TYPICAL REPRESENTATIONS FOR LEVEL ZERO BERNSTEIN COMPONENTS OF $\mathrm{GL}_n(F)$

SANTOSH NADIMPALLI

Abstract. Let F be a non-discrete non-Archimedean locally compact field. In this article for a level zero Bernstein component s, we classify those irreducible smooth representations of $\mathrm{GL}_n(\mathcal{O}_F)$ (called typical representations) whose appearance in a smooth irreducible representation π of $\mathrm{GL}_n(F)$ implies that the cuspidal support of π is s. These results extend, for level zero representations, the results of Henniart and Paškūnas on cuspidal representations. The results are independent of the characteristic of the base field.

1. Introduction

Let F be a non-discrete non-Archimedean locally compact field. The isomorphism classes of irreducible smooth complex representations of $\mathrm{GL}_n(F)$, denoted by $\mathcal{A}_n$, can be decomposed as a disjoint union

$$\mathcal{A}_n = \coprod_{s\in\mathcal{B}_n} \mathcal{A}_n(s),$$

where $\mathcal{A}_n(s)$ is defined in terms of parabolic induction and the parameter s is called the Bernstein component or inertial support. In the context of the local Langlands correspondence, the parameter s determines the isomorphism class of the restriction to the inertia subgroup I_F of the Weil–Deligne representation associated by the classical local Langlands correspondence.

The Local class field theory gives a natural isomorphism between I_F and $\mathcal{O}_F^{\times}$, the group of units of the ring of integers of F. It is natural to ask for a relation between the representations of I_F which can be extended to a Weil–Deligne representation and the representations of the maximal compact subgroup $\mathrm{GL}_n(\mathcal{O}_F)$. One natural way would be to understand the cuspidal support of a smooth irreducible representation from its restriction to $\mathrm{GL}_n(\mathcal{O}_F)$. Indeed in several arithmetic applications (see [BM02], [EG14]) it is desired to construct irreducible smooth representations τ_s of the maximal compact subgroup $\mathrm{GL}_n(\mathcal{O}_F)$ such that for any irreducible smooth representation π of $\mathrm{GL}_n(F)$,

$$\mathrm{Hom}_{\mathrm{GL}_n(\mathcal{O}_F)}(\tau_s, \pi) \neq 0 \Rightarrow \pi \in \mathcal{A}_n(s).$$

Such a representation τ_s is called a **typical representation** for s. In this article we completely classify typical representations τ_s for all level-zero Bernstein components (see section 3) of $\mathrm{GL}_n(F)$.

The existence of typical representation, for any s, follows from the theory of types developed by Bushnell and Kutzko. For all $s \in \mathcal{B}_n$, Bushnell and Kutzko constructed pairs (J_s, λ_s) such that for any irreducible smooth representation π of $\mathrm{GL}_n(F)$,

$$\mathrm{Hom}_{J_s}(\lambda_s, \pi) \neq 0 \Leftrightarrow \pi \in \mathcal{A}_n(s).$$

It follows from Frobenius reciprocity that any irreducible subrepresentation of

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_n(\mathcal{O}_F)} \lambda_s \tag{1}$$

is a typical representation for s. In general the representation (1) is not irreducible and it is not known if there are any other typical representations which are not subrepresentations of (1).

For $n = 2$, Henniart (see [BM02]) classified typical representations for all inertial classes. Later Paškūnas (see [Pas05]) classified typical representations occurring in cuspidal representation of $\mathrm{GL}_n(F)$ for $n \geq 3$. It turns out that there exists a unique typical representation occurring in a cuspidal representation. For a

Date: August 9, 2019.

general Bernstein component s, typical representations may not be unique. In this article we classify typical representations for a level zero Bernstein component s.

We now describe the main result of this article. Let s be a level zero Bernstein component and (J_s, λ_s) (see section 3 for a complete description of the type (J_s, λ_s)) be the Bushnell–Kutzko type for the component s. We will prove that

Theorem 1.1. *Any typical representation τ_s for a level-zero Bernstein component s occurs as a subrepresentation of*

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_n(\mathcal{O}_F)}(\lambda_s). \tag{2}$$

In our analysis we will also obtain a certain multiplicity result on the typical representations τ_s (see Corollary 3.3).

This article is based on chapter 3 of my thesis. In my thesis typical representations are classified for several Bernstein components. I would like to thank my thesis advisor Guy Henniart for suggesting this problem and numerous discussions. I thank Corinne Blondel for pointing out several corrections and improvements. I would like to express my deep gratitude to the referee for very useful suggestions and comments.

2. Preliminaries

2.1. **Basic notation.** Let F be a non-Archimedean local field with ring of integers $\mathcal{O}_F$, maximal ideal $\mathfrak{P}_F$ and a finite residue field k_F. All our representations are on vector spaces over $\mathbb{C}$.

Let G be a locally pro-finite group and H be a closed subgroup of G. For (τ, V) a smooth representation of H, we denote by $\mathrm{ind}_H^G(\tau)$ the induced smooth representation of G and by c-$\mathrm{ind}_H^G(\tau)$ the compactly induced representation. When G is the group of F-rational points of an algebraic reductive group, the group G is equipped with a locally profinite topology induced from F. For P the set of F-rational points of an F-parabolic subgroup of G and σ a smooth representation of a Levi subgroup M of P, we denote by $i_P^G(\sigma)$ the normalized parabolically-induced representation.

For any two groups H_1 and H_2 such that $H_2 \subset H_1$ and σ a representation of H_1, we denote by $\mathrm{res}_{H_2}(\sigma)$ the restriction of σ to H_2. We use $\boxtimes$ and $\otimes$ for the tensor product of representations of two different groups and the same group respectively. If H_2 is a subgroup of a group H_1, τ is a representation of H_2 and $h \in H_1$ then we denote by ${}^h\tau$ the representation of hH_2h^{-1} given by $h' \mapsto \tau(h^{-1}h'h)$ for all $h' \in hH_2h^{-1}$.

After recalling some general definitions we will restrict ourself to the case where $G = \mathrm{GL}_n(F)$ and the following notation will be used: We denote by G_n the group $\mathrm{GL}_n(F)$ and by K_n the maximal compact subgroup $\mathrm{GL}_n(\mathcal{O}_F)$. Let $K_n(m)$ the principal congruence subgroup of $\mathrm{GL}_n(\mathcal{O}_F)$ of level m.

Let $I = (n_1, n_2, n_3, \dots, n_r)$ be an ordered partition of a positive integer n. Let R be an $\mathcal{O}_F$ algebra. Let $P_I(R)$ be the group of invertible block upper triangular matrices of type $(n_1, n_2, \dots, n_r)$ with entries in R. We denote by $M_I(R)$ and $U_I(R)$ the subgroups of $P_I(R)$ consisting of block diagonal matrices of type I and the unipotent unipotent matrices of type I respectively. We use the notation P_I, M_I and U_I for $P_I(F)$, $M_I(F)$ and $U_I(F)$ respectively. We call P_I and M_I the standard parabolic subgroup and standard Levi subgroup of type I respectively.

