an attempt to understand the blocks of the category of \mathcal{H} -modules.

We compute the dimension of degree one extensions of characters of the affine pro- p Iwahori–Hecke algebra for a connected reductive group over a p -adic field. With the recent work of Abe on parabolic induction and their adjoint functors (see [Abe14a] and [Abe14b]) we know the dimension of degree one extensions between simple modules of \mathcal{H}^{aff} . In particular if the group is semi-simple and simply connected the dimension of degree one extensions between simple modules of pro- p Iwahori–Hecke algebra, denoted by \mathcal{H} , can be computed using the data of support of the supersingular characters (see section 2).

The structure of the algebra \mathcal{H} is studied in detail by Vigneras in the article [Vig05] when \mathbf{G} is split and in the article [Vig16] for any connected reductive group \mathbf{G} . Among other things she showed presentations of the algebra \mathcal{H} similar to that of Iwahori–Matsumoto and Bernstein as in the classical Iwahori–Hecke algebra. Ollivier defined the notion of supersingularity for a \mathcal{H} module in the split case (see [Oll14]) and later obtained a classification of simple supersingular modules of \mathcal{H} . These results were generalised to arbitrary groups by Vigneras (see [Vig15]). All simple modules are constructed from supersingular representations (see [Oll10b] and [Abe14a]). The simple supersingular modules are characterised completely by the results of Ollivier and Vigneras and we use their explicit description in our calculations.

The dimensions of extension spaces of simple modules over \mathcal{H} are computed for GL_2/K by Breuil and Paskunas (see [BP12]). The homological dimension of the algebra \mathcal{H} is investigated by Koziol in the article [Koz15]. When \mathbf{G} is split group Koziol showed that the homological dimension is usually infinite. In this regard the higher extensions always exist but the question of blocks for the category of modules over \mathcal{H} is to be determined. The work of Abdellatif and the author gives dimensions of extension spaces between simple modules of \mathcal{H} for SL_2 . For rank one groups, it was observed that the notion of L -packets and the blocks are closely related. To understand the blocks of Iwahori–Hecke modules we first compute the dimension of degree one extensions (see Theorem 3.6). For the case of SL_n the definition of L -packets was given by Koziol (see [Koz16b, Definition 6.4]). We observed that the notions of supersingular blocks and L -packets do not coincide if $n > 2$.

In this article we explicitly compute the blocks of simple \mathcal{H} modules for unramified unitary groups in 2 and 3 variables denoted by $U(1, 1)$ and $U(2, 1)$ respectively. The case of $U(1, 1)$ is similar to that of SL_2 and blocks and L -packets are the same. This paper is the authors attempt to understand the relationship between blocks and L packets for general reductive groups. Even for the case of SL_n the complete relationship between blocks and L -packets for $n > 3$ is not complete.

2 Preliminaries

Let K be a non-Archimedean non-discrete local field with ring of integers \mathfrak{o}_K , its maximal ideal \mathfrak{p}_K and residue field k of cardinality q a power of prime p . In this article all modules and representations are over a fixed algebraically closed field R of characteristic p . Let \mathbf{G} be a connected reductive group scheme over K . We denote by \mathfrak{X}_K the adjoint Bruhat–Tits building associated to (\mathbf{G}, K) and $j : \mathfrak{X}_K \hookrightarrow \mathfrak{X}'_K$ be the enlarged Bruhat–Tits building. For any facet F of \mathfrak{X}_K we denote by \mathbf{G}_F the Bruhat–Tits group scheme over \mathfrak{o}_K associated to F such that $\mathbf{G}_F(\mathfrak{o}_K)$ is isomorphic to the pointwise G stabiliser of the facet $j(F)$ and $\mathbf{G}_F \times_{\mathfrak{o}_K} K \simeq \mathbf{G}$. Let \mathbf{G}_F^0 be the connected component of \mathbf{G}_F and let P_F be the group $\mathbf{G}_F^0(\mathfrak{o}_K)$. The group P_F is the parahoric subgroup of $\mathbf{G}(K)$. Let $U_{F,k}$ be the unipotent radical of $\mathbf{G}^0 \times_{\mathfrak{o}_K} k$. Let I_F be the pro- p group

$$\{\mathbf{G}_F^0(\mathfrak{o}_K) \mid g \in U_{F,k}(k) \bmod \mathfrak{p}_K\}.$$

For any K -group scheme \mathbf{H} we denote by H the group $\mathbf{H}(K)$.

In the rest of the section the main reference is [Vig15, section 1.3]. Let \mathbf{T} be the maximal K -split torus contained in \mathbf{G} . We denote by \mathbf{N} and \mathbf{Z} the normaliser and centraliser of the torus \mathbf{T} . Let V be the space spanned by the set of coroots $\Phi(\mathbf{G}, \mathbf{T})^\vee \subset X_*(\mathbf{T}) \otimes \mathbb{R}$. Let $\mathcal{A}_{\mathbf{T}}$ be the apartment in \mathfrak{X}_K corresponding to \mathbf{T} and $\nu : Z \rightarrow V$ be the Bruhat–Tits homomorphism. The group Z acts on $\mathcal{A}_{\mathbf{T}}$ by translations via the map $\nu : Z \rightarrow V$ moreover this action extends to an action of N . Let Z_0 be the unique parahoric subgroup of Z and Z_1 be the maximal pro- p subgroup of Z_0 . We denote by Z_k the quotient Z_0/Z_1 . Let $W(1)$ be the group N/Z_1 and W be the extended affine Weyl group N/Z_0 . With these notations $W(1)$ fits in the following exact sequence.

$$0 \rightarrow Z_k \rightarrow W(1) \rightarrow W \rightarrow 0.$$

Let Λ be the group Z/Z_0 , the group W_0 normalizes Z/Z_0 and we have an isomorphism $W \simeq \Lambda \rtimes W_0$. The homomorphism $\nu : Z \rightarrow V$ factorizes through Z_0 and hence W acts on V .

We fix a chamber C contained in $\mathcal{A}_{\mathbf{T}}$. The group P_C is the Iwahori subgroup and I_C is the maximal pro- p subgroup of P_C . The maps $n \mapsto P_C n P_C$ and $n \mapsto I_C n I_C$ induces bijections

$W \simeq P_C \backslash G / P_C$ and $W(1) \simeq I_C \backslash G / I_C$ respectively. Let $\mathcal{S}(C)$ be the set of faces of \bar{C} and for every $F \in \mathcal{S}(C)$ we denote by s_F the affine reflection fixing the face F . Let W^{aff} be the group generated by $\{s_F \mid F \in \mathcal{S}(C)\}$, we denote this set of reflections by S^{aff} . The group W^{aff} is called the affine Weyl group. The pair $(W^{\text{aff}}, S^{\text{aff}})$ is a Coxeter system and the group W^{aff} is contained in $\nu(N)$ the image of N in the group of affine automorphisms of $\mathcal{A}_{\mathbf{T}}$. Let $l : W^{\text{aff}} \rightarrow \mathbb{Z}$ be the length function of the Coxeter system $(W^{\text{aff}}, S^{\text{aff}})$.

