Indian Institute of Technology Kanpur
Department of Mathematics and Statistics
Abstract Algebra (MTH 204A/B)
Exercise Set 3

(1) Let G be a finite group and p be a prime such that $p|n := |G|$. Consider the natural action of $\langle (1\,2\,\cdots\,p)\rangle$ on the following set:
$$\mathscr{S} := \{(g_1, g_2, \ldots, g_p) \in G^p : g_1g_2\ldots g_p = 1 \text{ and } (g_1, g_2, \cdots, g_p) \neq (1, 1, \cdots, 1)\},$$
i.e., by
$$\tau \cdot (g_1, g_2, \cdots, g_p) := (g_{\tau^{-1}(1)}, g_{\tau^{-1}(2)}, \cdots, g_{\tau^{-1}(p)}), \forall \tau \in \langle (1\,2\,\cdots\,p)\rangle, (g_1, g_2, \cdots, g_p) \in G^p.$$
Show that this action has a fixed point. Deduce the *Cauchy's theorem* that says G has an element of order p.

(2) Let $M \subseteq \mathbb{C} \setminus \{0\} \times \mathbb{C} \setminus \{0\}$ consisting of all pairs (ω_1, ω_2) such that $Im\left(\frac{\omega_1}{\omega_2}\right) > 0$. For each $(\omega_1, \omega_2) \in M$, we associate a lattice $\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ and denote that by $\Gamma(\omega_1, \omega_2)$. The set of all latices in $\mathbb{C}$ (viewed as a vector space over $\mathbb{R}$) is denoted by $\mathscr{R}$.

(a) Show that the map $M \longrightarrow \mathscr{R}$, $(\omega_1, \omega_2) \mapsto \Gamma(\omega_1, \omega_2)$, for all $(\omega_1, \omega_2) \in M$, is surjective.
(b) Show that $SL_2(\mathbb{Z})$ acts on M in the following way:
$$g \cdot (\omega_1, \omega_2) := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}, \text{ for all } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \text{ and } (\omega_1, \omega_2) \in M.$$
(c) Show that two pairs (ω_1, ω_2) and (ω_1', ω_2') in M are in the same $SL_2(\mathbb{Z})$ orbit of the action defined in (2b) if and only if $\Gamma(\omega_1, \omega_2) = \Gamma(\omega_1', \omega_2')$. Conclude that $\mathscr{R}$ can be identified to the orbit space $SL_2(\mathbb{Z})\diagdown M$.
(d) Show that the multiplicative group $\mathbb{C} \setminus \{0\}$ acts on $\mathscr{R}$ in the following way:
$$(\lambda, \Gamma) \mapsto \lambda\Gamma, \text{ for all } \lambda \in \mathbb{C} \setminus \{0\} \text{ and } \Gamma \in \mathscr{R}.$$
(e) Show that there is a bijective map between the following orbit spaces: $(\mathbb{C} \setminus \{0\})\diagdown\mathscr{R}$ and $PSL_2(\mathbb{Z})\diagdown\mathbb{H}$, where $PSL_2(\mathbb{Z})$ is the quotient group $SL_2(\mathbb{Z})/\{\pm I_2\}$.

(3) Let G be a finite and nontrivial group. Show the following:
(a) The natural action of $Aut(G)$ on $G \setminus \{1\}$ is transitive if and only if $G \cong (\mathbb{Z}/p\mathbb{Z})^n$ for some prime p and $n \in \mathbb{N}$.
(b) The natural action of $Aut(G)$ on $G \setminus \{1\}$ is 2-transitive if and only if $G \cong (\mathbb{Z}/2\mathbb{Z})^n$, for some $n \in \mathbb{N}$, or $G \cong \mathbb{Z}/3\mathbb{Z}$.
(c) The natural action of $Aut(G)$ on $G \setminus \{1\}$ is 3-transitive if and only if $G \cong (\mathbb{Z}/2\mathbb{Z})^2$.

(4) (a)$*$ For any field $\mathbb{F}$ and $n \in \mathbb{N}$, let U stand for the multiplicative group of all upper triangular $n \times n$ matrices over $\mathbb{F}$ with all diagnoal entries 1. Find the normalizer of U in $GL_n(\mathbb{F})$.
(b) Find the number of Sylow p-subgroups of $GL_n(\mathbb{Z}/p\mathbb{Z})$, where p is a prime.

(5) (a) For any field $\mathbb{F}$ and $n \in \mathbb{N}$, find $Z(GL_n(\mathbb{F}))$ and $Z(SL_n(\mathbb{F}))$.
(b) Identify the quotient group $SL_2(\mathbb{Z}/3\mathbb{Z})/Z(SL_2(\mathbb{Z}/3\mathbb{Z}))$.

(6) Show that every group of order 108 has a normal subgroup of order 9 or 27.

(7) (a) Construct an isomorphism from $Aut(S_4)$ to S_4.
(b) Show that all automorphisms of S_4 are inner.

(8)$*$ (a) Construct an injective group homomorphism $\varphi : S_5 \longrightarrow S_6$ such that the natural action of $\varphi(S_5)$ on $\{1, 2, 3, 4, 5, 6\}$ is transitive.
(b) Construct an automorphism of S_6 which sends the subgroup $\varphi(S_5)$ to a conjugate of S_5 in S_6.
(c) Show that there exists an automorphism of S_6 which is not inner.

(9) Find the values of $n \in \mathbb{N}$ for which D_{2n} is nilpotent.

(10) Let G be a nontrivial nilpotent group. Prove the following:

(a) $\{1\} \neq N \trianglelefteq G \Longrightarrow N \cap Z(G) \neq \{1\}$. In particular, every minimal normal subgroup lies in the center and has prime order. (Hint: Look at the least n such that $N \cap Z_n(G) \neq \{1\}$.)

(b) Every maximal subgroup is normal with prime index and contains G'.

(c) If $|G| < \infty$ then for any prime $p||G|$, there is a minimal normal subgroup of G of order p and a maximal normal subgroup of G of index p.

(d)$*$ If $H \leq G$ and $[G : H] = n \in \mathbb{N}$ then $g^n \in H$, for all $g \in G$. (Hint: $H \trianglelefteq N_G(H) \trianglelefteq N_G(N_G(H)) \trianglelefteq \ldots$ must stabilize with G.)

(11) Suppose that G be a finite solvable group.

(a) Show that if $\{1\} \neq N$ is a *minimal* normal subgroup of G, i.e., it contains no normal subgroup of G apart from $\{1\}$ and N, then N must be an abelian p-group for some prime p.

(b)$*$ Show that if H is a *maximal* proper subgroup of G, i.e., there is no proper $K \leq G$ with $K \supsetneq H$, then the index of H in G must be a power of some prime number. (Hint: Prove by induction on $|G|$. You can pick a minimal normal subgroup N of G, if necessary so that $|\frac{G}{N}| < |G|$. Now consider the two cases $N \leq H$ and $N \nleq H$ separately.)

(12) (a) Fix a prime p and let $\{H_1, \ldots, H_n\}$ and $\{K_1, \ldots, K_m\}$ be two collections of cyclic p-groups. Assume that $m > n$. Prove of disprove the following:

There exists an onto homomorphism from $H_1 \times \cdots \times H_n$ to $K_1 \times \cdots \times K_m$.

(b)$**$ Show that, if G is a finite abelian group and $H \leq G$ then, G contains a subgroup isomorphic to G/H.