2.2. **Bernstein decomposition and typical representations.** Let $B(G)$ be the set of pairs (M, σ) where M is a Levi subgroup of an F-parabolic subgroup P of G and σ is an irreducible supercuspidal representation of M. Recall that the inertial equivalence relation on $B(G)$ is defined by setting

$$(M_1, \sigma_1) \sim (M_2, \sigma_2)$$

if and only if there exist an element $g \in G$ and an unramified character χ of M_2 such that $M_1 = gM_2g^{-1}$ and $\sigma_1^g \simeq \sigma_2 \otimes \chi$. We denote by $\mathcal{B}_G$ the set of such equivalence classes called **inertial classes** or **Bernstein components**. Any irreducible smooth representation π of G occurs as a sub-representation of a parabolic induction $i_P^G(\sigma)$ where σ is an irreducible supercuspidal representation of a Levi subgroup M of P. The pair (M, σ) is well determined up to G-conjugation. We call the class $s = [M, \sigma]$ the **inertial support of** π. We will use the notation $I(\pi)$ for the inertial support of π.

Let $\mathcal{M}(G)$ be the category of all smooth representations of G. For an inertial class $s = [M, \sigma]$ we denote by $\mathcal{M}_s(G)$ the full sub-category consisting of smooth representations all of whose irreducible sub-quotients appear in the composition series of some $i_P^G(\sigma \otimes \chi)$, with χ an unramified character of M. It is shown by Bernstein (see [Ren10, VI.7.2, Theorem]) that the category $\mathcal{M}(G)$ decomposes as a direct product of $\mathcal{M}_s(G)$ in particular every smooth representation can be written as a direct sum of objects in the categories $\mathcal{M}_s(G)$. We denote by $\mathcal{A}_G(s)$ the set of isomorphism classes of simple objects in the category $\mathcal{M}_s(G)$. If $G = \mathrm{GL}_n(F)$ we use the notation $\mathcal{A}_n(s)$ for $\mathcal{A}_G(s)$ and $\mathcal{B}_n$ for $\mathcal{B}_{\mathrm{GL}_n(F)}$.

Given an irreducible smooth representation ρ of a maximal compact subgroup K of G the compact induction $\pi := \text{c-}\mathrm{ind}_K^G(\rho)$ is a finitely generated smooth representation of G and hence there exists an irreducible G-quotient of π. By Frobenius reciprocity [BH06, Proposition 2.5] we get that ρ occurs in a smooth irreducible representation of G. For a given inertial class, we are interested in the representations ρ of K which only occur in irreducible smooth representations with inertial support s.

Definition 2.1. *Let s be an inertial class for G. An irreducible smooth representation τ of a maximal compact subgroup K of G is called* **K-typical representation for** *s if, for any irreducible smooth representation π of G, $\mathrm{Hom}_K(\tau, \pi) \neq 0$ implies that $\pi \in \mathcal{A}_G(s)$.*

In this article we will confine ourselves to the cases where $G = \mathrm{GL}_n(F) = G_n$, $K = \mathrm{GL}_n(\mathcal{O}_F) = K_n$ and $n \geq 2$ and in these cases we call a K-typical representation for s a typical representation for s. An irreducible representation τ of K_n is called *atypical* if τ occurs in two irreducible smooth representations π_1 and π_2 such that $I(\pi_1) \neq I(\pi_2)$.

For any component $s \in \mathcal{B}_n$, the existence of a typical representation can be deduced from the theory of types developed by Bushnell and Kutzko in the articles [BK99] and [BK93]. Bushnell and Kutzko constructed a pair (J_s, λ_s), which we call a *Bushnell–Kutzko type*, where J_s is a compact open subgroup of $\mathrm{GL}_n(F)$ and λ_s is an irreducible representation of J_s such that for every irreducible smooth representation π of G_n,

$$\mathrm{Hom}_{J_s}(\pi, \lambda_s) \neq 0 \;\Leftrightarrow\; \pi \;\in\; \mathcal{A}_n(s).$$

The group J_s can be arranged to be a subgroup of $\mathrm{GL}_n(\mathcal{O}_F)$ by conjugating with an element of $\mathrm{GL}_n(F)$ and hence we assume that $J_s \subset \mathrm{GL}_n(\mathcal{O}_F)$. It follows from Frobenius reciprocity that any irreducible sub-representation of

$$\mathrm{ind}_{J_s}^{\mathrm{GL}_n(\mathcal{O}_F)}(\lambda_s) \tag{3}$$

is a typical representation. The irreducible sub-representations of (3) are classified by Schneider and Zink in [SZ99, Section 6, $T_{K,\lambda}$ functor].

For $s = [G_n, \sigma]$, Paškūnas in [Pas05, Theorem 8.1] showed that up to isomorphism there exists a unique typical representation for s. More precisely,

Theorem 2.2 (Paškūnas)**.** *Let n be a positive integer greater than one and σ be an irreducible supercuspidal representation of G_n. Let (J_s, λ_s) be a Bushnell–Kutzko type for the component $s = [G_n, \sigma]$ with $J_s \subset \mathrm{GL}_n(\mathcal{O}_F)$. The representation*

$$\mathrm{ind}_{J_s}^{K_n}(\lambda_s)$$

is the unique typical representation for the component $[G_n, \sigma]$ and occurs with multiplicity one in $\sigma \otimes \chi$, for all unramified characters χ of G_n.

We will consider the classification of typical representations for components $[M, \sigma]$ where M is a Levi subgroup of a proper parabolic subgroup of G_n.

Let $s = [M, \sigma]$ be an inertial class of G_n. We will choose a representative for s. Let P be a parabolic subgroup with M as its Levi subgroup. There exists a $g \in G_n$ such that $gPg^{-1} = P_I$ for some ordered partition $I = (n_1, n_2, \ldots, n_r)$ of n. The groups gMg^{-1} and M_I are two Levi subgroups of P_I hence we get an $u \in \mathrm{Rad}\, P_I$ such that $ugM(ug)^{-1} = M_I$. This shows that there exists an element $g' \in G_n$ such that $g'Mg'^{-1} = M_I$. Let J be a permutation of the ordered partition $(n_1, n_2, \ldots, n_r)$. We can choose a $g'' \in G_n$ such that M_I and M_J are conjugate so the two pairs (M, σ) and $(M_J, \sigma^{g'g''})$ are inertially equivalent. In certain cases it is convenient to choose a particular permutation. For example in the proof of the main theorem in this article we choose $J = (n'_1, n'_2, \dots n'_r)$ such that $n'_i \leq n'_j$ for all $i \leq j$. We denote by σ_I and σ_J the representations $\sigma^{g'}$ and $\sigma^{g'g''}$ respectively and hence $s = [M_I, \sigma_I] = [M_J, \sigma_J]$.

