FAMILIES OVER THE INTEGRAL BERNSTEIN CENTER AND TATE COHOMOLOGY OF LOCAL BASE CHANGE LIFTS FOR $\mathrm{GL}_n(F)$

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Abstract. Let p and l be distinct odd primes, and let F be a p-adic field. Let π be a generic smooth integral representation of $\mathrm{GL}_n(F)$ over an $\overline{\mathbb{Q}}_l$-vector space. Let E be a finite Galois extension of F with $[E:F]=l$. Let Π be the base change lift of π to the group $\mathrm{GL}_n(E)$. Let $\mathbb{W}^0(\Pi,\psi_E)$ be the lattice of $\overline{\mathbb{Z}}_l$-valued functions in the Whittaker model of Π, with respect to a standard $\mathrm{Gal}(E/F)$-equivaraint additive character $\psi_E : E \to \overline{\mathbb{Q}}_l^{\times}$. We show that the unique generic sub-quotient of the zero-th Tate cohomology group of $\mathbb{W}^0(\Pi,\psi_E)$ is isomorphic to the Frobenius twist of the unique generic sub-quotient of the mod-l reduction of π. We first prove a version of this result for a family of smooth generic representations of $\mathrm{GL}_n(E)$ over the integral Bernstein center of $\mathrm{GL}_n(F)$. Our methods use the theory of Rankin-selberg convolutions and simple identities of local γ-factors. The results of this article remove the hypothesis that l does not divide the pro-order of $\mathrm{GL}_{n-1}(F)$ in our previous work [DN22].

1. Introduction

Let G be a reductive group defined over a number field K. Let σ be an order l automorphism of G defined over K, and let G^σ be the connected component of the fixed points of σ. In the seminal paper [TV16], D.Treumann and A. Venkatesh established the functoriality lifting of a mod-l automorphic form on G^σ to a mod-l automorphic form on G; here mod-l automorphic form is defined as a Hecke eigenclass in the cohomology of congruence subgroup with k-coefficients. At the same time, they also made some conjectures for representation theory of p-adic groups, and these conjectures predict that local (Langlands) functoriality is compatible with Tate cohomology for the action of $\langle\sigma\rangle$ (see [TV16, Section 6.3]). After initial evidence in the context of local cyclic base change for depth-zero cuspidal representations due to Ronchetti ([Ron16]), Feng ([Fen24], [Fen23]) made remarkable progress towards these conjectures–using the recent advances due to the works of V. Lafforgue and Fargues-Scholze. In this article, using completely different line of arguments, we approach the conjectures in [TV16, Section 6.3] in the setting of the local Base change lifting. We use local Rankin-Selberg zeta functions and their functional equations to study the Tate cohomology and linkage principle. Our approach is in the spirit of local converse theorems. In our earlier work [DN22], we assumed that the prime l is large. In this article, exploiting the theory of smooth representations in families, we remove this hypothesis.

Let (Π, W) be an irreducible smooth $\overline{\mathbb{F}}_l$-representation of $G(F)$, where G is a connected reductive group over a p-adic field F. Say $\sigma\in\mathrm{Aut}(G)$ is of order l and that Π^σ is isomorphic to Π. The Tate cohomology group $\widehat{H}^i(\sigma, V)$ is a representation of $G^\sigma(F)$. Any irreducible sub-quotient π of $\widehat{H}^i(\sigma, W)$ is defined to be linked with Π. In the article [TV16], the authors discovered that linkage is compatible with local functoriality. In this paper, we prove the Linkage conjectures in the local Base change setting of GL_n for all odd l. In our previous work [DN22], we proved the main theorem under the restriction that l does not divide the pro-order of $\mathrm{GL}_{n-1}(F)$. By using the theory of Co-Whittaker modules and integral Bernstein center, developed by Emerton and Helm ([EH14], [Hel20]) in an essential way, we remove this hypothesis on l.

To state the main results of this article, some notations are in order. Let F be a finite extension of $\mathbb{Q}_p$ and let E/F be a finite Galois extension of prime degree l. Let $\overline{\mathcal{K}}$ be the algebraic closure of the fraction field of $W(\overline{\mathbb{F}}_l)$ with ring of integers $\mathcal{O}$. Let (π, V) be an irreducible smooth

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representation of $\mathrm{GL}_n(F)$, where V is a vector space over $\overline{\mathcal{K}}$. We assume that (π, V) is an integral representation, i.e., V admits a $\mathrm{GL}_n(F)$-invariant $\mathcal{O}$-lattice. Let (Π, W) be the base change lifting of π to $\mathrm{GL}_n(E)$. The representation Π is isomorphic to Π^γ, where γ is any generator of $\mathrm{Gal}(E/F)$. Let us fix an isomorphism $T : \Pi \to \Pi^\gamma$ such that $T^l = \mathrm{id}$. Note that Π is an integral representation of $\mathrm{GL}_n(E)$.

If π is generic, then Π is also a generic representation. Let us fix an additive character ψ_F of F and let ψ_E be the composition $\psi_F \circ \mathrm{Tr}$, where Tr is the trace map of the extension E/F. Let $\mathbb{W}(\pi, \psi_F)$ (resp. $\mathbb{W}(\Pi, \psi_E)$) be the Whittaker model of π (resp. Π). Vigneras showed that the $\mathcal{O}$-module $\mathbb{W}^0(\Pi, \psi_E)$ is a $\mathrm{GL}_n(E)$-stable lattice and by multiplicity one results, $\mathbb{W}^0(\Pi, \psi_E)$ is stable under the Galois action of $\mathrm{Gal}(E/F)$ induced by its natural action on $\mathrm{GL}_n(E)$. Note that there exists a unique generic sub-quotient of the mod-l reduction of a generic integral representation π of $\mathrm{GL}_n(K)$, where K is a p-adic field and this is denoted by $J_l(\pi)$. It is easy to show that the Tate cohomology of $\widehat{H}^0(\mathbb{W}^0(\Pi, \psi_E))$ has a unique generic sub-quotient as a representation of $\mathrm{GL}_n(F)$ ([DN22, Proposition 6.3]). Let V be a vector space over $\overline{\mathbb{F}}_l$, and let $V^{(l)}$ be the vector space $V \otimes_{\mathrm{Fr}} \overline{\mathbb{F}}_l$ where Fr be the Frobenius automorphism of $\overline{\mathbb{F}}_l$. Ronchetti ([Ron16, Theorem 6]) proved that the Tate cohomology group $\widehat{H}^1(\mathcal{L})$ is zero for any $\mathrm{GL}_n(E)$ and $\mathrm{Gal}(E/F)$ stable lattice $\mathcal{L}$ in a cuspidal representation. When l does not divide the pro-order of $\mathrm{GL}_n(F)$, the authors showed the same vanishing result for generic representations ([DN22, Corollary 8.2]). We prove the following theorem on the zeroth Tate cohomology:

Theorem 1.1. *Let l and p be two distinct odd primes. Let F be a p-adic field and let E/F be a Galois extension of degree l. Let π be an integral smooth generic representation of $\mathrm{GL}_n(F)$ over a $\overline{\mathbb{Q}}_l$-vector space. Let Π be a base change lifting of π to $\mathrm{GL}_n(E)$. The unique generic sub-quotient of the Tate cohomology group $\widehat{H}^0(\mathbb{W}^0(\Pi, \psi_E))$ is isomorphic to $J_l(\pi)^{(l)}$.*

In our previous work ([DN22]), we proved the above theorem when l does not divide the pro-order of $\mathrm{GL}_{n-1}(F)$. We use the theory of Rankin–Selberg zeta integrals and their functional equations. The hypothesis on the prime l in [DN22] is needed for the completeness of mod-l Whittaker models. In order to remove this hypothesis on l, we use the theory of smooth representations in families due to Emerton and Helm. This approach deals with the nilpotents which arise in the mod-l Zeta integrals. So, we first prove a version of the above theorem over families and use it to prove the above theorem. In order to define the correct family, we needed the work of Helm on local converse theorem, where he defines a map between the integral Bernstein center and a certain deformation ring. This gave us the notation of Base change lifting map between Bernstein centers.

We will begin with the description of our result on Tate cohomology of a family of smooth representations of $\mathrm{GL}_n(E)$. Let $\mathcal{M}_n(K)$ be the category of smooth $W(\overline{\mathbb{F}}_l)[\mathrm{GL}_n(K)]$-modules, where K is a p-adic field, and the center of the category $\mathcal{M}_n(K)$ is denoted by $\mathcal{Z}_n(K)$. The primitive idempotents of $\mathcal{Z}_n(K)$ corresponds to an inertial class $[L, \sigma]$, where σ is a supercuspidal $\overline{\mathbb{F}}_l$ representation of the Levi-subgroup L of $\mathrm{GL}_n(K)$ ([Hel16a, Section 12]). Let $e_\mathfrak{s}$ be a primitive idempotent of $\mathcal{Z}_n(F)$. Using mod-l semisimple Langlands correspondence of Vigneras ([Vig01, Theorem 1.6]), we get a smooth representation

$$\rho : W_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_l)$$

associated with (L, σ). Let $e_\mathfrak{r}$ be the primitive idempotent in $\mathcal{Z}_n(E)$ associated with $\mathrm{res}_{W_E} \rho$. Using Helm's construction of a homomorphism between irreducible component of Bernstein centre to a deformation ring, we obtain a map

$$z_{E/F} : e_\mathfrak{r}\mathcal{Z}_n(E) \to e_\mathfrak{s}\mathcal{Z}_n(F)$$

which interpolates the cuspidal support of Base change lifting ([AC89, Chapter 1]).