For any $F \in \mathcal{S}(C)$ we denote by $G_{F,k}$ the group P_F / I_F . The group $P_{F,k}$ is also the k -points of the reductive quotient of $\mathbf{G}^0 \times_{o_K} k$, denoted by $\mathbf{G}_{F,k}$. The group $\mathbf{G}_{F,k}$ is a connected reductive group of rank one. The image of I_F in $P_{F,k}$ is the group of k -rational points of the unipotent radical $\mathbf{U}_{F,k}$ of a Borel subgroup $T_{F,k} \mathbf{U}_{F,k}$. We denote by $\bar{\mathbf{U}}_{F,k}$ be the unipotent radical of the opposite Borel subgroup of $\mathbf{G}_{F,k}$. We denote by $Z_{F,k}$ the group $Z_k \cap \langle U_{F,k}, \bar{U}_{F,k} \rangle$ where $\langle U_{F,k}, \bar{U}_{F,k} \rangle$ is the group generated by the two opposite unipotent groups. Moreover for any $s \in S^{\text{aff}}$ there exists an $n_s \in N \cap P_s$ such that its image in $G_{k,s}$ belongs to the group $\langle U_{F,k}, \bar{U}_{F,k} \rangle$. The image of n_s in $W(1)$ is called an admissible lift of s .

Let Ω be the W stabiliser of C . The group W can be identified with $W^{\text{aff}} \rtimes \Omega$. The group Ω normalizes W^{aff} and the length function l extends to a function on W . We denote by l the inflation of l to $W(1)$. If \mathbf{G} is semi-simple simply-connected group Ω is trivial. The group Ω is trivial in some other interesting cases. Consider an unramified quadratic extension L of K and (W, h) be a pair consisting of a vector space W over L and h is a hermitian form. Assume that the dimension of the anisotropic part of W is less than one. The unitary group $\mathbf{U}(W)/K$ associated to the pair (W, h) is quasi-split and in this case $W = W^{\text{aff}}$. We may take L to be a separated ramified quadratic extension of K , when the dimension of W is odd and we have $W = W^{\text{aff}}$ for $\mathbf{U}(W)/K$.

Let \mathcal{H} be the algebra $\text{End}_G(\text{ind}_{I_C}^G \text{id})$ and we identify \mathcal{H} with the space of functions $f : G \rightarrow R$ such that $f(i_1 g i_2) = f(g)$ for all $i_1, i_2 \in I_C$ and $g \in G$. For any $w \in W(1)$ we denote by T_w the characteristic function on $I_C w I_C$. The elements $\{T_w\}_{w \in W(1)}$ form a basis for the Hecke algebra and \mathcal{H} admits the following presentation given by two sets of relations

1. (braid relations) For any $w, w' \in W(1)$ such that $l(w) + l(w') = l(ww')$ we have $T_w T_{w'} = T_{ww'}$
2. (quadratic relations) For any $s \in S^{\text{aff}}$ we have $T_s^2 = -c_s T_s$ where $c_s = 1/|Z_{k,s}| \sum_{z \in Z_{k,s}} T_z$.

We denote by \mathcal{H}^{aff} the subalgebra of \mathcal{H} generated by $\{T_w \mid w \in W^{\text{aff}}\}$. The algebra \mathcal{H} is called the pro- p Iwahori–Hecke algebra and \mathcal{H}^{aff} the affine pro- p Iwahori–Hecke algebra. The algebra \mathcal{H} is isomorphic to a certain twisted tensor product of $R[Z_k]$ and \mathcal{H}^{aff} . In this article we restrict to the characters of \mathcal{H}^{aff} and we will not need this description. Let ι be an involutive R automorphism of \mathcal{H} such that $\iota(T_{\bar{s}}) = T_s - c_{\bar{s}}$.

We restrict ourselves to characters of \mathcal{H} and \mathcal{H}^{aff} we do not recall the description of all simple modules. We first describe the set of characters of \mathcal{H}^{aff} (see [Vig15, Theorem 1.6]). Let λ be any character of Z_k and we denote by S_{λ} the set $\{s \in S^{\text{aff}} \mid \lambda(c_s) \neq 0\}$. For any subset I of S^{aff} we denote by W_I the subgroup of W generated by $s \in I$. The set of characters of the algebra \mathcal{H} are parametrised by pairs (λ, I) consisting of a character λ of Z_k and a subset I of S_{λ} . We denote by $\xi_{\lambda, I}$ the character corresponding to (λ, I) and is given by

$$\xi_{\lambda, I}(T_{wt}) = 0 \text{ for all } w \in W \setminus W_I \text{ and } t \in Z_k. \quad (1)$$

$$\xi_{\lambda, I}(T_{wt}) = \lambda(t)(-1)^{l(w)} \text{ for all } w \in W_I \text{ and } t \in Z_k. \quad (2)$$

For any character ξ of \mathcal{H}^{aff} we denote by S_ξ the set $\{s \in S^{\text{aff}} \mid c_s(T_{\tilde{s}})\}$ where \tilde{s} is an admissible lift of s . This definition of S_ξ does not depend on admissible lifts. The character ξ is called sign character if $S_\xi = S^{\text{aff}}$. If ξ is a sign character then $\xi \circ \iota$ is called the trivial character. Any character ξ is supersingular if and only if ξ is not a sign character or trivial character (see [Vig15, Theorem 1.6]).

3 Calculations of degree one extensions.

In this section we want to compute the dimension of the spaces $\text{Ext}_{\mathcal{H}^{\text{aff}}}^1(\xi_{\lambda_1, I_1}, \xi_{\lambda_2, I_2})$. We denote by \mathcal{H} the affine pro- p Iwahori–Hecke algebra by abuse of notation. The algebra \mathcal{H} is generated by T_t for $t \in Z_k$ and $T_{\tilde{s}}$ where $s \in S_{\text{aff}}$. For convenience we drop the \tilde{s} in the admissible lift of s . We use the generators and relations to calculate the dimension of the degree one extensions.

Let E be an extension of $\mathfrak{m} := \xi_{\lambda_2, I_2}$ by $\mathfrak{n} := \xi_{\lambda_1, I_1}$, i.e, we have

$$0 \rightarrow \mathfrak{n} \xrightarrow{p} E \xrightarrow{q} \mathfrak{m} \rightarrow 0.$$

We fix two non-zero vectors \mathbf{v}'_1 and \mathbf{v}'_2 in \mathfrak{n} and \mathfrak{m} respectively. Fix a Z_k equivariant section $s : \mathfrak{m} \rightarrow E$ of the map q . Let \mathbf{v}_1 and \mathbf{v}_2 be the vectors $p(\mathbf{v}'_1)$ and $s(\mathbf{v}'_2)$. For any $s \in S^{\text{aff}}$ let the action of T_s on \mathbf{v}_2 be

$$\begin{aligned} T_s \mathbf{v}_2 &= a_s \mathbf{v}_1 - \mathbf{v}_2 \quad \forall s \in I_2, \\ T_s \mathbf{v}_2 &= a_s \mathbf{v}_1, \quad \forall s \notin I_2. \end{aligned}$$

Moreover for any t in Z_k we have,

$$\begin{aligned} \lambda_2(t^s)(a_s \mathbf{v}_1 - \mathbf{v}_2) &= T_s T_{t^s} \mathbf{v}_2 = T_t T_s \mathbf{v}_2 = a_s \lambda_1(t) \mathbf{v}_1 - \lambda_2(t) \mathbf{v}_2 \quad \forall s \in I_2, \\ a_s \lambda_2(t^s) \mathbf{v}_1 &= \lambda_2(t^s) T_s \mathbf{v}_2 = T_s T_{t^s} \mathbf{v}_2 = T_t T_s \mathbf{v}_2 = \lambda(t) a_s \mathbf{v}_1, \quad \forall s \notin I_2. \end{aligned}$$

From the above relations we get that $a_s((s\lambda_2)(t) - \lambda_1(t)) = 0$ for all $s \in S_{\text{aff}}$ and $t \in Z_k$. Let I_E be the set $\{s \in S_{\text{aff}} \mid a_s \neq 0\}$. If E is non-split extension then the set I_E is non-empty and moreover we have $\lambda_2^s = \lambda_1$ for all $s \in I_E$. If $\lambda_1 \neq \lambda_2$ the values $(a_s)_{s \in S_{\text{aff}}}$ are determined by E and does not depend on the choice of $s : \mathfrak{m} \rightarrow E$. If $\lambda_1 = \lambda_2$ the section $s : \mathfrak{m} \rightarrow E$ is not unique we have to take this into consideration to identify the space of extensions. But for the present purpose let us fix a section $s : \mathfrak{m} \rightarrow E$.