Let τ be a typical representation for the component s. The representation τ occurs as a K_n sub-representation of a G_n-irreducible smooth representation π (see the reasoning given in the paragraph above Definition 2.1). From the above paragraph π occurs in the composition series of $i_{P_I}^{G_n}(\sigma_I)$ where σ_I is a supercuspidal representation of M_I. Hence to classify typical representations we fix a pair $(M_I, \sigma_I) \sim (M, \sigma)$ and examine the K_n-irreducible sub-representations of

$$\mathrm{res}_{K_n}(i_{P_I}^{G_n}(\sigma_I)),$$

looking for possible typical representations for s.

By the Iwasawa decomposition $G_n = K_n P_I$ we get that

$$\mathrm{res}_{K_n}(i_{P_I}^{G_n}(\sigma_I)) \simeq \mathrm{ind}_{P_I \cap K_n}^{K_n}(\sigma_I).$$

We write σ_I as $\boxtimes_{i=1}^r \sigma_i$ where σ_i is a supercuspidal representation of G_{n_i} for $1 \leq i \leq r$. We denote by τ_i the unique typical representation for the component $[G_{n_i}, \sigma_i]$ for $1 \leq i \leq r$ and let τ_I be the $M_I(\mathcal{O}_F)$-representation $\boxtimes_{i=1}^r \tau_i$. Will Conley observed in his thesis (see [Wil10, Theorem 2.28]) that the representation

$$\mathrm{ind}_{P_I \cap K_n}^{K_n}(\tau_I)$$

admits a complement in $\mathrm{ind}_{P_I \cap K_n}^{K_n}(\sigma_I)$ whose irreducible sub-representations are atypical for s. We prove a mild generalisation which will be used later in proofs by induction.

Let $t_i = [M_i, \lambda_i]$ be a Bernstein component of G_{n_i} for $1 \leq i \leq r$. Let σ_i be a smooth representation from $\mathcal{M}_{t_i}(G_{n_i})$. We suppose

$$\mathrm{res}_{K_{n_i}} \sigma_i = \tau_i^0 \oplus \tau_i^1$$

for $1 \leq i \leq r$ such that irreducible subrepresentations of τ_i^1 are atypical. We denote by t the Bernstein component

$$[M_1 \times M_2 \times \cdots \times M_r, \lambda_1 \boxtimes \lambda_2 \boxtimes \cdots \boxtimes \lambda_r]$$

of G_n. The component t is independent of the choice of representatives (M_i, λ_i). Let $\tau_I^0 = \boxtimes_{i=1}^r \tau_i^0$ and $\sigma_I = \boxtimes_{i=1}^r (\sigma_i)$.

Proposition 2.3. *The representation*

$$\mathrm{ind}_{P_I \cap K_n}^{K_n}(\tau_I^0)$$

admits a complement in $\mathrm{res}_{K_n} i_{P_I}^{G_n}(\sigma_I)$ *with all its irreducible subrepresentations atypical.*

Proof. Any K_n-irreducible subrepresentation of $\mathrm{res}_{K_n} i_{P_I}^{G_n}(\sigma_I)$ occurs as a subrepresentation of

$$\mathrm{ind}_{P_I \cap K_n}^{K_n}(\boxtimes_{i=1}^r \gamma_i) \tag{4}$$

where γ_i is a K_{n_i}-irreducible subrepresentation of σ_i. Suppose there exists $N \leq r$ such that γ_N occurs in τ_N^1. Thus there exists a component $t'_N \in \mathcal{B}_{n_N}$ such that t'_N is equal to $[M'_N, \lambda'_N] \neq t_N$ and γ_N occurs in the restriction $\mathrm{res}_{K_N} i_{P'_N}^{\mathrm{GL}_{n_N}(F)}(\lambda'_N)$. Hence the representation (4) occurs as a K_n-subrepresentation of

$$i_{P_I}^{G_n}\{i_{P_1}^{\mathrm{GL}_{n_1}(F)}(\lambda_1) \boxtimes \cdots \boxtimes i_{P'_N}^{\mathrm{GL}_{n_N}(F)}(\lambda'_N) \boxtimes \cdots \boxtimes i_{P_r}^{\mathrm{GL}_{n_r}(F)}(\lambda_r)\}$$

The inertial support t' of the above representation is

$$[M_1 \times \cdots \times M'_N \times \cdots \times M_r, \lambda_1 \boxtimes \cdots \boxtimes \lambda'_N \boxtimes \cdots \boxtimes \lambda_r].$$

We may assume that M_i is a standard Levi subgroup for $1 \leq i \leq r$. Now

$$[M_N = \prod_{j=1}^{p} \mathrm{GL}_{m_j}(F), \lambda_N = \boxtimes_{j=1}^p \zeta_j] \neq [M'_N = \prod_{j=1}^{p'} \mathrm{GL}_{m'_j}(F), \lambda'_N = \boxtimes_{j=1}^{p'} \zeta'_j]$$

implies that there exists a cuspidal component $[\mathrm{GL}_{m_k}(F), \zeta_k]$ occurring in the multi-set

$$\{[\mathrm{GL}_{m_1}(F), \zeta_1], [\mathrm{GL}_{m_2}(F), \zeta_2], \dots, [\mathrm{GL}_{m_p}(F), \zeta_p]\}$$

which has a different multiplicity in

$$\{[\mathrm{GL}_{m'_1}(F), \zeta'_1], [\mathrm{GL}_{m'_2}(F), \zeta'_2], \dots, [\mathrm{GL}_{m_{p'}}(F), \zeta'_{p'}]\}.$$

Adding cuspidal components with the same multiplicity to the above two multi-sets cannot make the multiplicities of the component $[\mathrm{GL}_k(F), \zeta_k]$ the same. This shows that $t' \neq t$ and hence the desired complement is the direct sum of the representations as in (4) such that γ_i occur in τ_i^1 for some $i \in \{1, 2, \dots, r\}$. □

Lemma 2.4. *Let $t_i = [G_{n_i}, \sigma_i]$ be a Bernstein component for G_{n_i} and τ_i be a typical representation for t_i and let τ_I be the representation $\tau_1 \boxtimes \tau_2 \boxtimes \cdots \boxtimes \tau_s$. The representation*

$$\mathrm{ind}_{P_I \cap K_n}^{K_n}(\tau_I)$$

admits a complement in $\mathrm{res}_{K_n} i_{P_I}^{G_n}(\sigma_I)$ whose irreducible sub-representations are atypical.