To interpolate those generic representations of $\mathrm{GL}_n(E)$ which arise as a base change lift of representations with a given cuspidal support, we define

$$\mathcal{V} = (e_\mathfrak{r} \,\mathrm{ind}_{N_n(E)}^{\mathrm{GL}_n(E)} \psi_E^{-1}) \otimes_{z_{E/F}} e_\mathfrak{s}\mathcal{Z}_n(F).$$

The module $\mathcal{V}$ is a co-Whittaker $e_{\mathfrak{s}}\mathcal{Z}_n(F)[\mathrm{GL}_n(F)]$-module. The theory of Rankin-selberg γ-factors in families show that the representation $\mathcal{V}$ is isomorphic to $\mathcal{V}^{\gamma}$, for all $\gamma \in \mathrm{Gal}(E/F)$. Thus, the space of Whittaker functions (containing functions on $\mathrm{GL}_n(E)/N_n(E)$ with values in $e_{\mathfrak{s}}\mathcal{Z}_n$) is invariant under the action of $\mathrm{Gal}(E/F)$. We then prove that the Tate cohomology group $\widehat{H}^0(\mathbb{W}(\mathcal{V}, \psi_E))$ realises

$$[e_{\mathfrak{s}}(\mathrm{ind}_{N_n(F)}^{\mathrm{GL}_n(F)} \psi_F^{-1}) \otimes_{W(\overline{\mathbb{F}}_l)} \overline{\mathbb{F}}_l]^{(l)}$$

as a sub-quotient. The proof of this result is similar to our previous result in [DN22, Theorem 6.7], where we use Rankin-Selberg local zeta integrals and their liftings from homogeneous spaces over F to the corresponding spaces over E.

Theorem 1.1 is proved by a specialization argument. The proofs of this theorem and the previous version in families are inspired from the proofs of local converse theorems, as in [JPSS79, Proposition 7.5.2] and [Hen93, Theorem 1.1]. Note that the above arguments in families and our specialization arguments essentially use Kirillov models and completeness of Whittaker models. The following is the essential content. Let X_F be the space $\mathrm{GL}_{n-1}(F)/N_{n-1}(F)$, and let $H \in C_c(X_F, \overline{\mathbb{F}}_l)$. Typically, for some fixed integer k and a suitably chosen Whittaker function W in the Kirrilov model of Π, the function H is equal to

$$[\widehat{H}^0(\pi_E)(w) - \pi_F^{(l)}(w)]W\left(\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right)$$

when $\det(g) = k$ and zero otherwise. Essentially we need to show that H is zero. If not, there exists a primitive idempotent $e_{\mathfrak{s}} \in \mathcal{Z}_n(F)$ and a $e_{\mathfrak{s}}\mathcal{Z}_n(F)$-valued Whittaker function

$$W' \in \mathbb{W}(e_{\mathfrak{s}}\, \mathrm{ind}_{N_{n-1}(F)}^{\mathrm{GL}_{n-1}(F)} \psi_F^{-1}, \psi_F^{-1})$$

such that

$$\int_{X_F} H(g) \otimes W'(g)\, d\mu \neq 0.$$

When the ring $e_{\mathfrak{s}}\mathcal{Z}_n(F) \otimes \overline{\mathbb{F}}_l$ is reduced, we can simply use Rankin-Selberg theory of mod-l representations to prove the vanishing of H and deduce our main theorem. This has been the main point of our work [DN22, Theorem 6.7]. In this article, to overcome the problems with the reducibility of the ring $e_{\mathfrak{s}}\mathcal{Z}_n(F) \otimes \overline{\mathbb{F}}_l$, we use our result on families.

2. Integral Bernstein Center and Base change

In this section, we recall the theory of integral Bernstein centers and define a base change map between Bernstein centers–which is compatible with the local base change lifting of irreducible smooth representations of GL_n. We essentially follow the works of Emerton–Helm ([EH14]) and Helm ([Hel20]).

2.1. For any finite extension K of $\mathbb{Q}_p$, we denote by $\mathfrak{o}_K$ its ring of integers and the maximal ideal of $\mathfrak{o}_K$ is denoted by $\mathfrak{p}_K$. Let q_K be the cardinality of the residue field $\mathfrak{o}_K/\mathfrak{p}_K$. The Weil group of K is denoted by W_K and its inertia subgroup is denoted by I_K. The prime to l-subgroup of I_K is denoted by $I_K^{(l)}$. We denote by $G_n(K)$ the group $\mathrm{GL}_n(K)$ and it is equipped with the natural topology induced from that of K. Let F be a finite extension of $\mathbb{Q}_p$ and let E be a finite Galois extension of prime degree l, where p and l are distinct odd primes. The Galois group $\mathrm{Gal}(E/F)$ is denoted by Γ. Let $\mathbb{F}$ be the algebraic closure of the finite field of l elements and let $W(\mathbb{F})$ be the ring of Witt vectors; we use the notation Λ for $W(\mathbb{F})$. Let $\overline{\mathcal{K}}$ be the algebraic closure of the field of fraction of Λ with ring of integers $\mathcal{O}$. Let $\psi_F : F \to \Lambda^{\times}$ be a non-trivial additive character and let ψ_E be the character $\psi_F \circ \mathrm{Tr}_{E/F}$, where $\mathrm{Tr}_{E/F}$ denotes the trace map of the extension E/F.

2.2. Let R be a commutative ring with unity and let G be the K-rational points of a reductive algebraic group defined over K. We denote by $\mathcal{M}_R(G)$ the category of smooth $R[G]$-modules (the stabilizer of every element under the action of G is open), and the Bernstein center of $\mathcal{M}_R(G)$ is denoted by $\mathcal{Z}_R(G)$. We assume that G is the group $G_n(K)$. We use the notation $\mathcal{Z}_n^K$ for the ring $\mathcal{Z}_\Lambda(G_n(K))$. Let P be a parabolic subgroup of G. Let M be a Levi-subgroup of P and let U be the unipotent radical of P. Let (σ, W) be a supercuspidal representation of M, where W is an $\mathbb{F}$-vector space. Let $\mathfrak{s} = [M, \sigma]$ be the inertial equivalence class containing the pair (M, σ). Let $\mathcal{M}_\mathfrak{s}(G)$ be the full subcategory of $\mathcal{M}_\Lambda(G)$ consisting of all smooth $\Lambda[G]$ modules such that any irreducible subquotient has mod-l (see [Hel16a, Definition 4.12]) inertial supercuspidal support equal to $\mathfrak{s}$. Given any primitive idempotent e of $\mathcal{Z}_n^K$, there exists a unique inertial equivalence class $\mathfrak{s}$ such that the Bernstein center of $\mathcal{M}_\mathfrak{s}(G)$ is equal to $e\mathcal{Z}_n^K$. The primitive idempotents of $\mathcal{Z}_n^K$ correspond to the inertial classes of the form $[M, \sigma]$, where σ is a supercuspidal representation of M over an $\mathbb{F}$-vector space.

2.3. Helm in his foundational work [Hel20] described integral Bernstein center in terms of local Galois deformation rings. He constructed a map between connected component of the integral Bernstein center and a local Galois deformation ring which interpolates the local Langlands correspondence. Helm's constructions naturally defines a map between Bernstein centers $\mathcal{Z}_n^E$ and $\mathcal{Z}_n^F$, which interpolates the compatibility of cuspidal support maps of base change lifting. For the cyclic extension E/F, the base change lifting of smooth irreducible representations of G over complex vector spaces is characterized via certain character identities (see [AC89, Chapter 3]). There is a relation between the local base change lifting of irreducible smooth $\overline{\mathcal{K}}$-representations of G and the local Langlands correspondence over $\overline{\mathcal{K}}$. Here, we use the local Langlands correspondence by fixing the normalizations as in [EH14, Section 4.2].

2.3.1. An l-inertial type is a representation $\nu : I_K^{(l)} \to \mathrm{GL}_n(\Lambda)$ that extends to a representation of $\mathcal{W}_F$. Let $\mathfrak{s}$ be the inertial equivalence class $[M, \sigma]$. where M is a Levi subgroup of G and σ is a supercuspidal representation of M over an $\mathbb{F}$-vector space. Vigneras' construction ([Vig01, Theorem 1.6]) of mod-l local Langlands correspondence attaches a unique semisimple representation $\overline{\rho} : W_K \to \mathrm{GL}_n(\mathbb{F})$ with the pair (M, σ)

$$(M, \sigma) \leftrightsquigarrow \overline{\rho}\ (\mathrm{mod} - l\ \mathrm{LLC})$$

Let $\overline{\tau}$ be an irreducible representation of $I_K^{(l)}$ over an $\mathbb{F}$-vector space, and let τ be the (unique) lift of $\overline{\tau}$ over Λ. Given any Λ-algebra A and a representation $\rho : W_K \to \mathrm{GL}_n(A)$, we have

$$\rho_A \simeq \bigoplus_{[\overline{\tau}]} (\mathrm{Hom}_A(\tau, \rho_A) \otimes \tau).$$

The direct sum is over the W_K-conjugacy classes of $I_K^{(l)}$-representations, denoted as $[\overline{\tau}]$. The space $\mathrm{Hom}_A(\tau, \rho)$ is a free A-module and a pseudoframing of ρ_A is a choice of a basis for each $\mathrm{Hom}_A(\tau, \rho_A)$. In the article [Hel20, Section 8], the author constructed a Λ-algebra R_ν corresponding to an l-inertial type ν–which is universal for the pseudoframed deformation $\rho_A : W_K \to \mathrm{GL}_n(A)$, i.e., for any choice of basis of $\mathrm{Hom}_{I_K^{(l)}}(\tau, \rho_A)$ which lifts a basis of $\mathrm{Hom}_{I_K^{(l)}}(\overline{\tau}, \overline{\rho})$ for each W_F-conjugacy classes of $I_K^{(l)}$-representations, denoted by $[\overline{\tau}]$, there exists a map $R_\nu^K \to A$ such that the pseudoframed deformation ρ_A is obtained as a base change of the universal pseudoframed deformation ρ_ν^K. The affine scheme corresponding to the deformation ring R_ν^K is equipped with an action of a group G_ν^K, defined in [Hel20, Section 8], which acts by change of framing; we recall the precise definition of G_ν^K in the proof of Lemma 2.1. This action gives a space of invariance $(R_\nu^K)^{\mathrm{inv}}$, which is infact a subalgebra of R_ν^K. The space $(R_\nu^K)^{\mathrm{inv}}$ is useful to construct the base change map between the integral Bernstein centers, as we will see in the next subsections.