Lemma 3.1. For a fixed basis $(\mathbf{v}_1, \mathbf{v}_2)$ as above we get that $a_s = 0$ for any $s \in I_1 \cap I_2$. If $s \in S_{\lambda_1}$, $s \notin I_1$ and $s \notin I_2$ then $a_s = 0$. If $s \notin S_{\lambda_1}$ and $s \in I_2$ then $a_s = 0$. With the above relations on a_s the quadratic relations are satisfied for all $s \in S_{\text{aff}}$.

Proof. In this lemma we only use the quadratic relations on the elements T_s for $s \in S_{\text{aff}}$. To begin with consider any $s \in I_1 \cap I_2$. Consider the case where $s \in I_1 \cap I_2$. In this case we have $T_s \mathbf{v}_2 = a_s \mathbf{v}_1 - \mathbf{v}_2$ and $T_s \mathbf{v}_1 = -\mathbf{v}_1$. Now the quadratic relation on T_s gives us

$$c_s(-a_s \mathbf{v}_1 + \mathbf{v}_2) = -c_s T_s \mathbf{v}_2 = T_s^2 \mathbf{v}_2 = -2a_s \mathbf{v}_1 - \mathbf{v}_2.$$

Now the reflection s belongs to I_1 and I_2 and hence $c_s \mathbf{v}_1 = c_s \mathbf{v}_2 = 1$. This concludes that $a_s = 0$. Now consider the case where $s \notin I_1$ and hence $s \notin I_2$. In this case we get that $T_s \mathbf{v}_1 = 0$ and

$T_s \mathbf{v}_2 = a_s \mathbf{v}_1$. The quadratic relations gives us $-c_s T_s \mathbf{v}_2 = 0$. Hence we get that $-a_s c_s \mathbf{v}_1 = 0$. Now if $s \in S_{\lambda_1}$ then $c_s \mathbf{v}_1 = \mathbf{v}_1$ which implies that $a_s = 0$.

Consider the case where $s \in I_1$ and $s \notin I_2$. We have $T_s \mathbf{v}_1 = -\mathbf{v}_1$ and $T_s \mathbf{v}_2 = a_s \mathbf{v}_1$. Now $T_s(T_s \mathbf{v}_2) = -a_s \mathbf{v}_1$. The element $-c_s T_s \mathbf{v}_2 = -a_s c_s \mathbf{v}_1$. Since $s \in I_1 \subset S_{\lambda_1}$ we get that $c_s \mathbf{v}_1 = \mathbf{v}_1$. This shows that T_s satisfies the required quadratic relation. Now consider the case $s \in I_2$ and $s \notin I_1$. In this case we have $T_s \mathbf{v}_2 = a_s \mathbf{v}_1 - \mathbf{v}_2$ and $T_s \mathbf{v}_1 = 0$. This shows that $T_s(T_s \mathbf{v}_2) = -T_s \mathbf{v}_2$. But $-c_s T_s \mathbf{v}_2$ is equal to $-a_s c_s \mathbf{v}_1 + c_s \mathbf{v}_2$. Now $s \in I_2 \subset S_{\lambda_2}$ we get that $c_s \mathbf{v}_2 = \mathbf{v}_2$. Hence we get that $a_s c_s \mathbf{v}_1 = a_s \mathbf{v}_1$. Now $s \in S_{\lambda_1}$ then the quadratic relation is satisfied but otherwise $a_s = 0$. \square

Remark 3.2. Note that a_s may be non-zero in either of the following cases. In the first case, $s \in I_1$ and $s \notin I_2$, in the second case $s \in S_{\lambda_1}$, $s \notin I_1$ and $s \in I_2$, the third case when $s \notin S_{\lambda_1}$, $s \notin I_2$. Now the existence of extensions and their isomorphism classes can be computed by examining the braid relations.

Lemma 3.3. For any s_1 and s_2 in $I_1 \setminus (I_1 \cap I_2)$ the constants a_{s_1} equals to a_{s_2} . If s_1 and s_2 belong to $I_2 \setminus (I_1 \cap I_2)$ then $a_{s_1} = a_{s_2}$.

Proof. In the first case the action of T_{s_1} and T_{s_2} is given by $T_{s_1} \mathbf{v}_1 = -\mathbf{v}_1$, $T_{s_1} \mathbf{v}_2 = a_{s_1} \mathbf{v}_1$ and $T_{s_2} \mathbf{v}_1 = -\mathbf{v}_1$ and $T_{s_2} \mathbf{v}_2 = a_{s_2} \mathbf{v}_1$. Now $(T_{s_i} T_{s_j})^m \mathbf{v}_2 = -a_{s_j} \mathbf{v}_1$ and $T_{s_j} (T_{s_i} T_{s_j})^m \mathbf{v}_2 = a_{s_j} \mathbf{v}_1$. Now by braid relation for T_{s_1} and T_{s_2} we get that $a_{s_i} = a_{s_j}$. In the second case the action of T_{s_1} and T_{s_2} on E are given by $T_{s_i} \mathbf{v}_1 = 0$ and $T_{s_i} \mathbf{v}_2 = a_{s_i} \mathbf{v}_1 - \mathbf{v}_2$. Now $(T_{s_i} T_{s_j})^m \mathbf{v}_2 = -a_{s_i} \mathbf{v}_1 + \mathbf{v}_2$ and $T_{s_j} (T_{s_i} T_{s_j})^m \mathbf{v}_2 = a_{s_j} \mathbf{v}_1 - \mathbf{v}_2$. Using braid relations we get that $a_{s_i} = a_{s_j}$. \square

Note that in the third case when $s \notin S_{\lambda_1}$ and $s \in I_2$, $a_s = 0$. Now if there is an s in the third case we get that for any $s \in I_2 \setminus (I_1 \cap I_2)$ the value $a_s = 0$. The next lemma concludes the verification of the remaining braid relations on T_s for $s \in S_{\text{aff}}$.

Hypothesis 3.1. We make the following hypothesis on the function $a : S^{\text{aff}} \rightarrow \bar{k}$ sending $s \mapsto a_s$.