Proof. We use the uniqueness of typical representations for supercuspidal representations (see [Pas05]) to decompose $\mathrm{res}_{K_{n_i}} \sigma_i$ as $\tau_i \oplus \tau_i^1$ such that irreducible sub-representations of τ_i^1 are atypical. The lemma follows as a consequence of Proposition 2.3. □

Given a component $s = [M_I, \sigma_I]$ of G_n the above lemma shows that typical representations only occur as sub-representations of

$$\mathrm{ind}_{P_I \cap K_n}^{K_n}(\tau_I).$$

The above representation is still an infinite dimensional representation of the compact group K_n. We write the above representation as an increasing union of finite-dimensional representations.

Let $\{H_i\}_{i \geq 1}$ be a decreasing sequence of compact open subgroups of the maximal compact subgroup K_n. Let $\bar{U}_I$ be the unipotent radical of the opposite parabolic subgroup $\bar{P}_I$ of P_I with respect to the Levi subgroup M_I. We assume that H_i has an Iwahori decomposition with respect to the parabolic subgroup P_I and Levi subgroup M_I for all $i \geq 1$ i.e. the product map

$$(H_i \cap \bar{U}_I) \times (H_i \cap M_I) \times (H_i \cap U_I) \to H_i$$

is a homeomorphism for any ordering of the factors on the left hand side and that $\bigcap_{i \geq 1} H_i = K_n \cap P_I$. Let τ be a finite dimensional smooth representation of the group $M_I(\mathcal{O}_F)$. We assume that τ extends to a representation of H_i for all $i \geq 1$ such that $H_i \cap U_I$ and $H_i \cap \bar{U}_I$ are contained in the kernel of τ. By definition the representation $\mathrm{ind}_{H_i}^{K_n}(\tau)$ is contained in $\mathrm{ind}_{K_n \cap P_I}^{K_n}(\tau)$.

Lemma 2.5. *The union of the representations*

$$\mathrm{ind}_{H_i}^{K_n}(\tau)$$

for all $i \geq 1$ is equal to the representation

$$\mathrm{ind}_{K_n \cap P_I}^{K_n}(\tau).$$

Proof. Let W be the underlying space for the representation τ. Any element f in the space

$$\mathrm{ind}_{K_n \cap P_I}^{K_n}(\tau)$$

is a function $f : K_n \to W$ such that

(1) $f(pk) = \tau(p) f(k)$ for all $p \in K_n \cap P_I$ and $k \in K_n$,
(2) here exists a positive integer m (depending on f) such that $f(gk) = f(g)$ for all $k \in K_n(m)$ and $g \in K_n$.

Now there exists a positive integer i such that $H_i \cap \bar{U}_I \subset K_n(m)$. For such a choice of i and $h \in H_i$ write $h = h^- h^+$ where $h^+ \in K_n \cap P$, $h^- \in H_i \cap \bar{U}_I$ which we can do so by Iwahori decomposition of H_i. We observe that $f(hk) = f(h^- h^+ k) = f(h^+ k (h^+ k)^{-1} h^- (h^+ k)) = f(h^+ k) = \tau(h^+) f(k)$ (since $(h^+ k)^{-1} h^- (h^+ k) \in K_n(m)$). Hence $f \in \mathrm{ind}_{H_i}^{K_n}(\tau)$. □

We shall need the following technical lemma for frequent reference. Let P be any parabolic subgroup of G_n with a Levi subgroup M and U be the unipotent radical of P. Let J_1 and J_2 be two compact open subgroups of K_n such that J_1 contains J_2. Suppose J_1 and J_2 both satisfy Iwahori decomposition with respect to the Levi subgroup M. With $J_1 \cap U = J_2 \cap U$ and $J_1 \cap \bar{U} = J_2 \cap \bar{U}$. Let λ be an irreducible smooth representation of J_2 which admits an Iwahori decomposition i.e. $J_2 \cap U$ and $J_2 \cap \bar{U}$ are contained in the kernel of λ.

Lemma 2.6. *The representation $\mathrm{ind}_{J_2}^{J_1}(\lambda)$ is the extension of the representation $\mathrm{ind}_{J_2 \cap M}^{J_1 \cap M}(\lambda)$ such that $J_1 \cap U$ and $J_1 \cap \bar{U}$ are contained in the kernel of the extension.*

Proof. From the Iwahori decomposition we get that $(J_1 \cap M)J_2 = J_1$ and from the Mackey decomposition we get that

$$\mathrm{res}_{J_1 \cap M} \mathrm{ind}_{J_2}^{J_1}(\lambda) \simeq \mathrm{ind}_{J_2 \cap M}^{J_1 \cap M}(\lambda).$$

We now verify that $J_1 \cap U$ and $J_1 \cap \bar{U}$ act trivially on $\mathrm{ind}_{J_2}^{J_1}(\lambda)$. Observe that

$$\mathrm{res}_{J_1 \cap P} \mathrm{ind}_{J_2}^{J_1}(\lambda) \simeq \mathrm{ind}_{J_2 \cap P}^{J_1 \cap P}(\lambda).$$

Since the double coset representatives for

$$\frac{J_1 \cap P}{J_2 \cap P}$$

can be chosen from $M \cap J_1$ the group $J_1 \cap U$ acts trivially on $\mathrm{ind}_{J_2}^{J_1}(\lambda)$. Similarly $J_1 \cap \bar{U}$ acts trivially on $\mathrm{ind}_{J_2}^{J_1}(\lambda)$. This concludes the lemma. □

Lemma 2.7. *Let G be the F-rational points of an algebraic reductive group and χ be a character of G. Let τ be a K-typical representation for the component $s = [M, \sigma]$. The representation $\tau \otimes \chi$ is a typical representation for the component $[M, \sigma \otimes \chi]$.*

Proof. Let $\mathrm{Hom}_K(\tau \otimes \chi, \pi) \neq 0$ for some irreducible smooth representation π of G. We now have $\mathrm{Hom}_K(\tau, \pi \otimes \chi^{-1}) \neq 0$. This implies that $\pi \otimes \chi^{-1}$ occurs in the composition series of

$$i_P^G(\sigma \otimes \eta)$$

for some parabolic subgroup P containing M as a Levi subgroup and η an unramified character of M. Now π occurs in the composition series for the representation

$$i_P^G(\sigma \otimes \chi \otimes \eta)$$

hence $\tau \otimes \chi$ is a K-typical representation for the component $[M, \sigma \otimes \chi]$. □

3. Level-Zero Bernstein components

Definition 3.1. *Let $I = (n_1, n_2, \dots, n_r)$ be an ordered partition of n. An inertial class $s = [M_I, \boxtimes_{i=1}^r \sigma_i]$ is called a level-zero inertial class if the $K_{n_i}(1)$ invariants of σ_i is non trivial, for $1 \leq i \leq r$.*