2.3.2. In [HM18, Section 7], the authors constructed a map

$$\mathbb{L}_\nu^K : e\mathcal{Z}_n^K \to R_\nu^K$$

which is compatible with the local Langlands correspondance, i.e., for any morphism

$$x : R_\nu^K \to \overline{\mathcal{K}},$$

the morphism

$$x \circ \mathbb{L}_\nu^K : e\mathcal{Z}_n^K \to \overline{\mathcal{K}}$$

gives the action of $e\mathcal{Z}_n^K$ on the representation π_x–associated with ρ_x via the local Langlands correspondence. The map $\mathbb{L}_\nu^K$ identifies the algebra $e\mathcal{Z}_n^K$ with the $\mathcal{G}_\nu$-fixed points of R_ν^K. This morphism plays a fundamental role in proving the local Langlands in families by Emerton and Helm ([EH14, Conjecture 1.3.1]). Let F be a finite extension of $\mathbb{Q}_p$ and let E be a finite extension of F of degree l with $l \neq p$. Let $\nu : I_F^{(l)} \to \mathrm{GL}_n(\Lambda)$ be an l-inertial type. As the group $I_F^{(l)}$ is equal to $I_E^{(l)}$, any pseudo-framed deformation $\rho_A : W_F \to \mathrm{GL}_n(\Lambda)$ also determines a pseudoframed deformation $\rho_A : W_E \to \mathrm{GL}_n(\Lambda)$ by restriction and hence we get a map

$$B_{E/F} : R_\nu^E \to R_\nu^F. \tag{2.1}$$

Lemma 2.1. *The map $B_{E/F}$ induces a map between $(R_\nu^E)^{\mathrm{inv}}$ and $(R_\nu^F)^{\mathrm{inv}}$.*

Proof. Let $X_{q,m}$ be the affine Λ-scheme parametrizing the pairs (g, h), where g, h are the invertible m by m matrices with $ghg^{-1} = h^q$. Let $R_{q,m}$ be the ring of functions of the connected compotent of $X_{q,m}$ containing the pair $(\mathrm{Id}_n, \mathrm{Id}_n)$. Note that the Λ-algebra R_ν^F is isomorphic to $\bigotimes_{[\overline{\tau}]} R_{q_{\overline{\tau}}, n_{\overline{\tau}}}$, where $n_{\overline{\tau}}$ is the dimension of the space $\mathrm{Hom}_{I_F^{(l)}}(\overline{\tau}, \overline{\nu})$. The group $\mathrm{GL}_{n_{\overline{\tau}}}$ acts on $R_{q_{\overline{\tau}}, n_{\overline{\tau}}}$ by change of frame. Let $S_E = \{\tau_1, \tau_2, \ldots, \tau_k\}$ (resp. $S_F \subset S_E$) be a set of representatives for the equivalence classes of $I_F^{(l)}$ representations for the action of W_E (resp. W_F). The group

$$\prod_{\overline{\tau} \in S_E} \mathrm{GL}_{n_{\overline{\tau}}}$$

denoted by $\mathcal{G}_\nu^E$, acts on R_ν^E via its action on $R_{q_{\overline{\tau}}, n_{\overline{\tau}}}$. Similarly the group $\mathcal{G}_\nu^F$, isomorphic to $\prod_{\overline{\tau} \in S_F} \mathrm{GL}_{n_{\overline{\tau}}}$ is contained in $\mathcal{G}_\nu^E$. Thus, the image of the restriction of $B_{E/F}$ to $(R_\nu^E)^{\mathrm{inv}}$ is contained in the space of invariance $(R_\nu^F)^{\mathrm{inv}}$. □

2.3.3. Let $e_{\mathfrak{s}}$ be a primitive idempotent of $\mathcal{Z}_n^F$ corresponding to the inertial equivalence class $\mathfrak{s}$. Fix a pair (L, σ) in the class $\mathfrak{s}$, where L is a Levi subgroup of $G_n(F)$ and σ is an irreducible supercuspidal $\mathbb{F}$-representation of L. Let ρ_F be the n-dimensional semisimple representation of W_F associated with σ via the mod-l semisimple local Langlands correspondence ([Vig01, Theorem 1.6]). Let $\nu : I_F^{(l)} \to \mathrm{GL}_n(\Lambda)$ be the l-inertial type such that its mod-l reduction is isomorphic to $\mathrm{res}_{I_F^{(l)}}(\rho_F)$. Note that the restriction $\mathrm{res}_{W_E}(\rho_F)$ via mod-l local Langlands correspondence defines a pair (L', σ') such that its inertial equivalence class, denoted by $\mathfrak{r}$, is independent of the choice of (L, σ). Let $e_{\mathfrak{r}}$ be the primitive idempotent of $\mathcal{Z}_n^E$ associated with $\mathfrak{r}$. The map $B_{E/F}$ induces a map $z_{E/F} : e_{\mathfrak{r}}\mathcal{Z}_n^E \to e_{\mathfrak{s}}\mathcal{Z}_n^F$ such that the following diagram commutes.

$$\begin{array}{ccc} e_{\mathfrak{r}}\mathcal{Z}_n^E & \xrightarrow{\mathbb{L}_\nu^E} & R_\nu^E \\ {\scriptstyle z_{E/F}}\downarrow & & \downarrow{\scriptstyle B_{E/F}} \\ e_{\mathfrak{s}}\mathcal{Z}_n^F & \xrightarrow{\mathbb{L}_\nu^F} & R_\nu^F \end{array}$$

Let $\rho_{x_F} : W_F \to \mathrm{GL}_n(\overline{\mathcal{K}})$ be the representation corresponding to $x_F : R_\nu^F \to \overline{\mathcal{K}}$ and let π_{x_F} be a smooth irreducible representation of $\mathrm{GL}_n(F)$ associated to ρ_{x_F} via, the local Langlands correspondence. Let x_E be the map $x_F \circ B_{E/F} : R_\nu^E \to \overline{\mathcal{K}}$, and the representation π_{x_E} is the base change lifting of π_{x_F}. The action of $\mathcal{Z}_n^E$ on the representation π_{x_E} factorises through $e_{\mathfrak{r}}\mathcal{Z}_n^E$, and it is given by the homomorphism $x_F \circ \mathbb{L}_\nu^F \circ z_{E/F}$.

3. Rankin–Selberg convolution in families

In this section, we review the theory of Rankin-Selberg convolution over arbitrary Noetherian Λ-algebras. We start by recalling the notion of co-Whittaker modules. For a precise reference, see [HM18], [Mos16a].

3.1. **Co-Whittaker modules.** Fix a non-trivial additive character $\psi_K : K \to \Lambda^\times$. For any Noetherian Λ-algebra R, we denote by $\psi_{K,R}$ the composition

$$K \xrightarrow{\psi_K} \Lambda^\times \to R^\times.$$

Let $N_n(K)$ be the group of all unipotent upper triangular matrices in $G_n(K)$. The character $\psi_{K,R} : K \to R^\times$ induces a character of $N_n(K)$, defined as

$$(x_{ij})_{i,j=1}^n \longmapsto \psi_{K,R}(x_{12} + x_{23} + \cdots x_{n-1,n}),$$

which is denoted by $\Psi_{K,R}$. Let (π, V) be a smooth $R[G_n(K)]$-module. We denote by $V^{(n)}$ the space of $\Psi_{K,R}$-coinvariants, which is defined as the quotient of V modulo the R-submodule generated by the set $\Big\{\pi(x)v - (\Psi_{K,R})(x)v : x \in N_n(K), v \in V\Big\}$. The R-module $V^{(n)}$ is the n-th Bernstein-Zelevinski derivative of V.

3.1.1. A smooth admissible $R[G_n(K)]$-module (π, V) is called *co-Whittaker* if

(1) the n-th derivative $V^{(n)}$ is a free R-module of rank one,
(2) for any quotient W of V with $W^{(n)} = 0$, we have $W = 0$.