1. the conditions on a_s satisfied by Lemmas 3.1 and 3.3,
2. If there exist $s_1 \in I_1 \setminus (I_1 \cap I_2)$ and $s_2 \in I_2 \setminus (I_1 \cap I_2)$ such that the order of $s_1 s_2$ is 2 then we have $a_{s_i} + a_{s_j} = 0$ for all $s_i \in I_1 \setminus (I_1 \cap I_2)$ and $s_j \in I_2 \setminus (I_1 \cap I_2)$,
3. Let $s \in S^{\text{aff}} \setminus (I_1 \cup I_2)$ and there exists an element $s' \in I_1 \setminus (I_1 \cap I_2)$ such that the order of ss' is 2 then $a_s = 0$,
4. Let $s \in S^{\text{aff}} \setminus (I_1 \cup I_2)$ and there exists an element $s' \in I_2 \setminus (I_1 \cap I_2)$ such that the order of ss' is 2 then $a_s = 0$,
5. Let $s \in S^{\text{aff}} \setminus (I_1 \cup I_2)$ and there exists an element $s' \in (I_1 \cap I_2)$ such that the order of ss' is 3 then $a_s = 0$.

Lemma 3.4. Let E be a 2 dimensional vector space and for any basis $(\mathbf{v}_1, \mathbf{v}_2)$ of E such that $T_t \mathbf{v}_1 = \lambda_1(t) \mathbf{v}_1$ and $T_t \mathbf{v}_2 = \lambda_2(t) \mathbf{v}_2$ for all $t \in Z_k$. Suppose the function $a : S^{\text{aff}} \rightarrow \bar{k}$ satisfy the following Hypothesis 3.1. The relations $T_s \mathbf{v}_1 = -\mathbf{v}_1$ for $s \in I_1$, $T_s \mathbf{v}_1 = 0$ for $s \notin I_1$ and $T_s \mathbf{v}_2 = a_s \mathbf{v}_1 - \mathbf{v}_2$ for $s \in I_2$, $T_s \mathbf{v}_2 = a_s \mathbf{v}_1$ for $s \notin I_2$ makes E a \mathcal{H} module.

Proof. From Lemma 3.1 we have to verify braid relations in \mathcal{H} . Note that Lemma 3.3 we get the braid relations for pairs (s_1, s_2) such that s_1, s_2 are both in $I_1 \setminus (I_1 \cap I_2)$ and $I_2 \setminus (I_1 \cap I_2)$. If s_1 and s_2 both belong to $I_1 \cap I_2$ then $a_s = 0$ and hence braid relations follow as E is a direct sum when restricted to the algebra generated by T_{s_1} and T_{s_2} . Let s_1 and s_2 belong to $S_{\text{aff}} \setminus (I_1 \cup I_2)$. In this case we have: $T_{s_1} \mathbf{v}_1 = 0$, $T_{s_1} \mathbf{v}_2 = a_{s_1} \mathbf{v}_1$, $T_{s_2} \mathbf{v}_1 = 0$ and $T_{s_2} \mathbf{v}_2 = a_{s_2} \mathbf{v}_1$. Now $(T_{s_i} T_{s_j})^m = 0$ for $m \geq 1$ from which the braid relations follow.

Fix any $s_1 \in I_1 \setminus (I_1 \cap I_2)$ and consider the case when $s_2 \in I_1 \cap I_2$ then we have $T_{s_1} \mathbf{v}_1 = -\mathbf{v}_1$, $T_{s_1} \mathbf{v}_2 = a_{s_1} \mathbf{v}_1$, $T_{s_2} \mathbf{v}_1 = -\mathbf{v}_1$ and $T_{s_2} \mathbf{v}_2 = -\mathbf{v}_2$. In this case we have $(T_{s_i} T_{s_j})^m \mathbf{v}_2 = -a_{s_1} \mathbf{v}_1$ and $T_{s_j} (T_{s_i} T_{s_j})^m \mathbf{v}_2 = a_{s_1} \mathbf{v}_1$. Hence the braid relations are satisfied. Now consider the case where $s_2 \in I_2 \setminus (I_1 \cap I_2)$. In this case we have the relations $T_{s_2} \mathbf{v}_1 = 0$ and $T_{s_2} \mathbf{v}_2 = a_{s_2} \mathbf{v}_1 - \mathbf{v}_2$. With these relations we get that $T_{s_2} T_{s_1} \mathbf{v}_2 = 0$, $T_{s_1} T_{s_2} \mathbf{v}_2 = -(a_{s_1} + a_{s_2}) \mathbf{v}_1$ and $T_{s_2} T_{s_1} T_{s_2} \mathbf{v}_2 = 0$. By Hypothesis 3.1, (2) we get the braid relations in this case. Now consider the case when $s_1 \in I_1 \setminus (I_1 \cap I_2)$ and $s_2 \in S_{\text{aff}} \setminus (I_1 \cup I_2)$. In this case we have $T_{s_2} \mathbf{v}_1 = 0$ and $T_{s_2} \mathbf{v}_2 = a_{s_2} \mathbf{v}_1$. Moreover $T_{s_2} T_{s_1} \mathbf{v}_2 = 0$, $T_{s_1} T_{s_2} \mathbf{v}_2 = -a_{s_2} \mathbf{v}_1$ and $T_{s_2} T_{s_1} T_{s_2} \mathbf{v}_2 = 0$. By Hypothesis 3.1, (3) we get the braid relations in this case.

Now fix any $s_1 \in (I_1 \cap I_2)$. We have $T_{s_1} \mathbf{v}_1 = -\mathbf{v}_1$ and $T_{s_1} \mathbf{v}_2 = -\mathbf{v}_2$. Consider the case where $s_2 \in I_2 \setminus (I_1 \cap I_2)$. In this case we have $T_{s_2} \mathbf{v}_1 = 0$ and $T_{s_2} \mathbf{v}_2 = a_{s_2} \mathbf{v}_1 - \mathbf{v}_2$. In this case we have $(T_{s_i} T_{s_j})^m = -a_{s_2} \mathbf{v}_1 + \mathbf{v}_2$ and $T_{s_j} (T_{s_i} T_{s_j})^m = a_{s_1} \mathbf{v}_1 - \mathbf{v}_2$ and hence the braid relations are satisfied. Consider the case when $s_2 \in S_{\text{aff}} \setminus (I_1 \cup I_2)$. In this case we have $T_{s_2} \mathbf{v}_1 = 0$ and $T_{s_2} \mathbf{v}_2 = a_{s_2} \mathbf{v}_1$. Which gives the relations $T_{s_1} T_{s_2} \mathbf{v}_2 = T_{s_2} T_{s_1} \mathbf{v}_2 = -a_{s_2} \mathbf{v}_1$, $T_{s_2} T_{s_1} T_{s_2} \mathbf{v}_2 = 0$ and $T_{s_1} T_{s_2} T_{s_1} \mathbf{v}_2 = a_{s_2} \mathbf{v}_1$. By Hypothesis 3.1, (5) we get the braid relations in this case.