We fix a level-zero inertial class $s = [M_I, \sigma_I]$. The subgroup K_{n_i} acts on the $K_{n_i}(1)$ invariants of σ_i and τ_i be this representation of K_{n_i} on $K_{n_i}(1)$ invariants of σ_i. The representation is τ_i is the inflation of a cuspidal representation of $\mathrm{GL}_{n_i}(k_F)$. **The pair (K_{n_i}, τ_i) is the Bushnell–Kutzko type for the inertial class $[G_{n_i}, \sigma_i]$.**

Let m be a positive integer and $P_I(m)$ be the inverse image of $P_I(\mathcal{O}_F/\mathfrak{P}_F^m)$ under the mod-$\mathfrak{P}_F^m$ reduction map

$$\pi_m : K_n \to \mathrm{GL}_n(\mathcal{O}_F/\mathfrak{P}_F^m).$$

The representation $\boxtimes_{i=1}^r \tau_i$ of $M_I(k_F)$ can be viewed as a representation of $P_I(k_F)$ by inflation via the quotient map

$$P_I(k_F) \to P_I(k_F)/U_I(k_F) \simeq M_I(k_F).$$

The representation $\boxtimes_{i=1}^r \tau_i$ of $P_I(k_F)$ is also a representation of $P_I(1)$ by inflation via the map π_1. We note that $P_I(1) \cap U_I$ and $P_I(1) \cap \bar{U}_I$ are contained in the kernel of this extension. **The pair** $(P_I(1), \tau_I)$

is the Bushnell–Kutzko type for the component s (see [BK99, Section 8.3.1]). The irreducible sub-representations of

$$\mathrm{ind}_{P_I(1)}^{K_n}(\tau_I)$$

are thus typical for s.

We note that the groups $P_I(m)$ have Iwahori decomposition with respect to P_I and M_I. The representation τ_I of $M_I(\mathcal{O}_F)$ extends to a representation of $P_I(m)$ such that $P_I(m)\cap U_I$ and $P_i(m)\cap \bar{U}_I$ are contained in the kernel of the extension. This shows that the sequence of groups $\{P_I(m) \mid m\geq 1\}$ and τ_I satisfy the hypothesis for the groups $\{H_m \mid m\geq 1\}$ and τ in Lemma 2.5 hence we have the isomorphism

$$\bigcup_{m\geq 1}\mathrm{ind}_{P_I(m)}^{K_n}(\tau_I)\simeq \mathrm{ind}_{P_I\cap K_n}^{K_n}(\tau_I).$$

We recall that the Lemma 2.4 shows that typical representations for the component s can only occur in the above representation.

Using Frobenius reciprocity we get that the representation τ_I occurs in $\mathrm{ind}_{P_I(m)}^{P_I(1)}(\tau_I)$ with multiplicity one. Let $m\geq 1$ and $U_m^0(\tau_I)$ be the $P_I(1)$-stable complement of the representation τ_I in $\mathrm{ind}_{P_I(m)}^{P_I(1)}(\tau_I)$. Let $U_m(\tau_I)$ be the representation

$$\mathrm{ind}_{P_I(1)}^{K_n}(U_m^0(\tau_I)).$$

We note that

$$\mathrm{ind}_{P_I(1)}^{K_n}(\tau_I)\oplus U_m(\tau_I)\simeq \mathrm{ind}_{P_I(m)}^{K_n}(\tau_I)$$

We will show that irreducible sub-representations of $U_m(\tau_I)$ are atypical.

Theorem 3.2 (Main)**.** *Let $m\geq 1$. The K_n-irreducible subrepresentations of $U_m(\tau_I)$ are atypical.*

Using this, the classification of typical representations for the inertial class s is given by the following corollary.

Corollary 3.3. *The irreducible sub-representations of $\mathrm{ind}_{P_I(1)}^{K_n}(\tau_I)$ are precisely the typical representations for the level-zero inertial class $[M_I,\sigma_I]$. Moreover if Γ is a typical representation then*

$$\dim_{\mathbb{C}}\ \mathrm{Hom}_{K_n}(\Gamma,\mathrm{ind}_{P_I(1)}^{K_n}(\tau_I))=\dim_{\mathbb{C}}\ \mathrm{Hom}_{K_n}(\Gamma, i_{P_I}^{G_n}(\sigma_I)).$$

Proof. Given a typical representation Γ for the inertial class s, the theorem shows that Γ is a sub-representation of $\mathrm{ind}_{P_I(1)}^{K_n}(\tau_I)$ and the multiplicity formula follows from Lemma 2.4 and the above theorem. Conversely if Γ is a sub-representation of $\mathrm{ind}_{P_I(1)}^{K_n}(\tau_I)$ then, by Frobenius reciprocity, we get that $\mathrm{Hom}_{P_I(1)}(\tau_I,\Gamma)\neq 0$. If Γ is contained as a K_n-irreducible sub-representation in an irreducible smooth representation π of G_n then the restriction of π to $P_I(1)$ contains the representation τ_I. The pair $(P_I(1),\tau_I)$ is the Bushnell–Kutzko type for the inertial class $s=[M_I,\sigma_I]$ hence the inertial support of π is s. Hence Γ is a typical representation and this proves the corollary. □

3.1. **Decomposition of an auxiliary representation.** We will need a few lemmas regarding the splitting of a certain representation for the proof of the main theorem. Let I be the ordered partition $(n_1,n_2,\dots,n_r)$ of the positive integer n as fixed at the beginning of this chapter. **Until the beginning of the section 4 we assume that** $r>1$ in other words M_I is a proper Levi subgroup. We denote by I' the ordered partition $(n_1,n_2,\dots,n_{r-1})$ of $n-n_r$. Let m be a positive integer and $P_I(1,m)$ be the following set

$$\left\{\begin{pmatrix} A & B\\ \varpi_F^m C & D\end{pmatrix} | A\in P_{I'}(1); B^{tr}, C\in M_{n_r\times(n-n_r)}(\mathcal{O}_F); D\in K_{n_r}\right\}.$$

Here tr denotes transpose. Note that $P_I(1,1)=P_I(1)$.