A co-Whittaker module (π, V) admits a central character denoted by $\varpi_\pi : F^\times \to R^\times$. By definition, the representation (π, V) admits an R-module isomorphism $V^{(n)} \simeq R$. By Frobenius reciprocity, this isomorphism induces a $G_n(K)$-equivariant homomorphism

$$V \longrightarrow \mathrm{Ind}_{N_n(K)}^{G_n(K)}(\Psi_{K,R}).$$

The image of V under the above map is called the *Whittaker* space of π. It is denoted by $\mathbb{W}(\pi, \psi_{K,R})$ and is independent of the choice of isomorphism $V^{(n)} \simeq R$. The identification $V^{(n)} \simeq R$, also induces a $P_n(K)$-equivariant map

$$V \longrightarrow \mathrm{Ind}_{N_n(K)}^{P_n(K)}(\Psi_{K,R}),$$

the image of the above map is called *Kirillov* space of π, and it is denoted by $\mathbb{K}(\pi, \psi_{K,R})$. The map $W \mapsto \mathrm{res}_{P_n(K)}(W)$ gives an isomorphism $\mathbb{W}(\pi, \psi_{K,R}) \xrightarrow{\sim} \mathbb{K}(\pi, \psi_{K,R})$ ([MM22, Section 4]). Moreover, the space $\mathbb{K}(\pi, \psi_{K,R})$ contains $\mathrm{ind}_{N_n(K)}^{P_n(K)}(\Psi_{K,R})$ as $R[P_n(K)]$-submodule.

3.1.2. For a co-Whittaker $R[G_n(K)]$-module (π, V), the endomorphism ring $\mathrm{End}_{R[G_n(K)]}(V)$ is equal to R, and the action of the integral Bernstein center $\mathcal{Z}_n^K$ on V induces a map $f_\pi : \mathcal{Z}_n^K \to R$. The map f_π is called the *supercuspidal support* of π. The supercuspidal supports of two co-Whittaker $R[G_n(K)]$-modules are the same if and only if they have the same Whittaker spaces (see [LM20, Lemma 2.4]). Moreover, if we have two co-Whittaker $R[G_n(K)]$-modules (π_1, V_1) and (π_2, V_2) with a $G_n(K)$-equivariant surjection $V_1 \to V_2$, then $f_{\pi_1} = f_{\pi_2}$.

3.1.3. Let W_n^K be the smooth $\Lambda[G_n(K)]$-module $\mathrm{ind}_{N_n(K)}^{G_n(K)}(\Psi_K^{-1})$. For a primitive idempotent e in the integral Bernstein center $\mathcal{Z}_n^K$, the space eW_n^K is a co-Whittaker $e\mathcal{Z}_n^K[G_n(K)]$-module. The following result is due to [Hel16b, Theorem 6.3], which specify the universal property of eW_n^K.

Theorem 3.1. *Let R be a Noetherian Λ-algebra, equipped with a $e\mathcal{Z}_n^K$-algebra structure. Then $eW_n^K \otimes_{e\mathcal{Z}_n^K} R$ is a co-Whittaker $R[G_n(K)]$-module. Conversely, if (π, V) is a co-Whittaker $R[G_n(K)]$-module in the category $\mathcal{M}_e(G_n(K))$, then there exists a surjection $\beta : eW_n^K \otimes_{e\mathcal{Z}_n^K} R \to V$ such that the induced map $\beta^{(n)} : (eW_n^K \otimes_{e\mathcal{Z}_n^K} R)^{(n)} \to V^{(n)}$ is an isomorphism.*

We end this subsection with a version of key vanishing result, usually known as completeness of Whittaker models, provided by G.Moss ([Mos21, Corollary 4.3]). This vanishing result will be crucially used in proving the main results of this article.

Theorem 3.2. *Let R be a Λ-algebra, and let ϕ be an element of* $\mathrm{ind}_{N_n(K)}^{G_n(K)}(\Psi_{K,R})$. *If*

$$\int_{N_n(K)\backslash G_n(K)} \phi(g)\otimes W(g)\,dg = 0,$$

for all $W \in \mathbb{W}(eW_n^K, \psi_K^{-1})$ and for all primitive idempotents e of $\mathcal{Z}_n^K$, then $\phi = 0$.

3.2. **Rankin-Selberg formal series.** In this subsection, we recall the Rankin-Selberg gamma factors over families, which provide the classical Rankin-Selberg gamma factors as a specialization on $\overline{\mathcal{K}}$-points. For a reference, see [Mos16a, Section 3].

3.2.1. We now introduce some notations. Let w_n be the matrix of $G_n(K)$ of the form:

$$w_n = \begin{pmatrix} 0 & & 1 \\ & \cdot & \\ & \cdot & \\ 1 & & 0 \end{pmatrix}$$

For $r \in \mathbb{Z}$, set $G_{n-1}(K)_r = \{g \in G_{n-1}(K) : v_K(\det(g)) = r\}$, where v_K denote the normalised discrete valuation of K. Let X_K be the coset space $N_{n-1}(K) \setminus G_{n-1}(K)$. For an integer r, we denote by X_r^K the set of the form $\{N_{n-1}(K)g : g \in G_{n-1}(K)_r\}$.

3.2.2. Let A and B be two Noetherian Λ-algebras. Let π (resp. π') be the co-Whittaker $A[G_n(K)]$ (resp. $B[G_{n-1}(K)]$)-module. For any $W \in \mathbb{W}(\pi, \psi_{K,A})$ and $W' \in \mathbb{W}(\pi', \psi_{K,B}^{-1})$, the integral

$$c_r^K(W, W') := \int_{X_r^K} W\begin{pmatrix} g & 0 \\ 0 & 1\end{pmatrix} \otimes W'(g)\,dg$$

is well-defined for all integers r and it is zero for $r << 0$ (see [Mos16a, Section 3]). The formal Laurent series

$$\sum_{r\in\mathbb{Z}} c_r^K(W, W')X^r,$$

is an element of $S^{-1}((A\otimes_\Lambda B)[X, X^{-1}])$, where S is the multiplicative set consisting of polynomials $\sum_{r=t}^{s} a_r X^r$ with a_t and a_s being units in $A\otimes_\Lambda B$. Now, let us consider the functions $\widetilde{W}$ and $\widetilde{W'}$, defined as

$$\widetilde{W}(g) := W(w_n(g^t)^{-1})$$

and

$$\widetilde{W'}(g) := W'(w_{n-1}(x^t)^{-1}),$$

for all $g \in G_n(K)$ and $x \in G_{n-1}(K)$. Making change of variables, we have the identity:

$$c_r^K(\widetilde{W}, \widetilde{W'}) = c_{-r}^K\big(\pi(w_n)W, \pi'(w_{n-1})W'\big). \tag{3.1}$$

3.2.3. *Functional Equation.* Given π and π' as above, there is a unique element $\gamma(X, \pi, \pi', \psi_K)$ in the ring of fraction $S^{-1}((A\otimes_\Lambda B)[X, X^{-1}])$ such that

$$\sum_{r\in\mathbb{Z}} c_r^K(\widetilde{W}, \widetilde{W'})X^{-r} = \varpi_{\pi'}(-1)^{n-1}\gamma(X, \pi, \pi', \psi_K)\sum_{r\in\mathbb{Z}} c_r^K(W, W')X^r,$$

for all $W \in \mathbb{W}(\pi, \psi_{K,A})$ and $W' \in \mathbb{W}(\pi', \psi_{K,B}^{-1})$.

Let e (resp. e') be the primitive idempotent in $\mathcal{Z}_n^K$ (resp. $\mathcal{Z}_{n-1}^K$) such that the supercuspidal support map f_π (resp. $f_{\pi'}$) factors through the center $e\mathcal{Z}_n^K$ (resp. $e'\mathcal{Z}_{n-1}^K$). The gamma factor $\gamma(X, eW_n^K, e'W_{n-1}^K, \psi_K)$ corresponding to the pair $(eW_n^K, e'W_{n-1}^K)$ admits the following universal property ([Mos16b, Theorem 5.4]) in the families of co-Whittaker modules.

Theorem 3.3. *Let A and B be two Noetherian Λ-algebras. Let π and π' be two co-Whittaker $A[G_n(K)]$ and $B[G_{n-1}(K)]$-modules respectively. Let e (resp. e') be the primitive idempotent of $\mathcal{Z}_n^K$ (resp. $\mathcal{Z}_{n-1}^K$) such that supercuspidal support f_π (resp. $f_{\pi'}$) factors through $e\mathcal{Z}_n^K$ (resp. $e'\mathcal{Z}_{n-1}^K$). Then*

$$\gamma(X, \pi, \pi', \psi_K) = (f_\pi \otimes f_{\pi'})\big(\gamma(X, eW_n^K, e'W_{n-1}^K, \psi_K)\big).$$

3.3. **Frobenius twist.** Let R be a Noetherian $\mathbb{F}$-algebra. Let (π, V) be a co-Whittaker $R[G_n(K)]$-module. Let $\mathrm{Fr} : R \to R$ be the map $x \mapsto x^l$. We denote by $V^{(l)}$ the R module $V \otimes_{\mathrm{Fr}} R$. The module $(\pi^{(l)}, V^{(l)})$ is called the Frobenius twist of (π, V). Let $\mathfrak{s}$ be the inertial equivalence class such that $\pi \in \mathcal{M}_{\mathfrak{s}}(G_n(K))$, and let e be the primitive idempotent of $\mathcal{Z}_n^K$, associated with $\mathfrak{s}$. The composition

$$e\mathcal{Z}_n^K \xrightarrow{f_\pi} R \xrightarrow{\lambda \mapsto \lambda^l} R,$$

denoted by $f_{\pi^{(l)}}$, is the supercuspidal support of the Frobenius twist $\pi^{(l)}$. Let R' be a Noetherian $\mathbb{F}$-algebra, and let (π', V') be the co-Whittaker $R'[G_{n-1}(K)]$-module. Let $\mathfrak{s}'$ be the inertial equivalence class such that $\pi' \in \mathcal{M}_{\mathfrak{s}'}(G_{n-1}(K))$, and let e' be the primitive idempotent in $\mathcal{Z}_{n-1}^K$, associated with $\mathfrak{s}'$. Using Theorem 3.3 and the properties of supercuspidal supports $f_{\pi^{(l)}}$ and $f_{\pi'^{(l)}}$, we get the following identity of gamma factors

$$\gamma(X, \pi, \pi', \psi_K)^l = \gamma(X^l, \pi^{(l)}, \pi'^{(l)}, \psi_K^l). \tag{3.2}$$

In the next subsection, we set up an idenitity of the integrals on the homeogeneous space of F with those on the homogeneous space of E. This is crucial for the main results.