Finally we have to consider the case where $s_1 \in I_2 \setminus (I_2 \cap I_1)$ and $s_2 \in S_{\text{aff}} \setminus (I_1 \cup I_2)$. In this case we have $T_{s_1} \mathbf{v}_1 = 0$, $T_{s_1} \mathbf{v}_2 = a_{s_1} \mathbf{v}_1 - \mathbf{v}_2$, $T_{s_2} \mathbf{v}_1 = 0$ and $T_{s_2} \mathbf{v}_2 = a_{s_2} \mathbf{v}_1$. This shows that $T_{s_1} T_{s_2} \mathbf{v}_2 = 0$, $T_{s_2} T_{s_1} \mathbf{v}_2 = -a_{s_2} \mathbf{v}_1$ and $T_{s_1} T_{s_2} T_{s_1} \mathbf{v}_2 = 0$. By Hypothesis 3.1, (4) we get the braid relations in this case. \square

We will investigate the structure constants with respect to Baer sum. Let \mathfrak{n} and \mathfrak{m} be the \mathcal{H} modules ξ_{λ_1, I_1} and ξ_{λ_2, I_2} . Assume that $\lambda_1 \neq \lambda_2$. Fix a basis vectors $\mathbf{v}_1^0 \in \mathfrak{n}$ and $\mathbf{v}_2^0 \in \mathfrak{m}$. There is a canonical basis of E given by $p(\mathbf{v}_1^1)$ and $q(\mathbf{v}_2^1)$ and we denote them by \mathbf{v}_1 and \mathbf{v}_2 . Consider any two extensions $\mathfrak{n} \hookrightarrow E_1 \twoheadrightarrow \mathfrak{m}$ and $\mathfrak{n} \hookrightarrow E_2 \twoheadrightarrow \mathfrak{m}$ and the Baer sum $E_1 \dot{+} E_2$ is given by the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{n} \oplus \mathfrak{n} & \xrightarrow{p_1 \oplus p_2} & E_1 \oplus E_2 & \xrightarrow{q_1 \oplus q_2} & \mathfrak{m} \oplus \mathfrak{m} & \longrightarrow & 0 \\
& & \uparrow f_1 = \text{id} & & \uparrow f_2 & & \uparrow \Delta & & \\
0 & \longrightarrow & \mathfrak{n} \oplus \mathfrak{n} & \xrightarrow{p'} & E' & \xrightarrow{q'} & \mathfrak{m} & \longrightarrow & 0 \\
& & \downarrow \Sigma & & \downarrow g_2 & & \downarrow g_3 = \text{id} & & \\
0 & \longrightarrow & \mathfrak{n} & \xrightarrow{p_3} & E_1 \dot{+} E_2 & \xrightarrow{q_3} & \mathfrak{m} & \longrightarrow & 0
\end{array} \tag{3}$$

Here Δ and Σ are diagonal and the sum maps respectively. The two rows are pullback and push-out by Δ and Σ respectively.

We denote by a_s, a'_s and a''_s by the structure constants of E_1, E_2 and $E_1 \dot{+} E_2$ respectively. Let \mathbf{v}'_2 be a vector in E' pulled back via q' . Since $\lambda_1 \neq \lambda_2$ the vector \mathbf{v}'_2 is unique. Let $f_2(\mathbf{v}'_2) = (\mathbf{v}'_2 + \mathbf{v}_2^1)$ for \mathbf{v}_1^1 and \mathbf{v}_2^2 in the first and second summand of $E_1 \oplus E_2$. Let \mathbf{v}'_1 and \mathbf{v}''_1 be two vectors in each summand of $\mathfrak{n} \oplus \mathfrak{n}$. Now $T_s \mathbf{v}'_2 = b_s \mathbf{v}'_1 + d_s \mathbf{v}''_1 - \delta_{I_2}(s) \mathbf{v}_2$ and

$$f_2(T_s \mathbf{v}_2) = b_s f_2(\mathbf{v}'_1) + c_s f_2(\mathbf{v}''_1) - \delta_{I_2}(s) \mathbf{v}_2^1 - \delta_{I_2}(s) \mathbf{v}_2^2.$$

Comparing the action of $T_s \mathbf{v}_2^1$ and $T_s \mathbf{v}_2^2$ we get that $a_s = b_s$ and $a'_s = c_s$. Finally considering $g_2(T_s \mathbf{v}'_2) = (b_s + c_s) \mathbf{v}_1 - \delta_{I_2} \mathbf{v}_2$ we get that $a''_s = a_s + a'_s$. This shows that the map sending E to $(a_s)_{s \in S_{\text{aff}}}$ is injective and surjective onto functions $(a_s)_{s \in S_{\text{aff}}}$ satisfying the conditions of Lemmas 3.1 and 3.3.

Now we consider the case where $\lambda_1 = \lambda_2$. In this case there is no canonical basis of E stable under the action of T_t for $t \in Z_k$. If we choose a non-canonical basis then the structure constants are determined only upto translation by a certain function. Fix two vectors \mathbf{v}_1 and \mathbf{v}_2 of \mathfrak{n} and \mathfrak{m} respectively. Now choose a section $s : \mathfrak{m} \rightarrow E$ of the map q . Let \mathbf{v}' be the vector $s(\mathbf{v}_2)$. Now we note that $T_s \mathbf{v}' = a_s \mathbf{v}_1 - \delta_{I_2}(s) \mathbf{v}'$. Now if s' is another section of the map q then image of $s - s'$ is contained in \mathfrak{n} hence $s'(\mathbf{v}_2) = s(\mathbf{v}_2) + k \mathbf{v}_1$. Let $\mathbf{v}'' = s'(\mathbf{v}_2)$ and we have $T_s \mathbf{v}'' = (a_s - k \delta_{I_2}(s) + k \delta_{I_1}(s)) \mathbf{v}_1 - \delta_{I_2}(s) \mathbf{v}''$. Hence the map sending E to $s \mapsto a_s$ gives a map from $\text{Ext}_{\mathcal{H}}^1(\mathfrak{m}, \mathfrak{n})$ to functions on S_{aff} , denoted by $\bar{k}^{S_{\text{aff}}}$, modulo the function spanned by $\delta_{I_2} - \delta_{I_1}$. We denote by θ this map

$$\theta : \text{Ext}_{\mathcal{H}}^1(\xi_{\lambda_1, I_1}, \xi_{\lambda_2, I_2}) \rightarrow \frac{\bar{k}^{S_{\text{aff}}}}{\langle \delta_{I_2} - \delta_{I_1} \rangle}. \quad (4)$$

The map θ is non-canonical and depends on the choice of \mathbf{v}_1 and \mathbf{v}_2 , but these vectors are determined upto a scalar.

Lemma 3.5. *The map θ is a linear map and moreover is injective.*

Proof. Let us fix a section $s : \mathfrak{m} \rightarrow E'$ of the map q in (3). We also fix sections $s_i : \mathfrak{m} \rightarrow E_i$ of q_i . This also gives a section s_1 to the map q_3 . Let $\mathbf{v}'_2 = s(\mathbf{v}_2)$ and we denote by \mathbf{v}_2^1 and \mathbf{v}_2^2 vectors in each summand of $E_1 \oplus E_2$ such that $f_2(\mathbf{v}'_2) = \mathbf{v}_2^1 + \mathbf{v}_2^2$. Let \mathbf{v}'_1 and \mathbf{v}''_1 be two vectors in each summand of $\mathfrak{n} \oplus \mathfrak{n}$. Now $T_s \mathbf{v}'_2 = b_s p'(\mathbf{v}'_1) + d_s p'(\mathbf{v}''_1) - \delta_{I_2}(s) \mathbf{v}_2$ and

$$f_2(T_s \mathbf{v}_2) = b_s f_2(\mathbf{v}'_1) + c_s f_2(\mathbf{v}''_1) - \delta_{I_2}(s) \mathbf{v}_2^1 - \delta_{I_2}(s) \mathbf{v}_2^2.$$