Lemma 3.4. *The set $P_I(1,m)$ is a subgroup of $P_I(1)$.*

Proof. The group K_n acts on the set of lattices of F^n contained in the lattice ${\mathcal{O}_F}^n$. If $r-1=1$ the set $P_I(1,m)$ is the K_n-stabilizer of the lattice $(\mathcal{O}_F)^{n_1} \oplus (\varpi_F^m\,\mathcal{O}_F)^{n_2}$. In the case $r-1>1$ the set $P_I(1,m)$ is the K_n-stabilizer of the set of lattices $\{L_k \mid 1<k\leq r-1\}$ defined by:

$$L_k = (\mathcal{O}_F)^{n_1} \oplus \cdots \oplus (\mathcal{O}_F)^{n_{k-1}} \oplus (\varpi_F\,\mathcal{O}_F)^{n_k} \oplus \cdots \oplus (\varpi_F\,\mathcal{O}_F)^{n_{r-1}} \oplus (\varpi_F^m\,\mathcal{O}_F)^{n_r}.$$

This shows that $P_I(1,m)$ is a subgroup and is contained in $P_I(1)$ from the definition. □

The structure of the representation

$$\mathrm{ind}_{P_I(1,m+1)}^{P_I(1,m)}(\mathrm{id})$$

will be used in the proof of the main theorem. Using Clifford theory we decompose the above representation. Let $K_I(m)$ be the group $K_n(m)U_{(n-n_r,n_r)}(\mathcal{O}_F)$. We note that this group only depends on n and n_r, rather than the whole partition I.

Lemma 3.5. *The group $K_I(m)$ is a normal subgroup of $P_I(1,m)$ and $K_I(m)\cap P_I(1,m+1)$ is a normal subgroup of $K_I(m)$.*

Proof. The groups $K_I(m)$ and $P_I(1,m)$ satisfy Iwahori decomposition with respect to $U_{(n-n_r,n_r)}$, $\bar{U}_{(n-n_r,n_r)}$ and $M_{(n-n_r,n_r)}$. We also note that

$$K_I(m)\cap U_{(n-n_r,n_r)} = P_I(1,m)\cap U_{(n-n_r,n_r)}$$

and

$$K_I(m)\cap \bar{U}_{(n-n_r,n_r)} = P_I(1,m)\cap \bar{U}_{(n-n_r,n_r)}.$$

Hence $P_I(1,m)\cap U_{(n-n_r,n_r)}$ and $P_I(1,m)\cap \bar{U}_{(n-n_r,n_r)}$ normalize $K_I(m)$. Since $K_I(m)$ is a product of the group $K_n(m)$ and $U_{(n-n_r)}(\mathcal{O}_F)$ the group
$P_I(1,m)\cap M_{(n-n_r,n_r)}$ normalizes the group $K_I(m)$. This shows the first part.

Notice that $K_I(m)\cap U_{(n-n_r,n_r)}$ is equal to $K_I(m)\cap P_I(1,m+1)\cap U_{(n-n_r,n_r)}$ and $K_I(m)\cap M_{(n-n_r,n_r)}$ is equal to $K_I(m)\cap P_I(1,m+1)\cap M_{(n-n_r,n_r)}$ hence it is enough to check that $K_I(m)\cap \bar{U}_{(n-n_r,n_r)}$ normalizes the group $K_I(m)\cap P_I(1,m+1)$. Since $K_I(m)\cap P_I(1,m+1)\cap \bar{U}_{(n-n_r,n_r)}$ is abelian and is contained in $K_I(m)\cap \bar{U}_{(n-n_r,n_r)}$ hence we need to check that $u^-j(u^-)^{-1}$ and $u^-u^+(u^-)^{-1}$ are contained in $K_I(m)\cap P_I(1,m+1)$ for all u^-, j and u^+ in

$$\begin{aligned} &K_I(m)\cap \bar{U}_{(n-n_r,n_r)},\\ &K_I(m)\cap P_I(1,m+1)\cap M_{(n-n_r,n_r)} \text{ and}\\ &K_I(m)\cap P_I(1,m+1)\cap U_{(n-n_r,n_r)} = U_{(n-n_r,n_r)}(\mathcal{O}_F)\end{aligned}$$

respectively. Let u^+, u^- and j be three elements from $U_{n-n_r,n_r}(\mathcal{O}_F)$, $K_I(m)\cap \bar{U}_{(n-n_r,n_r)}$ and $K_I(m)\cap P_I(1,m+1)\cap M_{(n-n_r,n_r)}$ respectively. We write them in their block form as:

$$u^+ = \begin{pmatrix} 1_{n-n_r} & B \\ 0 & 1_{n_r}\end{pmatrix}$$

where $B\in M_{(n-n_r)\times n_r}(\mathcal{O}_F)$,

$$u^- = \begin{pmatrix} 1_{n-n_r} & 0 \\ \varpi_F^m C & 1_{n_r}\end{pmatrix}$$

where $C\in M_{n_r\times(n-n_r)}(\mathcal{O}_F)$ and

$$j = \begin{pmatrix} J_1 & 0 \\ 0 & J_2\end{pmatrix}.$$

We observe that $u^-j(u^-)^{-1} = j\{j^{-1}u^-j(u^-)^{-1}\}$ and the commutator
$\{j^{-1}u^-j(u^-)^{-1}\}$ in its block form is as follows:

$$\begin{pmatrix} 1_{n-n_r} & 0 \\ J_2^{-1}(\varpi_F^m C J_1^{-1} - \varpi_F^m C) & 1_{n_r}\end{pmatrix}.$$

We note that $J_2\in K_{n_r}(m)$ and $J_1\in K_{n-n_r}(m)$ hence $J_2^{-1}(\varpi_F^m C J_1^{-1}-\varpi_F^m C)$ belongs to

$$\varpi_F^{m+1}M_{(n-n_r)\times n_r}(\mathcal{O}_F).$$

This shows that

$$\{j^{-1}u^{-}j(u^{-})^{-1}\} \in K_I(m) \cap P_I(m+1)$$

Now the element $(u^{-})u^{+}(u^{-})^{-1}$ is of the form

$$\begin{pmatrix} 1_{n-n_r} - \varpi_F^m BC & B \\ -\varpi_F^{2m} CBC & 1_{n_r} + \varpi_F^m CB \end{pmatrix}. \tag{5}$$

Since $2m \geq m+1$ the matrix in (5) is contained in the group $K_I(m) \cap P_I(1, m+1)$. □

We now observe that $K_I(m)P_I(1, m+1) = P_I(1, m)$. From Mackey decomposition we get that

$$\mathrm{res}_{K_I(m)} \mathrm{ind}_{P_I(1,m+1)}^{P_I(1,m)}(\mathrm{id}) \simeq \mathrm{ind}_{K_I(m)\cap P_I(1,m+1)}^{K_I(m)}(\mathrm{id}).$$

Hence the above restriction decomposes into a direct sum of representations of the group

$$\frac{K_I(m)}{K_I(m) \cap P_I(1, m+1)}. \tag{6}$$

The inclusion map of $K_I(m) \cap \bar{U}_{(n-n_r,n_r)}$ in $K_I(m)$ induces the natural homomorphism

$$\tilde{\theta}_I : \frac{K_I(m) \cap \bar{U}_{(n-n_r,n_r)}}{P_I(1, m+1) \cap \bar{U}_{(n-n_r,n_r)}} \to \frac{K_I(m)}{K_I(m) \cap P_I(1, m+1)}.$$