3.4. **Integrals on homogeneous space.** As before, let E be a finite Galois extension of a p-adic field F with degree of extension l, where l is coprime to p. We denote by Γ the Galois group $\mathrm{Gal}(E/F)$. Let R be an $\mathbb{F}$-algebra with p being invertible R. There is a natural action of Γ on the coset space X_E and hence on the space $C_c^\infty(X_E, R)$, consisting of smooth and compactly supported R-valued functions on X_E, given by $(\gamma.\varphi)(g) := \varphi(\gamma^{-1}g)$, for all $\gamma \in \Gamma$, $f \in C_c^\infty(X_E, R)$ and for all $g \in X_E$. We denote by $C_c^\infty(X_E, R)^\Gamma$ the space of Γ-fixed elements in $C_c^\infty(X_E, R)$. With these, we now deduce:

Proposition 3.4. *Let $d\mu_E$ and $d\mu_F$ be the Haar measures on X_E and X_F respectively. Then, there exists a non-zero scalar $\alpha \in \mathbb{F}$ such that for all $\varphi \in C_c^\infty(X_E, R)^\Gamma$, we have*

$$\int_{X_E} \varphi \, d\mu_E = \alpha \int_{X_F} \varphi \, d\mu_F.$$

Proof. The proof is immediate by following the arguments of [DN22, Proposition 5.2] mutatis-mutandis. □

Remark 3.5. The Haar measures $d\mu_E$ and $d\mu_F$ on X_E and X_F, respectively, are now choosen in a way that ensures $\alpha = 1$. Moreover, if e is the ramification index of the extension E over F, then for all $k \notin \{re : r \in \mathbb{Z}\}$, we have

$$\int_{(X_E^k)^\Gamma} \varphi \, d\mu_F = 0,$$

and for all $k \in \{re : r \in \mathbb{Z}\}$, we have

$$\int_{(X_E^k)^\Gamma} \varphi \, d\mu_F = \int_{X_F^{\frac{r}{e}}} \varphi \, d\mu_F.$$

4. Tate cohomology of co-Whittaker modules

In this section, we study the compatibility of Tate cohomology with local base change for universal co-Whittaker modules.

4.1. Let E be a finite Galois extension of a p-adic field F with $[E : F] = l$, where l and p are distinct primes. We fix a generator γ of the cyclic group $\mathrm{Gal}(E/F)$. For any Λ-algebra R and any smooth $R[G_E]$-module V, we denote by V^γ the smooth $R[G_E]$-module, where the underlying set is V but the G_E-action on V^γ is twisted by γ. Note that the functor $V \mapsto V^\gamma$ is exact and covariant.

4.2. **Galois invariance.** Let $e_{\mathfrak{s}}$ be a primitive idempotent of $\mathcal{Z}_n^F$, and let ν be the l-inertial type corresponding to $e_{\mathfrak{s}}$. Then there exists a primitive idempotent $e_{\mathfrak{r}}$ of $\mathcal{Z}_n^E$, which corresponds to the same l-inertial type ν. We have the base change maps $z_{E/F} : e_{\mathfrak{r}}\mathcal{Z}_n^E \to e_{\mathfrak{s}}\mathcal{Z}_n^F$ and $B_{E/F} : R_\nu^E \to R_\nu^F$, as defined in the subsection (2.3.3), with the following identity

$$\mathbb{L}_\nu^F \circ z_{E/F} = B_{E/F} \circ \mathbb{L}_\nu^E. \tag{4.1}$$

We denote by A_E and A_F the Noetherian Λ-algebras $e_{\mathfrak{r}}\mathcal{Z}_n^E$ and $e_{\mathfrak{s}}\mathcal{Z}_n^E$ respectively. Note that the ring A_F is considered as an A_E-module via the map $z_{E/F}$.

Lemma 4.1. *Let $\mathcal{V}$ be the co-Whittaker $A_F[G_n(E)]$-module $e_{\mathfrak{r}}W_n^E \otimes_{A_E} A_F$. Then we have*

$$\mathbb{W}(\mathcal{V}, \psi_{E,A_F}) = \mathbb{W}(\mathcal{V}^\gamma, \psi_{E,A_F}).$$

Proof. We use the local converse therorem ([LM20, Theorem 1.1]) for co-Whittaker modules. It is enough to show that

$$\gamma(X, \mathcal{V}, \tau, \psi_E) = \gamma(X, \mathcal{V}^\gamma, \tau, \psi_E),$$

for τ varying over all irreducible generic integral $\overline{\mathcal{K}}$-representation of $G_t(E)$, for $1 \leq t \leq [\frac{n}{2}]$. For a specialization $x : A_F \to \overline{\mathcal{K}}$, the $G_n(E)$ representation $(e_{\mathfrak{r}}W_n^E \otimes_{A_E} A_F) \otimes_x \overline{\mathcal{K}}$, denoted by Π_x, is the base change lift of a smooth representation π_x of $G_n(F)$. In particular, $\Pi_x^\gamma \simeq \Pi_x$, for all $\gamma \in \Gamma$. Since

$$x(\gamma(X, \mathcal{V}, \tau, \psi_E)) = \gamma(X, \Pi_x, \tau, \psi_E),$$

and

$$x(\gamma(X, \mathcal{V}^\gamma, \tau, \psi_E)) = \gamma(X, \Pi_x^\gamma, \tau, \psi_E),$$

we have

$$x(\gamma(X, \mathcal{V}, \tau, \psi_E)) = x(\gamma(X, \mathcal{V}^\gamma, \tau, \psi_E)),$$

for all $\overline{\mathcal{K}}$-points of A_F. The lemma now follows since A_F is reduced and flat over Λ. □

From these, we deduce:

Lemma 4.2. *Let $\mathcal{V}$ be the co-Whittaker module $e_{\mathfrak{r}}W_n^E \otimes_{A_E} A_F$. Then the Whittaker space $\mathbb{W}(\mathcal{V}, \psi_{E,A_F})$ is invariant under the action of* $\mathrm{Gal}(E/F)$.

Proof. Since $\mathcal{V}$ is co-Whittaker, we have the A_F-module isomorphism $\mathcal{V}^{(n)} \simeq A_F$. Precomposing this isomorphism with the quotient map $\mathcal{V} \to \mathcal{V}^{(n)}$ induces the A_F-linear map $\mathcal{W} : \mathcal{V} \to A_F$ with

$$\mathcal{W}(n.v) = \Psi_{E,A_F}(n)\mathcal{W}(v),$$

for all $v \in \mathcal{V}$ and $n \in N_n(E)$. Moreover, for any $\gamma \in \mathrm{Gal}(E/F)$, we have

$$\mathcal{W}(\gamma(n).v) = \Psi_{E,A_F}(\gamma(n))\mathcal{W}(v) = \Psi_{E,A_F}(n)\mathcal{W}(v).$$

Therefore, the Whittaker spaces of $\mathcal{V}$ and $\mathcal{V}^\gamma$ with respect to the character ψ_E are induced by the same linear map $\mathcal{W}$. Let W_v be an element of $\mathbb{W}(\mathcal{V}, \psi_{E,A_F})$. Then

$$(\gamma^{-1}.W_v)(g) = \mathcal{W}(\gamma(g).v).$$

This shows that $\gamma^{-1}.W_v \in \mathbb{W}(\mathcal{V}^\gamma, \psi_{E,A_F})$. But $\mathbb{W}(\mathcal{V}^\gamma, \psi_{E,A_F}) = \mathbb{W}(\mathcal{V}, \psi_{E,A_F})$, by Lemma 4.1. This completes the proof. □

We now prove the following theorem.

Theorem 4.3. *Let $\mathcal{V}$ be the co-Whittaker $A_F[G_n(E)]$-module $e_{\mathfrak{r}}W_n^E \otimes_{A_E} A_F$, and let $\mathcal{V}_F$ be the $\mathbb{F}[G_n(F)]$-module $e_{\mathfrak{s}}W_n^F \otimes_\Lambda \mathbb{F}$. Let R be the $\mathbb{F}$-algebra $A_F \otimes_\Lambda \mathbb{F}$. Then there is a $G_n(F)$-stable subspace $\mathcal{M}$ of the Tate cohomology group $\widehat{H}^0(\mathbb{W}(\mathcal{V}, \psi_{E,A_F}))$ with $G_n(F)$-equivariant surjection*

$$\mathcal{M} \longrightarrow \mathbb{W}(\mathcal{V}_F^{(l)}, \psi_{F,R}^l).$$

Proof. We prove the above theorem using induction on the integer n.