Now the vectors \mathbf{v}_2^1 and \mathbf{v}_2^2 differ from $s_1(\mathbf{v}_2)$ and $s_2(\mathbf{v}_2)$ by $k_1 p_1(\mathbf{v}_1)$ and $k_2 p_2(\mathbf{v}_1)$. Comparing the action of $T_s \mathbf{v}_2^1$ and $T_s \mathbf{v}_2^2$ we get that $a_s = b_s + k_1(\delta_{I_2} - \delta_{I_1})$ and $a'_s = c_s + k_2(\delta_{I_2} - \delta_{I_1})$. This shows that $g_2(T_s \mathbf{v}'_2)$ is equal to the difference of $(b_s + c_s) \mathbf{v}_1 - \delta_{I_2} \mathbf{v}_2$ by $(k_1 + k_2)(\delta_{I_2} - \delta_{I_1})$. Hence the map θ is linear map. The injectivity is clear from the definition since the vanishing of the function $s \mapsto a_s$ for all $s \in S_{\text{aff}}$ implies E splits. \square

With this we are ready to state the main result of this article. We introduce some notations for the main results. Let $I(\lambda_1, \lambda_2)$ be the subset of S_{aff} such that $\lambda_1^s = \lambda_2$. Let $I(\lambda_1, I_2)$ be the intersection of $I(\lambda_1, \lambda_2)$ and

$$\{s \in S_{\text{aff}} \setminus (S_{\lambda_1} \cup I_2) \mid s \text{ does not satisfy the Hypothesis 3.1(3) (4) and (5)}\}.$$

Note that for any $s \notin I(\lambda_1, \lambda_2)$ we know that $a_s = 0$. Let $\delta_1 = 1$ if $I_1 \setminus (I_1 \cap I_2) \neq \emptyset$ and $I_1 \setminus (I_1 \cap I_2) \subset I(\lambda_1, \lambda_2)$ if otherwise, we set $\delta_1 = 0$. We set $\delta_2 = 1$ if $I_2 \subset S_{\lambda_1}$, $I_2 \not\subset I_1$ and $I_2 \setminus (I_1 \cap I_2) \subset I(\lambda_1, \lambda_2)$ if otherwise we set $\delta_2 = 0$.

Theorem 3.6. *Assume that $\lambda_1 \neq \lambda_2$ and $I_1 \neq I_2$ then the dimension of the space $\text{Ext}_{\mathcal{H}}^1(\xi_{\lambda_1, I_1}, \xi_{\lambda_2, I_2})$ is $|I(\lambda_1, I_2)| + \delta_1 + \delta_2$ if I_1 and I_2 does not satisfy the Hypothesis 3.1 (2) and is $\text{Ext}_{\mathcal{H}}^1(\xi_{\lambda_1, I_1}, \xi_{\lambda_2, I_2})$ is $|I(\lambda_1, I_2)| + \delta_1 + \delta_2 - 1$ otherwise. If $\lambda_1 \neq \lambda_2$ and $I_1 = I_2$ then the dimension of the space $\text{Ext}_{\mathcal{H}}^1(\xi_{\lambda_1, I_1}, \xi_{\lambda_2, I_2})$ is equal to $|I(\lambda_1, I_2)|$.*

Assume that $\lambda_1 = \lambda_2$ and $I_1 \neq I_1$ the dimension of the space $\text{Ext}_{\mathcal{H}}^1(\xi_{\lambda_1, I_1}, \xi_{\lambda_2, I_2})$ is equal to $|I(\lambda_1, I_2)| + \delta_1 + \delta_2 - 1$ if I_1 and I_2 does not satisfy the Hypothesis 3.1 (2) and $|I(\lambda_1, I_2)|$ otherwise. Now if $\lambda_1 = \lambda_2$ and $I_1 = I_2$ then the dimension of $\text{Ext}_{\mathcal{H}}^1(\xi_{\lambda_1, I_1}, \xi_{\lambda_2, I_2})$ is $|I(\lambda_1, I_2)|$.

Proof. If $\lambda_1 \neq \lambda_2$ then we have the map $E \mapsto (a_s)_{s \in S_{\text{aff}}}$ is injective linear map onto the image determined by Lemmas 3.1 and 3.3. If $\lambda_1 = \lambda_2$ then we use the map θ with the same conditions as in Lemmas 3.1 and 3.3 but now we have to quotient the image with span of the function $\delta_{I_2} - \delta_{I_1}$. \square

Corollary 3.7. Let λ_1 and λ_2 be two trivial characters and I_1 and I_2 are disjoint and they do not satisfy the Hypothesis 3.1 (2) then the dimension of $\text{Ext}_{\mathcal{H}}^1(\xi_{\text{id}, I_1}, \xi_{\text{id}, I_2})$ is 1.

Example 3.8. We verify this calculation for $\text{SL}_2(\mathbb{Q}_p)$. These results are proved by other methods in [Nad16]. Let χ_r be the character of Z_k sending $t \mapsto t^r$ for $0 \leq r < p - 1$. Let S_{aff} be the set $\{s_0, s_1\}$, the generators of the affine Weyl group W . Here $s_0 s_1$ has infinite order hence the only relevant conditions in Hypothesis 3.1 is the condition (1). We use the notation $\xi_{r, I}$ for the character $\xi_{\chi_r, I}$. The characters of the affine Hecke algebra are given by $\xi_{r, \emptyset}$ for $0 < r < p - 1$, ξ_{0, s_0} , ξ_{0, s_1} , $\xi_{0, \emptyset}$ and $\xi_{0, S_{\text{aff}}}$. The set of characters $\{\xi_{0, \emptyset}, \xi_{0, S_{\text{aff}}}\}$ are not supersingular and rest of the characters are supersingular. We first consider the regular case ($r \neq 0$). Consider the case when $\mathfrak{m} = \xi_{r_1, \emptyset}$ and $\mathfrak{n} = \xi_{r_2, \emptyset}$. The set $I(\chi_{r_1}, \chi_{r_2}) \neq \emptyset$ if and only if $r_1 + r_2 = p - 1$. In which case $I(\chi_{r_1}, \chi_{r_2}) = S_{\text{aff}}$. We may and do assume that $0 < r_i = (p - 1)/2$ for $i \in \{1, 2\}$. The sets $S_{\lambda_1} = I_1 = I_2 = S_{\lambda_2} = \emptyset$. Hence $I(S_{\lambda_1}, I_{\lambda_2}) = S_{\text{aff}}$. This shows that the space of extensions $\text{Ext}_{\mathcal{H}}^1(\xi_{r_1, \emptyset}, \xi_{p-1-r_2, \emptyset})$ has dimension 2. If $r_1 = r_2 = 1$ we have $I_1 = I_2$ and $\lambda_1 = \lambda_2$ case of 3.6 and hence the dimension of the space $\text{Ext}_{\mathcal{H}}^1(\xi_{r_1, \emptyset}, \xi_{p-1-r_2, \emptyset})$ has dimension 2.

Consider the case when $r_1 = 0$ (the Iwahori–case) then the set $I(\chi_{r_1}, \chi_{r_2}) \neq \emptyset$ if and only if $r_2 \in \{p - 1, 0\}$. We may assume that $r_2 = 0$. In this case the set $I(\chi_0, \chi_0)$ is S_{aff} . The set $S_{\chi_0} = S_{\text{aff}}$. Now consider the case when $\mathfrak{m} = \xi_{\chi_0, s_i}$ and $\mathfrak{n} = \xi_{\chi_0, s_j}$. The set $I(\chi_0, \{s_j\}) = \emptyset$ and note that $\delta_1 = 1$ and $\delta_2 = 1$ if $s_i \neq s_j$. If $s_i = s_j$ then $\delta_1 = 0$ and $\delta_2 = 0$ since $I_2 \not\subset I_1$ condition is not satisfied. This shows that the dimension of the space $\text{Ext}_{\mathcal{H}}^1(\xi_{0, s_i}, \xi_{0, s_j}) = 1$ if $i \neq j$ and is zero otherwise.