Lemma 3.6. *The map $\tilde{\theta}_I$ is an $M_{(n-n_r,n_r)} \cap P_I(1, m)$ equivariant isomorphism.*

Proof. The map is clearly injective and surjectivity follows from the Iwahori decomposition of $K_I(m)$ with respect to the Levi subgroup M_I. The inclusion of $K_I(m) \cap \bar{U}_{(n-n_r,n_r)}$ in $K_I(m)$ is an $M_{n-n_r,n_r} \cap P_I(1, m)$ equivariant map. □

Let u^{-} be an element of the group $K_I(m) \cap \bar{U}_{(n-n_r,n_r)}$ and its block form be given by

$$\begin{pmatrix} 1_{(n-n_r,n_r)} & 0 \\ U^{-} & 1_{n_r} \end{pmatrix}.$$

The map $u^{-} \mapsto \varpi_F^{-m} U^{-}$ induces an isomorphism between the groups

$$K_I(m) \cap \bar{U}_{(n-n_r,n_r)}$$

and $M_{n_r \times (n-n_r)}(\mathcal{O}_F)$. Let $\bar{U^{-}}$ be the image of U^{-} in the mod-$\mathfrak{P}_F$ reduction of $M_{n_r\times(n-n_r)}(\mathcal{O}_F)$. The map $u^{-} \mapsto \overline{\varpi_F^{-m} U^{-}}$ induces an isomorphism of the quotient (6) with the group of matrices $M_{n_r\times(n-n_r)}(k_F)$. We note that $M_{(n-n_r,n_r)}(\mathcal{O}_F) = K_{n-n_r} \times K_{n_r}$ acts on the group $M_{n_r\times(n-n_r)}(k_F)$ through its mod-$\mathfrak{P}_F$ reduction $\mathrm{GL}_{n-n_r}(k_F) \times \mathrm{GL}_{n_r}(k_F)$, the action is given by $(g_1, g_2)U = g_2 U g_1^{-1}$ for all g_1 in $\mathrm{GL}_{n-n_r}(k_F)$, g_2 in $\mathrm{GL}_{n_r}(k_F)$ and U in $M_{n_r\times(n-n_r)}(k_F)$. The map $u^{-} \mapsto \overline{\varpi_F^{-m} U^{-}}$ is hence an $M_{(n-n_r,n_r)}(\mathcal{O}_F)$-equivariant map between the quotient (6) and $M_{n_r\times(n-n_r)}(k_F)$. Moreover the action of $M_{(n-n_r,n_r)}(\mathcal{O}_F)$ factors through its quotient $M_{(n-n_r,n_r}(k_F)$.

The space $M_{n\times m}(k_F)$ is equipped with an action of $G := \mathrm{GL}_m(k_F) \times \mathrm{GL}_n(k_F)$ given by $(g_1, g_2)U = g_2 U g_1^{-1}$. We also have a G action on the set of matrices $M_{m\times n}(k_F)$ by setting $(g_1, g_2)V = g_1 V g_2^{-1}$. Let ψ be a non-trivial character of the additive group k_F. We define a pairing B between $M_{m\times n}(k_F)$ and $M_{n\times m}(k_F)$ by defining $B(V, U) = \psi \circ \mathrm{tr}(VU)$. Let T be the map from $M_{m\times n}(k_F)$ and $M_{n\times m}(k_F)^\wedge$ defined by

$$T(V)(U) = B(V, U).$$

Lemma 3.7. *The map T is a G-equivariant isomorphism.*

Proof. That the map T is G equivariant can be verified from the identity

$$(g_1, g_2)T(V)(U) = \psi \circ \mathrm{tr}(V g_2^{-1} U g_1) = \psi \circ \mathrm{tr}(g_1 V g_2^{-1} U) = T((g_1, g_2)V)(U).$$

It remains to show that B is non-degenerate. Let V_{ij} (U_{ij}) be a matrix whose ij-th entry is v_{ij} (u_{ij}) and all other entries are zero. We observe that $B(U_{ij}, V_{ij})$ is equal to $\psi(u_{ij}v_{ij})$. This shows that B is non-degenerate. □

The above two lemmas gives an $M_{(n-n_r,n_r)} \cap P_I(1,m)$ equivariant isomorphism

$$\theta_I : \left\{ \frac{K_I(m)}{K_I(m) \cap P_I(1,m+1)} \right\}^{\wedge} \to M_{(n-n_r)\times n_r}(k_F). \tag{7}$$

Since the group $K_I(m)$ is a normal subgroup of $P_I(1,m)$, we have an action of this group $P_I(1,m)$ on the set of characters of the abelian group

$$\frac{K_I(m)}{K_I(m) \cap P_I(1,m+1)}.$$

If η is one such character we denote by $Z(\eta)$ the $P_I(1,m)$-stabilizer of this character η. Clifford theory now gives the decomposition

$$\mathrm{ind}_{P_I(1,m+1)}^{P_I(1,m)}(\mathrm{id}) \simeq \bigoplus_{\eta} \mathrm{ind}_{Z(\eta)}^{P_I(1,m)}(U_\eta)$$

where η runs over a set of representatives for the orbits under the action of $P_I(1,m)$ and U_η is some irreducible representation of the group $Z(\eta)$. We also note that $Z(\mathrm{id}) = P_I(1,m)$ and the identity character occurs with multiplicity one (which follows from Frobenius reciprocity) and hence

$$\mathrm{ind}_{P_I(1,m+1)}^{P_I(1,m)}(\mathrm{id}) \simeq \mathrm{id} \oplus \bigoplus_{\eta\neq \mathrm{id}} \mathrm{ind}_{Z(\eta)}^{P_I(1,m)}(U_\eta). \tag{8}$$

Observe that

$$Z(\eta) = (Z(\eta) \cap M_{(n-n_r,n_r)})K_I(m).$$

Let $\theta_I(\eta) = A$. Since θ_I is $M_{(n-n_r,n_r)} \cap P_I(1,m)$ equivariant we get that

$$Z(\eta) \cap M_{(n-n_r,n_r)} = Z_{M_{(n-n_r,n_r)}\cap P_I(1,m)}(A)$$

for some matrix A in $M_{(n-n_r)\times n_r}(k_F)$. The group $M_{(n-n_r,n_r)} \cap P_I(1,m)$ acts on the group of matrices $M_{(n-n_r,n_r)}(k_F)$ through its mod-$\mathfrak{P}_F$ reduction. The mod-$\mathfrak{P}_F$ reduction of the group $P_I(1,m) \cap M_{(n-n_r,n_r)}$ is equal to the group $P_{I'}(k_F) \times \mathrm{GL}_{n_r}(k_F)$. In the next lemma we will bound the mod $\mathfrak{P}_F$ reduction of the group $Z(\eta) \cap M_I$ for the proof of the Main theorem. Let $\mathcal{O}_A$ be an orbit for the action of $P_{I'}(k_F) \times \mathrm{GL}_{n_r}(k_F)$ on the set of matrices $M_{(n-n_r)\times n_r}(k_F)$. Let p_j be the projection onto j^{th} factor of $M_I(k_F) = \prod_{i=1}^r \mathrm{GL}_{n_i}(k_F)$.