4.2.1. Let us consider the case $n = 1$. Recall that $\mathcal{Z}_1^K$ is the convolution Λ-algebra $C_c^\infty(K^\times, \Lambda)$. Let P_K be the prime to l part of $\mathfrak{o}_K^\times$. For any character $\eta : P_K \to \Lambda^\times$, we get an idempotent e_η of $\mathcal{Z}_1^K$, and the Λ-algebra $e_\eta \mathcal{Z}_1^K$ is equal to $\mathcal{H}(K^\times, P_K, \eta)$–the endomorphism algebra of the $\Lambda[E^\times]$ representation $\mathrm{ind}_{P_K}^{K^\times} \eta$. Local class field theory gives a character ν of $I_K^{(l)}$ assocaited with η. The ring R_ν^K is isomorphic to $\Lambda[E^\times/P_K]$ and $\mathbb{L}_\eta^K$ is the identity map. Let $\chi : F^\times \to \mathbb{F}$ be a character and let $\tilde{\chi}$ be the character $\chi \circ \mathrm{Nr}_{E/F}$. The characters χ and $\tilde{\chi}$ give rise to idempotents $e_{\mathfrak{s}}$ and $e_{\mathfrak{r}}$ of $\mathcal{Z}_1^F$ and $\mathcal{Z}_1^E$ respectively. From local class field theory the map $B_{E/F}$ is induced by the norm map $\mathrm{Nr}_{E/F} : E^\times \to F^\times$. The module $e_{\mathfrak{r}}\mathcal{Z}_1^E$ also has a natural $E^\times$-action and the identification

$$e_{\mathfrak{r}}\mathcal{Z}_1^E \otimes_{z_{E/F}} e_{\mathfrak{s}}\mathcal{Z}_1^F \simeq e_{\mathfrak{s}}\mathcal{Z}_1^F,$$

makes $e_{\mathfrak{s}}\mathcal{Z}_1^F$ an $e_{\mathfrak{s}}\mathcal{Z}_1^F[E^\times]$-module. Thus, we have

$$\widehat{H}^0(e_{\mathfrak{s}}\mathcal{Z}_1^F) \xrightarrow{\sim} (e_{\mathfrak{s}}\mathcal{Z}_1^F \otimes_\Lambda \mathbb{F})^{(l)}.$$

4.2.2. So now, we assume that the result is true for $n-1$. We denote by τ_E (resp. τ_F) the action of $G_n(E)$ (resp. $G_n(F)$) on the space $\mathcal{V}$ (resp. $\mathcal{V}_F$). Recall that $\mathbb{W}(\mathcal{V}, \psi_{E,A_F})$ is invariant under Γ-action. Let Φ_n be the composite map

$$\mathbb{K}(\mathcal{V}, \psi_{E,A_F})^\Gamma \xrightarrow{\theta_\ell} \mathrm{Ind}_{N_n(E)}^{P_n(E)}(\Psi_{E,R}) \xrightarrow{\mathrm{res}_{P_n(F)}} \mathrm{Ind}_{N_n(F)}^{P_n(F)}(\Psi_{F,R}^l),$$

where the map θ_ℓ is induced by the morphism $A_F \to R$, sending x to $x \otimes 1$. Note that the map Φ_n factorizes through the Tate cohomology space $\widehat{H}^0(\mathbb{K}(\mathcal{V}, \psi_{E,A_F}))$. Let $\mathcal{M}(\psi_F)$ be the inverse image of the Kirillov space $\mathbb{K}(\mathcal{V}_F^{(l)}, \psi_{F,R}^l)$ under Φ_n. Note that $\mathcal{M}(\psi_F)$ is non-zero, and it is stable under the action of $P_n(F)$ with non-zero $P_n(F)$-equivariant map

$$\Phi_n : \mathcal{M}(\psi_F) \longrightarrow \mathbb{K}(\mathcal{V}_F^{(l)}, \psi_{F,R}^l).$$

Let V be an element of $\mathcal{M}(\psi_F)$. We will show that

$$\Phi_n(\overline{\tau_E(w_n)}V) = \tau_F^{(l)}(w_n)\Phi_n(V), \tag{4.2}$$

where $\overline{\tau_E(w_n)}$ is the induced action of $\tau_E(w_n)$ on $\widehat{H}^0(\mathbb{K}(\mathcal{V}, \psi_{E,A_F}))$.

4.2.3. Let $e_{\mathfrak{s}'}$ be an arbitrary primitive idempotent of $\mathcal{Z}_{n-1}^F$ and let ν' be the l-inertial type corresponding to $e_{\mathfrak{s}'}$. Then there exists a primitive idempotent $e_{\mathfrak{r}'}$ of $\mathcal{Z}_{n-1}^E$, which corresponds to the same l-inertial type ν'. If A'_E (resp. A'_F) denotes the Noetherian Λ-algebras $e_{\mathfrak{r}'}\mathcal{Z}_{n-1}^E$ (resp. $e_{\mathfrak{s}'}\mathcal{Z}_{n-1}^F$), then we have the base change map $z'_{E/F} : A'_E \to A'_F$ with

$$\mathbb{L}_{\nu'}^F \circ z'_{E/F} = B'_{E/F} \circ \mathbb{L}_{\nu'}^E. \tag{4.3}$$

We denote by $\mathcal{V}'$ the co-Whittaker $A'_F[G_{n-1}(E)]$-module $e'_{\mathfrak{r}}W_{n-1}^E \otimes_{A'_E} A'_F$. Let R' be the $\mathbb{F}$-algebra $A'_F \otimes_\Lambda \mathbb{F}$, and let $\mathcal{V}'_F$ the co-Whittaker $R'[G_{n-1}(F)]$-module $e_{\mathfrak{s}'}W_{n-1}^F \otimes_\Lambda \mathbb{F}$. Using induction hypothesis, there is a $G_{n-1}(F)$-stable subspace $\mathcal{M}'(\psi_F)$ of the Tate cohomology group $\widehat{H}^0(\mathbb{W}(\mathcal{V}', \psi_{E,A'_F}^{-1}))$ such that the non-zero $P_n(F)$-equivariant map

$$\Phi_{n-1} : \mathbb{K}(\mathcal{V}', \psi_{E,A'_F}^{-1})^\Gamma \xrightarrow{\theta'_\ell} \mathrm{Ind}_{N_{n-1}(E)}^{P_{n-1}(E)}(\Psi_{E,R'}^{-1}) \xrightarrow{\mathrm{res}_{P_{n-1}(F)}} \mathrm{Ind}_{N_{n-1}(F)}^{P_{n-1}(F)}(\Psi_{F,R'}^{-l}),$$

gives the $G_n(F)$-equivariant surjection

$$\mathcal{M}'(\psi_F) \longrightarrow \mathbb{W}(\mathcal{V}'_F, \psi_{F,R'}^{-l}),$$

which we also denote by Φ_{n-1}. Here, the map θ'_ℓ is induced by the morphism $A'_F \to R'$, sending α to $\alpha \otimes 1$. We denote by τ'_E (resp. τ'_F) the action of $G_{n-1}(E)$ (resp. $G_{n-1}(F)$) on the space $\mathcal{V}'$ (resp. $\mathcal{V}'_F$).

4.2.4. Let V' be an element of $\mathbb{W}(\mathcal{V}'^{(l)}_F, \psi^{-l}_{F,R'})$. Then there exists an element W' in the Whittaker space $\mathbb{W}(\mathcal{V}', \psi_{E,A'_F})^\Gamma$ such that $\mathrm{res}_{P_{n-1}(E)}(W')$ is mapped to $\mathrm{res}_{P_{n-1}(F)}(V')$ under Φ_{n-1}. Let W be the element of $\mathbb{W}(\mathcal{V}, \psi_{E,A_F})^\Gamma$ such that $\mathrm{res}_{P_n(E)}(W) = V$. Then, using functional equation over E, we get

$$\sum_{k\in\mathbb{Z}} c^E_k(\widetilde{W}, \widetilde{W'})X^{-fk} = \varpi_{\tau'_E}(-1)^{n-1}\gamma(X, \mathcal{V}, \mathcal{V}', \psi_E)\sum_{k\in\mathbb{Z}} c^E_k(W, W')X^{fk},$$

where f is the residue degree of the extension E/F. Applying the morphism $(\theta_\ell \otimes \theta'_\ell)$, and using the identity (3.1) and Remark 3.5, the above relation becomes

$$\begin{aligned}\sum_{k\in\mathbb{Z}} & c^F_{-k}\big(\overline{\tau_E(w_n)}\theta_\ell(W), \overline{\tau'_E(w_{n-1})}\theta'_\ell(W')\big)X^{-lk} \\ &= \varpi_{\tau'_E}(-1)^{n-1}(\theta_\ell \otimes \theta'_\ell)(\gamma(X, \mathcal{V}, \mathcal{V}', \psi_E))\sum_{k\in\mathbb{Z}} c^F_k\big(\theta_\ell(W), V'\big)X^{lk}.\end{aligned} \tag{4.4}$$

