Indian Institute of Technology Kanpur
Department of Mathematics and Statistics

Abstract Algebra (MTH 204A/B)

Exercise Set 2

(1) Let $\mathbb{F} = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$. $A \in M_n(\mathbb{F})$, where $n \in \mathbb{N}$ is said to be *unipotent* if its all eigenvalues are 1. Consider the following collection consisting of all 2×2 unipotent upper and lower triangular matrices:

$$\left\{ A \in SL_2(\mathbb{F}) : A = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \text{ for some } b \in \mathbb{F} \right\}. \qquad (* \ 1)$$

   (a) Let $a \in \mathbb{F} \setminus \{1\}$. Show that the matrix $\mathrm{diag}(a, a^{-1})$ can be written in the following form:
   $$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix},$$
   for some x, b, c and $d \in \mathbb{F}$. (Hint: It is straightforward to select some $x \neq 0$ and then solve for b, c and d successively.)

   (b) Show that the subset given above in ($*$ 1) generates $SL_2(\mathbb{F})$. (Hint: Observe that every $A \in SL_2(\mathbb{F})$ can be brought to the form $\mathrm{diag}(a, a^{-1})$ by row and column operations).

(2) Let the set up be as in (1). For $a \in \mathbb{F} \setminus \{0\}$ and $b \in \mathbb{F}$, denote $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ and $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ by $s(a)$ and $u(b)$.

   (a) $\forall a \in \mathbb{F} \setminus \{0\}$ and $b \in \mathbb{F}$, compute $s(a)u(b)s(a)^{-1}u(b)^{-1}$.

   (b) Let $U := \{u(b) : b \in \mathbb{F}\}$. Show that $B' = U$, where B is the standard Borel subgroup of $SL_2(\mathbb{F})$, i.e., the subgroup that consists of all upper triangular 2×2 matrices having determinant 1.

   (c) Let $w := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Find wUw^{-1} and wBw^{-1}.

   (d) Show that $SL_2(\mathbb{F})' = SL_2(\mathbb{F})$.

(3) Consider the action of $SO_3(\mathbb{R})$ on itself by conjugation. Show that the intersection of every orbit with the following set is singleton:

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} : 0 \leq \theta \leq \pi \right\}.$$

(4) Let $\mathbb{R}[x_1, x_2, x_3, x_4]$ denote the set of all polynomials in $x_1, \cdots, x_4$ with real coefficients. Show that the following defines an action of S_4 on $\mathbb{R}[x_1, x_2, x_3, x_4]$:

$$\sigma \cdot f(x_1, x_2, x_3, x_4) := f(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}),$$

for all $\sigma \in S_4$ and polynomial $f(x_1, x_2, x_3, x_4) \in \mathbb{R}[x_1, x_2, x_3, x_4]$.

   (a) Describe the polynomials which are fixed by every permutation in S_4.

   (b) Identify the stabilizers of the polynomials $x_1x_2 + x_3x_4$ and $(x_1 + x_2)(x_3 + x_4)$.

   (c) Find the orbits of the polynomials mentioned in (4b).

(5) Let $G \curvearrowright X$. For $k \in \mathbb{N}$, the action is said to be *k-transitive* if for any two k tuples of <u>distinct</u> elements $(x_1, x_2, \cdots, x_k)$ and $(y_1, y_2, \cdots, y_k)$, there is $g \in G$ such that $gx_i = y_i$, for all $i = 1, 2, \cdots, k$.

   (a) Show that, for $n \geq 3$ the action of A_n on $\{1, 2, \cdots, n\}$ is $(n-2)$ transitive.

   (b) Show that, for $n \geq 4$, the action of the Dihedral group of order $2n$ on the vertices of a regular n-gon is not doubly transitive.

   (c) Find the largest $k \in \mathbb{N}$ such that the action of $GL_n(\mathbb{R})$ on $\mathbb{R}^n \setminus \{\mathbf{0}\}$ is k-transitive.

   (d) Suppose $G \curvearrowright X$ doubly transitively. Then show that any normal subgroup $N \trianglelefteq G$ will act on X either trivially or transitively.

(6) Let X be a set with at least 2 elements and $G \curvearrowright X$. Define an action of G on $X \times X$ by the following:

$$g \cdot (x, y) := (gx, gy), \text{ for all } g \in G \text{ and } x, y \in X. \tag{$*$ 2}$$

Show that $G \curvearrowright X$ is 2-transitive if and only if it is transitive and the action defined above in ($*$ 2) has exactly two orbits.

(7) Let X be a transitive G-space and H is the stabilizer of a point $y \in X$ in G. An equivalence relation $\sim$ on X is said to be a *G-equivalence relation* if it satisfies the following property: whenever $x, x' \in X$ are such that $x \sim x'$, one has $gx \sim gx'$ for all $g \in G$.

(a) Show that if $\sim$ is a G-equivalence relation on X then $K := \{g \in G : gy \sim y\}$ is a subgroup of G containing H.

(b) Show that, in that situation mentioned in (7a), every equivalence class in $\sim$ has the cardinality $[K : H]$.

(c) Suppose that K is a subgroup of G containing H. Show that there is a G-equivalence relation $\sim$ on X such that $K := \{g \in G : gy \sim y\}$.

(d) Hence conclude that the G-equivalence relations on X are in a bijective correspondence with the subgroups K of G such that $H \subseteq K \subseteq G$. Find the G-equivalence relations corresponding to the subgroups G and H.

(8) (a) Construct $\sigma \in Aut\,(\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z})$ such that $\sigma^2 = -I$, where I stands for the identity automorphism of $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

(b) Let σ be as in (8a). Fix a generator, say x, of $\mathbb{Z}/4\mathbb{Z}$ and consider the homomorphism $\varphi : \mathbb{Z}/4\mathbb{Z} \longrightarrow Aut(\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z})$ that sends x to σ. Denote the semidirect product $(\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}) \rtimes_\varphi \mathbb{Z}/4\mathbb{Z}$ by G and the copy of $\mathbb{Z}/4\mathbb{Z}$ in G by L. Show that the number of elements of order 2 in G can not exceed the number of conjugates of L.

(c) Show that the conjugacy class of x^2 in G contains precisely all elements of order 2 of G.

(d) Show that G does not have an element of order 6.

(9) Let $\mathbb{F}$ be as in (1) and $n \in \mathbb{N}$. Denote by B the standard Borel subgroup B of $GL_n(\mathbb{F})$, i.e., the subgroup consisting of all invertible upper triangular matrices.

(a) Show that $B \simeq U \rtimes_\varphi A$, for some homomorphism φ, where U is the subgroup of all unipotent upper triangular matrices and A is the subgroup of all invertible diagonal matrices.

(b)* Can the above observation be generalized to any parabolic subgroup of $GL_n(\mathbb{F})$?

(c) If now B denotes the standard Borel subgroup in $SL_n(\mathbb{F})$, i.e., the subgroup consisting of all invertible upper triangular matrices with determinant 1, U and A are the subgroups of B of all unipotent matrices in B and all diagonal matrices in B respectively, then is it still true in this case that $B \simeq U \rtimes_\varphi A$, for some homomorphism φ?