Remark 3.9. In the case of SL_n the L -packets are defined by Koziol as conjugation by $\text{PGL}_n(K)$ (see [Koz16b, Definition 6.4]). For $n = 3$ the Hypothesis 3.1 (2) is not relevant. If ξ_{λ_1, I_1} and ξ_{λ_2, I_2} are in the same L -packets then the sets $|I_1| = |I_2|$. Now consider the simple example for SL_3 the set $S^{\text{aff}} = \{s_1, s_2, s_3\}$ and assume $I_1 = \{s_1\}$ and $I_2 = \{s_2, s_3\}$. The above corollary shows that extensions exist among distinct L -packets. The notion of blocks and L -packets in the supersingular case of higher rank groups are different and the relationship is not clear in the higher rank cases.

4 Blocks for unramified unitary groups in 2 and 3 variables.

As an application we deduce the extensions of simple supersingular \mathcal{H} modules of unramified groups $U(2, 1)$ and $U(1, 1)$. In these cases we will try to precisely understand the relation between extensions and L -packets. The Iwahori–Hecke module structure of $U(2, 1)$ and $U(1, 1)$ are studied by Abdellatif, Koziol-Xu and Koziol in the articles (see [Abd11], [KX15] and [Koz16a]).

Let L be a unramified quadratic extension of K and (W, h) be a pair consisting of a 3-dimensional vector space W over L and h be a non-degenerate hermitian form on W . We denote by k_L the residue field of L which is a quadratic extension of k . Let \mathbf{G} be the isometry group scheme over K associated to the pair (W, h) . In this case the maximal K -split torus \mathbf{T} is isomorphic to (\mathbb{G}_m/K) . The normaliser of \mathbf{Z} of \mathbf{T} is isomorphic to $\text{Res}_{L/K} \mathbb{G}_m \times \mathbf{U}(1)(L/K)$. The group \mathbf{Z}_k is isomorphic to $\text{Res}_{k_L/k} \mathbb{G}_m \times \mathbf{U}(1)(k_L/k)$ such that the determinant map is the second projection $\mathbf{Z}_k \rightarrow \mathbf{U}(1)(k_L/k)$. Let us fix a chamber C and the set $S^{\text{aff}} = \{s_1, s_2\}$ where s_1 and s_2 are two affine reflections in the walls of C . The order of $s_1 s_2$ is infinite and hence the relevant conditions in Hypothesis 3.1 is the condition (1).

For quadratic relations we need to describe the groups Z_{k, s_1} and Z_{k, s_2} . By abuse of notation we identify the faces fixed by s_i with s_i . With out loss of generality we assume that $\mathbf{G}_{s_1, k}$ is isomorphic to $\mathbf{U}(2, 1)(k_L/k)$ and $\mathbf{G}_{s_2, k}$ is isomorphic to $\mathbf{U}(1, 1)(k_L/k) \times U(1)(k_L/k)$. This shows that the group $\langle \mathbf{U}_{s_1, k}, \overline{\mathbf{U}}_{s_1, k} \rangle \cap \mathbf{Z}_k$ is isomorphic to $\text{Res}_{k_L/k} \mathbb{G}_m$ and $\langle \mathbf{U}_{s_2, k}, \overline{\mathbf{U}}_{s_2, k} \rangle \cap \mathbf{Z}_k$ is isomorphic to \mathbb{G}_m/k . The group \mathbf{Z}_{k, s_2} embeds in \mathbf{Z}_k and is isomorphic to the first factor. Similarly the group \mathbf{Z}_{k, s_2} also embeds into the first factor of \mathbf{Z}_k . The groups $Z_{s_0, k} \simeq k_L^\times$ and $Z_{s_1, k} \simeq k^\times$. Since Λ is commutative the group $W(1)$ acts on Z_k by the quotient $W(1) \rightarrow W_0 \simeq \{\text{id}, s_1\}$.

Let $\zeta : k_E^\times \rightarrow \bar{k}$ and $\eta : U(1) \rightarrow \bar{k}$ be any two characters then we denote by χ the character $\zeta \otimes \eta$. Let $x \mapsto \bar{x}$ be the nontrivial Galois automorphism on k_L . The character χ^{s_1} is given by $\bar{\zeta} \otimes \eta$ where $\bar{\zeta}(x) = \zeta(\bar{x}^{-1})$. Note that the character $\chi = \zeta \otimes \eta$ is trivial on Z_{k, s_1} if and only if ζ is trivial and χ is trivial on Z_{k, s_2} if and only if $\zeta^{q+1} = \text{id}$. These cases are called as trivial-Iwahori and hybrid respectively by Koziol–Xu in these cases $\chi^{s_1} = \chi$. If χ is non-trivial on Z_{k, s_1} and Z_{k, s_2} then the character χ is called as regular and $\chi^{s_1} \neq \chi$. Now we list the various characters of $\mathcal{H} = \mathcal{H}^{\text{aff}}$. We use the description of supersingular characters given by Vigneras but we point out that these are also described by Koziol–Xu.

1. If $\chi = \zeta \otimes \eta$ is a trivial-Iwahori type then we have $S_\chi = S^{\text{aff}}$. So we have two supersingular characters ξ_{χ, s_1} and ξ_{χ, s_2} . The characters $\xi_{\chi, S^{\text{aff}}}$ and $\xi_{\chi, \emptyset}$ are not supersingular.
2. If $\chi = \zeta \otimes \eta$ is hybrid then $S_\chi = \{s_2\}$ and in this case we have two supersingular characters ξ_{χ, s_2} and $\xi_{\chi, \emptyset}$,
3. If $\chi = \zeta \otimes \eta$ is a regular character then we have only one supersingular character $\xi_{\chi, \emptyset}$,

Note that if χ is a trivial or hybrid type character then $I(\chi, \chi')$ is not an empty set if and only if $\chi = \chi'$.

Proposition 4.1. *Let χ be a trivial character then the dimension of $\text{Ext}_{\mathcal{H}}^1(\xi_{\chi, s_i}, \xi_{\chi, s_j})$ is 1 if $i \neq j$ and is 0 otherwise. If χ is a hybrid character then the dimension of $\text{Ext}_{\mathcal{H}}^1(\xi_{\chi, I_1}, \xi_{\chi, I_2})$ is 1 for all $I_1, I_2 \subset \{s_2\}$. If χ is regular the dimension of the space $\text{Ext}_{\mathcal{H}}^1(\xi_{\chi, \emptyset}, \xi_{\chi', \emptyset})$ is 2 when $\chi^{s_0} = \chi'$ and zero otherwise.*

Proof. Let ξ_{λ_1, I_1} and ξ_{λ_2, I_2} be the characters ξ_{χ, s_i} and ξ_{χ, s_j} respectively. We observed that $I(\chi, \chi) = S^{\text{aff}}$ and we have $I(\chi, \{s_i\}) = \emptyset$. If $i \neq j$ then we have $I_2 \not\subset I_1$ and hence $\delta_2 = 1$ and $I_1 \setminus (I_1 \cap I_2)$ is nonempty which gives $\delta_1 = 1$. If $i = j$ we have $\delta_1 = 0$. The subsets $I_1 = I_2$ hence $\delta_2 = 0$. Applying Theorem 3.6 we get that the dimension of the extension space $\text{Ext}_{\mathcal{H}}^1(\xi_{\chi, s_i}, \xi_{\chi, s_j})$ is 1 if $i \neq j$ and the dimension of $\text{Ext}_{\mathcal{H}}^1(\xi_{\chi, s_i}, \xi_{\chi, s_i})$ is 0.

