Indian Institute of Technology Kanpur
Department of Mathematics and Statistics

Abstract Algebra (MTH 204A/B)

Exercise Set 4

(1) Let D be a square free integer. Show that, $a + b\sqrt{D} \in \mathbb{Z}[\sqrt{D}]$ is a unit if and only if $|a^2 - Db^2| = 1$. Find all units in each of the following rings:
  (a) $\mathbb{Z}[\sqrt{2}]$.
  (b) $\mathbb{Z}[\sqrt{5}]$.
  (c) $\mathbb{Z}[\sqrt{7}]$.
  (d) $\mathbb{Z}[\sqrt{-D}]$, where D is a positive square free integer.
  (e) $\mathbb{Q}[\sqrt{D}]$, where D is a square free integer.

(2) Let A be a commutative ring. An element $x \in A$ is said to be *nilpotent* if $x^n = 0$, for some $n \in \mathbb{N}$. Show the following:
  (a) $\{x \in A : x \text{ is nilpotent}\}$ is an ideal (this is called the *nilradical* of A), and the quotient ring does not have a nonzero nilpotent element.
  (b) $\{1 + x : x \text{ is nilpotent}\}$ is a subgroup of the multiplicative group of all units in A.

(3)∗ Let A be as above in (2). Show that the nilradical is the intersection of all the prime ideals of A. (Hint: Suppose that $a \in A$ is not nilpotent. Consider the collection of all ideals I with the property that $a^n \notin I$, for all $n \in \mathbb{N}$. By Zorn's lemma, pick a maximal such ideal, say $\mathfrak{p}$. Now show that this $\mathfrak{p}$ is a prime ideal.)

(4) Let A be as above in (2). The *Jacobson radical* $\mathfrak{R}$ of A is defined to be the intersection of all the maximal ideals of A. Show that, $x \in \mathfrak{R} \iff 1 - xy$ is a unit, $\forall y \in A$.

(5) Let A be any commutative ring and I, J be two ideals of A. Show that $(I + J)(I \cap J) \subseteq IJ$. Give a sufficient condition so that equality occurs.

(6) (a)∗ Suppose that $I_1, I_2, \dots, I_n$ are prime ideals in a commutative ring A. Show that, if an ideal J of A is contained in $\bigcup_{i=1}^{n} I_i$ then $J \subseteq I_i$, for some i.
  (b) Assume A is commutative ring and $I_1, I_2, \dots, I_n$ are ideals of A. Show that, if P be a prime ideal in A such that $\bigcap_{i=1}^{n} I_i \subseteq P$ then $I_i \subseteq P$, for some i. Consequently, if $P = \bigcap_{i=1}^{n} I_i$ then $P = I_i$, for some i.

(7) Suppose A is a commutative ring and $I \subseteq A$ is an ideal. Show that
  (a) $r(I) := \{x \in A : x^n \in I, \text{ for some } n \in \mathbb{N}\}$ is an ideal of A.
  (b) $r(I)$ is the intersection of all prime ideals of A containing I.
  (c) $r(IJ) = r(I \cap J) = r(I) \cap r(J)$, for any two ideals I, J of A.
  (d) $r(P^n) = P$ for any prime ideal P of A and $n \in \mathbb{N}$.

(8) Let A, B be commutative rings and $f : A \longrightarrow B$ be a ring homomorphism. For any ideal J of B, the ideal $f^{-1}(J)$ of A is called the *contraction* of J, and denoted by J^c. The extension of an ideal I of A, denoted by I^e, is defined to be the ideal of B generated by $f(I)$.
  (a) J is a prime ideal of $B \Longrightarrow J^c$ is a prime ideal of A.
  (b) Prove or disprove: I^c is always a prime ideal of B for any prime ideal I of A.
  (c) Show that there is a bijective correspondence between the set of all contracted ideals in A and the set of all extended ideals in B. scr
  (d) Prove or disprove:
    (i) $(I_1 I_2)^e = I_1^e I_2^e$, for any two ideals I_1, I_2 of A, and
    (ii) $(J_1 J_2)^c = J_1^c J_2^c$, for any two ideals J_1, J_2 of B.

(9) Let A be a commutative ring and $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in A[x]$. Show the following:

(a) $f(x)$ is a unit in $A[x] \iff a_0$ is an unit in A and $a_1, \ldots, a_n$ are nilpotent. (Hint: If $b_0 + b_1 x + \cdots + b_m x^m$ is the inverse of $f(x)$, then by induction show that $a_n^{r+1} b_{m-r} = 0$. This yields that a_n is nilpotent and then use (2b).)
(b) f is nilpotent if and only if all of its coefficients are nilpotent.
(c) f is a zero divisor $\iff \exists a \in A \setminus \{0\}$ such that $af = 0$. (Hint: Choose a polynomial g with least degree such that $fg - 0$. If a is the leading coefficient of g then one has $aa_n = 0$. It follows that $a_n g = 0$. Proceeding by induction show that $a_i g = 0$ holds for each i.)
(d) f is said to be *primitive* if $(a_0, a_1, \ldots, a_n) = (1)$. For any $f(x), g(x) \in A[x]$, show that $f(x)g(x)$ is primitive if and only if $f(x)$ and $g(x)$ are primitive.

(10)∗ Generalize the results of (9) to $A[x_1, \ldots, x_n]$.

(11) Show that in $A[x]$, the Jacobson radical and nilradical coincide. (Hint: Use (4).)

(12) Let A be as in (9) and $f(x) = \sum_{n=0}^{\infty} a_n x^n \in A[[x]]$. Show the following:
(a) $f(x)$ is nilpotent $\Longrightarrow$ each a_n is nilpotent.
(b) $f(x)$ is in the Jacobson radical of $A[[x]]$ if and only if a_0 lies in the Jacobson radical of A.
(c) If I is a maximal ideal of $A[[x]]$ then I^c is a maximal ideal of A and $I = I^c + (x)$.
(d) Every prime ideal of A is the contraction of a proime ideal of $A[[x]]$.

(13) Let A be a commutative ring and $\mathfrak{R}$ be its nilradical. Show that the following are equivalent:

(a) A has exactly one prime ideal,
(b) every element of A is either a unit or nilpotent;
(c) $A/\mathfrak{R}$ is a field.

(14) Suppose that A is a commutative ring and $\mathfrak{S}$ is the set of all ideals I of A such that every $a \in I$ is a zero divisor. Show that $\mathfrak{S}$ has maximal elements and every maximal element of $\mathfrak{S}$ is a prime ideal. Deduce that the set of zero divisors in A is a union of prime ideals.