By induction hypothesis, we have

$$\Phi_{n-1}(\overline{\tau'_E(w_{n-1})}W') = \tau'^{(l)}_F(w_{n-1})V'.$$

Using this, it follows from (4.4) that

$$\begin{aligned}\sum_{k\in\mathbb{Z}} & c^F_{-k}\big(\overline{\tau_E(w_n)}\theta_\ell(W), \tau'^{(l)}_F(w_{n-1})V'\big)X^{-lk} \\ &= \varpi_{\tau'_F}(-1)^{l(n-1)}(\theta_\ell \otimes \theta'_\ell)(\gamma(X, \mathcal{V}, \mathcal{V}', \psi_E))\sum_{k\in\mathbb{Z}} c^F_k(\theta_\ell(W), V')X^{lk}.\end{aligned} \tag{4.5}$$

4.2.5. Recall that $\Phi_n(V)$ is an element of the Kirillov space $\mathbb{K}(\mathcal{V}^{(l)}_F, \psi^l_{F,R})$. Let U be the element of $\mathbb{W}(\mathcal{V}^{(l)}_F, \psi^l_{F,R})$ such that $\mathrm{res}_{P_n(F)}(U) = \Phi_n(V)$. By Theorem 3.2, the assertion (4.2) is equivalent to the following identity:

$$\begin{aligned}\sum_{k\in\mathbb{Z}} & c^F_{-k}\big(\Phi_n(\overline{\tau_E(w_n)}\theta_\ell(W)), \tau'^{(l)}_F(w_{n-1})V'\big)X^{-lk} \\ &= \sum_{k\in\mathbb{Z}} c^F_{-k}\big(\tau^{(l)}_F(w_n)U, \tau'^{(l)}_F(w_{n-1})V'\big)X^{-lk}\end{aligned} \tag{4.6}$$

From functional equation over F, we get

$$\sum_{k\in\mathbb{Z}} c^F_k(\widetilde{U}, \widetilde{V'})X^{-k} = \varpi_{\tau'_F}(-1)^{l(n-1)}\gamma(X, \mathcal{V}^{(l)}_F, \mathcal{V}'^{(l)}_F, \psi^l_F)\sum_{k\in\mathbb{Z}} c^F_k(U, V')X^k.$$

Using the relation (3.1) and replacing the variable X by X^l to the above equality, we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}} & c^F_{-k}\big(\tau^{(l)}_F(w_n)U,\, \tau'^{(l)}_F(w_{n-1})V'\big)X^{-lk} \\ &= \varpi_{\tau'_F}(-1)^{l(n-1)}\gamma(X, \mathcal{V}^{(l)}_F, \mathcal{V}'^{(l)}_F, \psi^l_F)\sum_{k\in\mathbb{Z}} c^F_k(U, V')X^{lk}.\end{aligned}$$

4.2.6. Comparing the above equation with (4.5), the equality (4.6) is now equivalent to the following identity of gamma factors

$$(\theta_\ell \otimes \theta'_\ell)(\gamma(X, \mathcal{V}, \mathcal{V}', \psi_E)) = \gamma(X, \mathcal{V}^{(l)}_F, \mathcal{V}'^{(l)}_F, \psi^l_F).$$

First, note that the base change maps $B_{E/F}$ and $B'_{E/F}$ together with the universal property of the pairs (R^E_ν, ρ^E_ν) and $(R^E_{\nu'}, \rho^F_{\nu'})$ induces the isomorphisms

$$\mathrm{res}_{\mathcal{W}_E}(\rho^F_\nu) \simeq \rho^E_\nu \otimes_{R^E_\nu} R^F_\nu \text{ and } \mathrm{res}_{\mathcal{W}_E}(\rho^F_{\nu'}) \simeq \rho^E_{\nu'} \otimes_{R^E_{\nu'}} R^F_{\nu'}.$$

Using these isomorphisms and the commutativity relations (4.1) and (4.3), we have

$$\begin{aligned}\gamma(X,\mathcal{V},\mathcal{V}',\psi_E) &= (z_{E/F}\circ(\mathbb{L}^E_\nu)^{-1})\otimes(z'_{E/F}\circ(\mathbb{L}^E_{\nu'})^{-1})\big(\gamma(X,\rho^E_\nu\otimes\rho^E_{\nu'},\psi_E)\big)\\ &= ((\mathbb{L}^F_\nu)^{-1}\circ B_{E/F})\otimes((\mathbb{L}^F_{\nu'})^{-1}\circ B'_{E/F})(\gamma(X,\rho^E_\nu\otimes\rho^E_{\nu'},\psi_E))\\ &= ((\mathbb{L}^F_\nu)^{-1}\otimes(\mathbb{L}^F_{\nu'})^{-1})\big(\gamma(X,(\rho^E_\nu\otimes_{R^E_\nu}R^F_\nu)\otimes(\rho^E_{\nu'}\otimes_{R^E_{\nu'}}R^F_{\nu'}),\psi_E)\big)\\ &= ((\mathbb{L}^F_\nu)^{-1}\otimes(\mathbb{L}^F_{\nu'})^{-1})\big(\gamma(X,\rho^F_\nu\otimes\rho^F_{\nu'}\otimes\mathrm{ind}^{\mathcal{W}_F}_{\mathcal{W}_E}(1_E),\psi_F)\big)\\ &= \prod_\eta\gamma(X,e_{\mathfrak{s}}W^F_n,e_{\mathfrak{s}'}W^F_{n-1}\otimes\eta,\psi_F),\end{aligned}$$

where η runs over the characters of $\mathrm{Gal}(E/F)$. Finally, applying the morphism $(\theta_\ell\otimes\theta'_\ell)$ and using the identity (3.2), we get

$$(\theta_\ell\otimes\theta'_\ell)(\gamma(X,\mathcal{V},\mathcal{V}',\psi_E))=\gamma(X^l,\mathcal{V}^{(l)}_F,\mathcal{V}'^{(l)}_F,\psi^l_F).$$

This shows that Φ_n is a non-zero $G_n(F)$-equivariant map. Since $\mathcal{V}^{(l)}_F$ is co-Whittaker $R[G_n(F)]$-module, the map Φ_n is infact a surjection. This completes the proof. □

5. Tate cohomology of generic representations

In this section, we prove our main result (Theorem 1.1) using Theorem 4.3. We will continue with the notations of the preceding section.

5.1. Let π_F be an integral generic representation of $G_n(F)$ over a $\overline{\mathcal{K}}$-vector space whose mod-l inertial supercuspidal support is $\mathfrak{s}$. Let π_E be the base change lifting of π_F with mod-l inertial supercuspidal support $\mathfrak{r}$. Recall that we have the base change map

$$z_{E/F}:e_{\mathfrak{r}}\mathcal{Z}^E_n\longrightarrow e_{\mathfrak{s}}\mathcal{Z}^F_n.$$

If f_{π_F} (resp. f_{π_E}) denotes the supercuspidal support of π_F (resp. π_E), then we have

$$f_{\pi_E}=f_{\pi_F}\circ z_{E/F}.$$

Let $J_\ell(\pi_F)$ be the unique generic sub-quotient of the mod-l reduction $r_\ell(\pi_F)$. The supercuspidal support of the $\mathbb{F}$-representation $J_\ell(\pi_F)$ is equal to $\mathfrak{s}$ ([Hel16a, Proposition 4.13]). We denote by A_E and A_F the Noetherian Λ-algebras $e_{\mathfrak{r}}\mathcal{Z}^E_n$ and $e_{\mathfrak{s}}\mathcal{Z}^F_n$ respectively.

Theorem 5.1. *Let F be a p-adic field and let E be a finite Galois extension of F with $[E:F]=l$, where l and p are distinct odd primes. Let π_F be an integral generic $\overline{\mathcal{K}}$-representation of $G_n(F)$, and let π_E be the base change lifting of π_F to $G_n(E)$. Let $\mathbb{W}^0(\pi_E,\psi_{E,\overline{\mathcal{K}}})$ be the space of all $\mathcal{O}$-valued functions in $\mathbb{W}(\pi_E,\psi_{E,\overline{\mathcal{K}}})$. Then the $\mathbb{F}$-representation $J_\ell(\pi_F)^{(l)}$ is a subquotient of the Tate cohomology group $\widehat{H}^0(\mathbb{W}^0(\pi_E,\psi_{E,\overline{\mathcal{K}}}))$.*

Proof. The proof relies on the completeness of Whittaker models (Theorem 3.2). Although the proof follows from the same line of arguments as in Theorem 4.3, we provide it in detail for the purpose of the completeness. We follow the same notations and terminologies as in Theorem 4.3.