Assume that χ is hybrid character. Let ξ_{λ_1, I_1} and ξ_{λ_2, I_2} be the characters ξ_{χ, s_2} and $\xi_{\chi, \emptyset}$ respectively. We note that $I(\lambda_1, \lambda_2) = S^{\text{aff}}$, the set $I(\lambda_1, I_2) = \{s_1\}$. Now if $I_1 = \{s_2\}$ and $I_2 = \{s_2\}$ we

have $\delta_1 = 0$ and $\delta_2 = 0$. The dimension of the space $\text{Ext}_{\mathcal{H}}^1(\xi_{\chi,s_2}, \xi_{\chi,s_2})$ is 1. Assume that $I_1 = \{\emptyset\}$ and $I_2 = \{s_2\}$ then $\delta_1 = 0$ and $\delta_2 = 1$ and hence the dimension of $\text{Ext}_{\mathcal{H}}^1(\xi_{\chi,s_2}, \xi_{\chi,\emptyset})$ is 1. If $I_1 = \{s_2\}$ and $I_2 = \{\emptyset\}$ then we have $\delta_1 = 1$ and $\delta_2 = 0$ hence the dimension of the space $\text{Ext}_{\mathcal{H}}^1(\xi_{\chi,s_2}, \xi_{\chi,\emptyset})$ is 1. Finally assume that $I_1 = I_2 = \emptyset$ in this case $\delta_1 = 0$ and $\delta_2 = \emptyset$ the dimension of $\text{Ext}_{\mathcal{H}}^1(\xi_{\chi,\emptyset}, \xi_{\chi,\emptyset})$ is 1.

Finally we consider the case when χ is regular. Assume that ξ_{λ_1, I_1} and ξ_{λ_2, I_2} be the characters $\xi_{\lambda_1, \emptyset}$ and $\xi_{\lambda_2, \emptyset}$ and assume that $I(\lambda_1, \lambda_2) \neq \emptyset$. In this case we have $S_{\lambda_1} = I_1 = I_2 = S_{\lambda_2} = \emptyset$ and $|I(\lambda_1, I_2)| = 2$. Moreover we have $\delta_1 = 0$ and $\delta_2 = 0$ and the dimension of the space $\text{Ext}_{\mathcal{H}}^1(\xi_{\lambda_1, \emptyset}, \xi_{\lambda_2, \emptyset})$ is 2. \square

Now we will consider the case of unitary group $U(1, 1)(L/K)$ where L is unramified over K . Let (W, h) be a pair consisting of a 2 dimensional vector space over L and h be a non-degenerate hermitian form on W . Let $\mathbf{U}(1, 1)$ be the unitary group scheme over F attached to (W, h) . In this case the maximal K -split torus \mathbf{T} is isomorphic to \mathbb{G}_m/K , its normalier is isomorphic to $\text{Res}_{L/K} \mathbb{G}_m$. Hence the group \mathbf{Z}_k is isomorphic to $\text{Res}_{k_L/k} \mathbb{G}_m$. Lets fix a chamber C and the set S^{aff} is given by $\{s_1, s_2\}$. The group schemes $\mathbf{G}_{k,s} \simeq \mathbf{U}(1, 1)(k_L/k)$ for $s \in \{s_1, s_2\}$ as s_1 and s_2 are conjugate in $GU(1, 1)$. This shows that \mathbf{Z}_{k,s_0} and \mathbf{Z}_{k,s_1} are both isomorphic to \mathbb{G}_m/k as $\langle \mathbf{U}_{k,s}, \overline{\mathbf{U}}_{k,s} \rangle = \text{SL}_2/k$.

Now consider any character χ on k_E^\times and the character χ is trivial on Z_{k,s_i} if and only if $\chi^{q+1} = \text{id}$. Following the above case we use the terminology that χ is trivial type if χ is trivial and hybrid type if $\chi^{q+1} = 1$ and $\chi \neq \text{id}$. We also note that $\chi^{s_1} = \chi$ if and only if $\chi^{q+1} = \text{id}$. Now the characters of the algebra $\mathcal{H} = \mathcal{H}^{\text{aff}}$ are given by

1. If χ is either trivial or hybrid character then $S_\chi = S^{\text{aff}}$ and ξ_{χ,s_1} and ξ_{χ,s_2} are supersingular characters. The characters $\xi_{\chi,\emptyset}$ and $\xi_{\chi,S^{\text{aff}}}$ are not supersingular characters.
2. If χ is a regular character then $S_\chi = \emptyset$ and the only characters of \mathcal{H} are $\xi_{\chi,\emptyset}$.

To begin with the calculation of extensions, for a character $\chi : k_L^\times \rightarrow R^\times$ such that $\chi^{q+1} = \text{id}$, the set $I(\chi, \chi') \neq \emptyset$ if and only if $(\chi')^{q+1} = \text{id}$.

Proposition 4.2. *Let χ be a character such that $\chi^{q+1} = \text{id}$ then the dimension of the space $\text{Ext}_{\mathcal{H}}^1(\xi_{\chi,s_i}, \xi_{\chi,s_j})$ is 1 if $i \neq j$ and is zero otherwise. If χ is a regular character then the dimension of the space $\text{Ext}_{\mathcal{H}}^1(\xi_{\chi,\emptyset}, \xi_{\chi',\emptyset})$ is 2 if $\chi' = \chi^{s_1}$ and is trivial otherwise.*

Proof. This situation is similar to SL_2 . Assume that $\chi^{q+1} = \text{id}$ and ξ_{λ_1, I_1} and ξ_{λ_2, I_2} be the characters ξ_{χ,s_i} and ξ_{χ,s_j} . In this case we have $I(\lambda_1, I_2) = S^{\text{aff}}$ has cardinality two. If $i \neq j$ we have $\delta_1 = 1$ and $\delta_2 = 1$ and since $I_1 \neq I_2$ we get that the dimension of $\text{Ext}_{\mathcal{H}}^1(\xi_{\chi,s_i}, \xi_{\chi,s_j})$ is 1 if $i \neq j$ and is zero otherwise. Assume that χ and ξ_{λ_1, I_1} and ξ_{λ_2, I_2} be the characters $\xi_{\chi,\emptyset}$ and $\xi_{\chi^{s_1}, \emptyset}$. The set $I(\lambda_1, \lambda_2) \neq \emptyset$ if and only if $\lambda_1 = \lambda_2^{s_1}$. If $I(\lambda_1, \lambda_2) \neq \emptyset$ then we have $I(\lambda_1, \lambda_2) = S^{\text{aff}}$. The sets $I(\lambda_1, \lambda_2) = \emptyset$, $S_{\lambda_1} = I_1 = I_2 = S_{\lambda_2} = \emptyset$. This shows that $\delta_1 = 0$ and $\delta_2 = 0$. The set $I(\lambda_1, I_2)$ has cardinality 2. This shows that $\text{Ext}_{\mathcal{H}}^1(\xi_{\chi,\emptyset}, \xi_{\chi^{s_1}, \emptyset})$ has dimension 2. \square

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