Indian Institute of Technology Kanpur
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Abstract Algebra (MTH 204A/B)

Exercise Set 1

(1) Show that the collection all continuous endomorphisms of the group $(\mathbb{C}, +)$ forms a vector space over $\mathbb{R}$ naturally. Find the dimension of this vector space.

(2) Find all continuous homomorphisms from $\mathbb{R}$ to $\mathbb{R} \setminus \{0\}$.

(3) Suppose that $\varphi : \mathbb{R} \longrightarrow \mathbb{S}^1$ is a continuous homomorphism. Show the following:

   (a) $\exists \varepsilon > 0$ such that $Re(z) > 0$ for every $z \in \varphi([-\varepsilon, \varepsilon])$,

   (b) $\exists y \in [-\frac{1}{4\varepsilon}, \frac{1}{4\varepsilon}]$ such that $\varphi(\varepsilon) = e^{2\pi i y \varepsilon}$,

   (c) $\forall r \in \mathbb{Z}[\frac{1}{2}] := \left\{ \frac{p}{2^n} : p, n \in \mathbb{Z} \text{ and } n \geq 0 \right\}$, one has $\varphi(r\varepsilon) = e^{2\pi i y r \varepsilon}$.

   (d) $\forall x \in \mathbb{R}$, $\varphi(x) = e^{2\pi i y x}$.

   Hence, find all continuous homomorphisms from $\mathbb{R}$ to $\mathbb{S}^1$.

(4) Using the previous exercise or otherwise, find all continuous homomorphisms from $\mathbb{S}^1$ to $\mathbb{S}^1$.

(5) Find all continuous homomorphisms from $\mathbb{R}$ to $\mathbb{C} \setminus \{0\}$.

(6) For $n \in \mathbb{N}$, let S_n denote the symmetric group of $\{1, 2. \cdots, n\}$.

   (a) Show that S_n is generated by the following subsets:

$$\{(1\,2), (1\,3), \cdots, (1\,n)\} \text{ and } \{(i\ i+1) : 1 \leq i \leq n-1\}.$$

   (b) Let H be a subgroup of S_n such that the action of H (by permutation) on $\{1, 2. \cdots, n\}$ is transitive, i.e., for any $i, j \in \{1, \ldots, n\}, \exists \sigma \in H$ such that $\sigma(i) = j$. If H is generated by transpositions then show that $H = S_n$.

(7) Show that, for $n, m \geq 3$, there is an injective homomorphism from D_{2n} to D_{2m} if and only if $n|m$.

(8) Classify all finite subgroups of $O_2(\mathbb{R})$.

(9) Let H be a group with the following two properties:

   (a) all automorphisms of H are inner, and

   (b) $Z(H) = \{1\}$.

   Then show that there exists a group G such that $G' = H$ if and only if $H' = H$. (Hint: For $H \leq G$, the subgroup $\{g \in G : ghg^{-1} = h, \forall h \in H\}$ might be useful).

(10) Let G be a group. Show the following:

   (a) $G' \leq H \Longrightarrow H \trianglelefteq G$.

   (b) If G is solvable and $\{1\} \neq H \trianglelefteq G$ then there exists a nontrivial abelian $A \leq H$ such $A \trianglelefteq G$.

(11) Show that every subgroup and homomorphic image of a nilpotent group must be nilpotent.

(12) (a) Let G be a group and $H \leq G$ have finite index in G. Show that there exists $N \trianglelefteq G$ such that $N \leq H$ and $[G : N]$ is finite.

   (b) Let G be a group and $H_1, H_2 \leq G$ have finite index. Then show that $[G : H_1 \cap H_2]$ is finite.