5.1.1. Note that the lattice $\mathbb{W}^0(\pi_E,\psi_{E,\overline{\mathcal{K}}})$ is stable under the action of $G_n(E)$ ([Vig04, Theorem]). Let Φ_n be the composite map

$$\mathbb{K}^0(\pi_E,\psi_{E,\overline{\mathcal{K}}})^\Gamma\xrightarrow{r_\ell}\mathrm{Ind}^{P_n(E)}_{N_n(E)}(\Psi^l_{E,\mathbb{F}})\xrightarrow{\mathrm{res}_{P_n(F)}}\mathrm{Ind}^{P_n(F)}_{N_n(F)}(\Psi^l_{F,\mathbb{F}}),$$

where r_ℓ denotes the pointwise mod-l reduction. It is clear that the map Φ_n factorizes through the space $\widehat{H}^0(\mathbb{K}^0(\pi_E,\psi_{E,\overline{\mathcal{K}}}))$. Let $\mathcal{N}(\psi_F)$ be the inverse image of the Kirillov space $\mathbb{K}(J_\ell(\pi_F)^{(l)},\psi^l_{F,\mathbb{F}})$ under Φ_n. Then $\mathcal{N}(\psi_F)$ is a non-zero $P_n(F)$-stable subspace of $\widehat{H}^0(\mathbb{K}^0(\pi_E,\psi_{E,\overline{\mathcal{K}}}))$ with a non-zero $P_n(F)$-equivariant map

$$\Phi_n:\mathcal{N}(\psi_F)\longrightarrow\mathbb{K}(J_\ell(\pi_F)^{(l)},\psi^l_{F,\mathbb{F}}).$$

We will prove that $\mathcal{N}(\psi_F)$ is $G_n(F)$-stable and the map Φ_n is $G_n(F)$-equivariant. To be precise, let V be an element of $\mathcal{N}(\psi_F)$. We will show that

$$\Phi_n(\overline{\pi_E(w_n)}V) = J_\ell(\pi_F)^{(l)}(w_n)\Phi_n(V),$$

where $\overline{\pi_E(w_n)}$ is the linear operator on $\widehat{H}^0(\mathbb{K}(\pi_E, \psi_{E,\overline{\mathcal{K}}}))$ induced by $\pi_E(w_n)$. Let W be an element of $\mathbb{W}^0(\pi_E, \psi_{E,\overline{\mathcal{K}}})^\Gamma$ with $\mathrm{res}_{P_n(E)}(W) = V$. By Theorem 3.2, the above assertion is equivalent to the following identity:

$$\sum_{k\in\mathbb{Z}} c_k^F\big(\Phi_n(\overline{\pi_E(w_n)}V), V'\big)X^k = \sum_{k\in\mathbb{Z}} c_k^F\big(J_\ell(\pi_F)^{(l)}(w_n)\Phi_n(V), V'\big)X^k, \tag{5.1}$$

for all $V' \in \mathbb{W}\big((e_{\mathfrak{s}'}W_{n-1}^F \otimes_\Lambda \mathbb{F})^{(l)}, \psi_{F,R'}^{-l}\big)$ and for all primitive idempotents $e_{\mathfrak{s}'}$ of $\mathcal{Z}_{n-1}^F$. Here R' denotes the ring $e_{\mathfrak{s}'}\mathcal{Z}_{n-1}^F \otimes_\Lambda \mathbb{F}$.

5.1.2. Let $e_{\mathfrak{s}'}$ be a primitive idempotent of $\mathcal{Z}_{n-1}^F$. Then there exists a primitive idempotent $e_{\mathfrak{r}'}$ of $\mathcal{Z}_{n-1}^E$ such that we have the base change map $z'_{E/F} : A'_E \to A'_F$ with

$$\mathbb{L}_{\nu'}^F \circ z'_{E/F} = B'_{E/F} \circ \mathbb{L}_{\nu'}^E,$$

where ν' is the l-inertial type, which corresponds to both $e_{\mathfrak{r}'}$ and $e_{\mathfrak{s}'}$. As before, we denote by $\mathcal{V}'$ and $\mathcal{V}'_F$ the co-Whittaker modules $e_{\mathfrak{r}'}W_{n-1}^E \otimes_{A'_E} A'_F$ and $e_{\mathfrak{s}'}W_{n-1}^F \otimes_\Lambda \mathbb{F}$ respectively. The action of $G_n(E)$ (resp. $G_n(F)$) on the space $\mathcal{V}'$ (resp. $\mathcal{V}'_F$) is denoted by π'_E (resp. π'_F).

5.1.3. Let V' be an element of $\mathbb{W}(\mathcal{V}_F'^{(l)}, \psi_{F,R'}^{-l})$. Following the arguments of Theorem 4.3, we get an element W' in $\mathbb{W}(\mathcal{V}', \psi_{E,A'_F}^{-1})^\Gamma$ such that W' is mapped to V' under the composition

$$\Phi_{n-1} : \mathbb{W}(\mathcal{V}', \psi_{E,A'_F}^{-1})^\Gamma \xrightarrow{\theta'_\ell} \mathrm{Ind}_{N_{n-1}(E)}^{P_{n-1}(E)}(\Psi_{E,R'}^{-1}) \xrightarrow{\mathrm{res}_{G_{n-1}(F)}} \mathrm{Ind}_{N_{n-1}(F)}^{G_{n-1}(F)}(\Psi_{F,R'}^{-l}),$$

with the following identity

$$\Phi_{n-1}(\pi'_E(w_{n-1})W') = \pi_F'^{(l)}(w_{n-1})V'. \tag{5.2}$$

Recall that the map θ'_ℓ is induced by the morphism $A'_F \to R'$, sending x to $x \otimes 1$. From functional equation over E, we get

$$\sum_{k\in\mathbb{Z}} c_{-k}^E\big(\overline{\pi_E(w_n)}r_l(W), \pi'_E(w_{n-1})\theta'_\ell(W')\big)X^{-fk}$$
$$= \varpi_{\pi'_E}(-1)^{n-1}(r_\ell \otimes \theta'_\ell)\big(\gamma(X, \pi_E, \mathcal{V}', \psi_E)\big) \sum_{k\in\mathbb{Z}} c_k^E\big(r_l(W), V'\big)X^{fk}.$$

Using Remark 3.5 and the relation (5.2), it follows from the above identity that

$$\begin{aligned}\sum_{k\in\mathbb{Z}} c_{-k}^F\big(\overline{\pi_E(w_n)}r_l(W), \pi_F'^{(l)}(w_{n-1})V'\big)X^{-lk}\\ = \varpi_{\pi'_F}(-1)^{l(n-1)}(r_\ell \otimes \theta'_\ell)\big(\gamma(X, \pi_E, \mathcal{V}', \psi_E)\big) \sum_{k\in\mathbb{Z}} c_k^F(r_l(W), V')X^{lk}.\end{aligned} \tag{5.3}$$

5.1.4. Let U be an element of $\mathbb{W}(J_\ell(\pi_F)^{(l)}, \psi_{F,\mathbb{F}}^{-l})$ such that $\mathrm{res}_{P_n(F)}(U) = \Phi_n(V)$. The functional equation over F gives

$$\sum_{k\in\mathbb{Z}} c_{-k}^F\big(J_\ell(\pi_F)^{(l)}(w_n)U, \pi_F'^{(l)}(w_{n-1})V'\big)X^{-k}$$
$$= \varpi_{\pi'_F}(-1)^{l(n-1)}\gamma(X, J_\ell(\pi_F)^{(l)}, \mathcal{V}_F'^{(l)}, \psi_F^l) \sum_{k\in\mathbb{Z}} c_k^F(U, V')X^k$$

Replacing X by X^l to the above equation, we get

$$\sum_{k\in\mathbb{Z}} c^F_{-k}\big(J_\ell(\pi_F)^{(l)}(w_n)U, \pi_F'^{(l)}(w_{n-1})W'\big)X^{-lk} = \varpi_{\pi_F'}(-1)^{l(n-1)}\gamma(X^l, J_\ell(\pi_F)^{(l)}, \mathcal{V}_F'^{(l)}, \psi_F^l)\sum_{k\in\mathbb{Z}} c^F_k(U, W')X^{lk}. \tag{5.4}$$

5.1.5. Comparing the relations (5.3) and (5.4), the assertion (5.1) is now equivalent to the following identity of gamma factors

$$(r_\ell\otimes\theta_\ell')\big(\gamma(X,\pi_E,\mathcal{V}',\psi_E)\big) = \gamma(X^l, J_\ell(\pi_F)^{(l)}, \mathcal{V}_F'^{(l)}, \psi_F^l).$$

This follows from similar type of computation of gamma factors as we did in Theorem 4.3. First, note that

$$\gamma(X,\pi_E,\mathcal{V}',\psi_E) = (f_{\pi_E}\otimes z'_{E/F})(\gamma(X, e_{\mathfrak{r}}W_n^E, e_{\mathfrak{r}'}W_{n-1}^E, \psi_E)).$$

Using the relations (4.1) and (4.3) and the fact that $f_{\pi_E} = f_{\pi_F}\circ z_{E/F}$, we get the following equality:

$$\gamma(X,\pi_E,\mathcal{V}',\psi_E) = \prod_\eta \gamma(X,\pi_F, e_{\mathfrak{s}'}W_{n-1}^F\otimes\eta, \psi_F),$$

where η runs over the characters of $\mathrm{Gal}(E/F)$. Applying the morphism $(r_\ell\otimes\theta_\ell')$ to the above identity, we get

$$(r_\ell\otimes\theta_\ell')(\gamma(X,\pi_E,\mathcal{V}',\psi_E)) = \gamma(X, J_\ell(\pi_F), \mathcal{V}_F', \psi_F)^l.$$

Finally, the identity (3.3) gives

$$(r_\ell\otimes\theta_\ell')(\gamma(X,\pi_E,\mathcal{V}',\psi_E)) = \gamma(X^l, J_\ell(\pi_F)^{(l)}, \mathcal{V}_F'^{(l)}, \psi_F^l).$$

Thus, the space $\mathcal{N}(\psi_F)$ is stable under the action of $G_n(F)$ and the map Φ_n is surjective. Now, using [DN22, Proposition 6.3], we get that there is a unique generic subquotient of $\widehat{H}^0(\mathbb{W}(\pi_E, \psi_{E,\overline{\mathcal{K}}})$, and this is necessarily equal to $J_\ell(\pi_F)^{(l)}$. This completes the proof. □

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