Lemma 3.8. *Let $\mathcal{O}_A$ be an orbit consisting of non-zero matrices in*

$$M_{(n-n_r)\times n_r}(k_F).$$

We can choose a representative A such that the $P_{I'}(k_F) \times \mathrm{GL}_{n_r}(k_F)$-stabilizer of A,

$$Z_{P_{I'}(k_F)\times \mathrm{GL}_{n_r}(k_F)}(A)$$

satisfies one of the following conditions.

(1) *There exists a positive integer j, $j \le r$ such that the image of*

$$p_j : Z_{P_{I'}(k_F)\times \mathrm{GL}_{n_r}(k_F)}(A) \cap M_I(k_F) \to \mathrm{GL}_{n_j}(k_F)$$

is contained in a proper parabolic subgroup of $\mathrm{GL}_{n_j}(k_F)$.

(2) *There exists an i with $1 \le i \le r-1$ such that $p_i(g) = p_r(g)$ for all g in*

$$Z_{P_{I'}(k_F)\times \mathrm{GL}_{n_r}(k_F)}(A) \cap M_I(k_F).$$

Proof. Let $A = [U_1, U_2, \dots, U_{(r-1)}]^{tr}$ be the block form (U_k is a matrix of size $n_r \times n_k$ for $1 \le k \le r-1$) of a representative m for an orbit $\mathcal{O}_m$ consisting of non-zero matrices. If $((M_{ij}), B) \in Z_{P_{I'}(k_F)\times \mathrm{GL}_{n_r}(k_F)}(A)$ (M_{ii} is a matrix of size $n_i \times n_i$) then we have

$$(M_{ij})[U_1, U_2, ..., U_{(r-1)}]^{tr} = [U_1, U_2, ..., U_{(r-1)}]^{tr}B. \tag{9}$$

Since $(M_{ij}) \in P_{I'}(k_F)$, we have $M_{ij} = 0$ for all $i > j$. Let $l \le r-1$ be the maximal positive integer such that U_l is non-zero and such an l exists since $m \neq 0$. From (9) we get that $M_{ll}U_l^{tr}t = U_l^{tr}B$ where. There exist matrices $P \in \mathrm{GL}_{n_r}(k_F)$ and $Q \in \mathrm{GL}_{n_l}(k_F)$ such that $PU_l^{tr}Q$ is a matrix of the form

$$\begin{pmatrix} 1_t & 0 \\ 0 & 0 \end{pmatrix} \tag{10}$$

where t is the rank of the matrix U_l^{tr}. Now we may change the representative A to $A' = [U_1', U_2', \dots, U_r']^{tr}$ by the action of the element

$$(\mathrm{diag}(1_{n_1}, \dots, P, \dots, 1_{n_{r-1}}), Q^{-1})$$

in $P_{I'}(k_F) \times \mathrm{GL}_{n_r}(k_F)$ such that $U_l'^{tr}$ is the matrix (10). If $t = n_l = n_r$ then condition (2) is satisfied. Consider the maps $T_1 : k_F^{n_l} \to k_F^{n_r}$ and $T_2 : k_F^{n_r} \to k_F^{n_l}$ given by

$$(a_1, a_2, \dots, a_{n_l}) \mapsto (a_1, a_2, \dots, a_{n_l}) U_l^{tr}$$

and

$$(a_1, a_2, \dots, a_{n_r}) \mapsto U_l^{tr}(a_1, a_2, \dots, a_{n_r})^{tr}$$

respectively. If $t = n_l = n_r$ does not hold then either of T_1 or T_2 has a non-trivial proper kernel (since $U_l \neq 0$). If T_1 has a non-trivial proper kernel then M_{ll} preserves this kernel and hence belongs to a proper parabolic subgroup of $\mathrm{GL}_{n_r}(k_F)$. If T_2 has a non-trivial proper kernel then B preserves this kernel and hence belongs to a proper parabolic subgroup of $\mathrm{GL}_{n_l}(k_F)$. Hence if $t = n_l = n_r$ does not hold then condition (1) is satisfied. □

The following lemma is due to Paškūnas but we give a mild modification for our applications (see [Pas05, Proposition 6.8]).

Lemma 3.9. *Let G be a reductive algebraic group over k_F and U be the unipotent radical of a proper parabolic subgroup of G. For any subgroup H of G such that $H \cap U = \{\mathrm{id}\}$ and ξ any irreducible subrepresentation of H, there exists an irreducible non-cuspidal representation σ such that ξ occurs in $\mathrm{res}_H\, \sigma$.*

Proof. Suppose the lemma is false then

$$\mathrm{ind}_H^G(\xi) \simeq \oplus_{k=1}^{t} \sigma_k$$

such that σ_k is cuspidal representation for all $k \leq t$. Since $U \cap H = \{\mathrm{id}\}$, using Mackey decomposition we deduce that,

$$\mathrm{Hom}_U(\mathrm{id}, \mathrm{ind}_H^G(\gamma)) \neq 0.$$

Now by our assumption we have $\mathrm{Hom}_U(\mathrm{id}, \sigma_k) \neq 0$ for some $k \leq t$ and hence a contradiction. □

The following lemma is similar to Proposition 2.3. The lemma is just a modified version of the Proposition 2.3 for our present use.

Lemma 3.10. *Let Γ be a K_n-irreducible sub-representation of*

$$\mathrm{ind}_{P_{(n-n_r, n_r)}(m)}^{K_n} \{U_m(\tau_{I'}) \boxtimes \tau_r\}.$$

If the irreducible sub-representations of $U_m(\tau_{I'})$ are atypical for the component $s = [M_{I'}, \sigma_{I'}]$, then the representation Γ is atypical for the component $s = [M_I, \sigma_I]$.

Proof. Let ρ be an irreducible sub-representation of $U_m(\tau_{I'})$. If ρ is not typical then, there exists another Bernstein component $[M_J, \lambda_J]$ of $\mathrm{GL}_{n-n_r}(F)$ such that

$$[M_{I'}, \sigma_{I'}] \neq [M_J, \lambda_J]$$

and ρ is contained in

$$\mathrm{res}_{K_{n-n_r}}\, i_{P_J}^{G_{n-n_r}}(\lambda_J)$$

where $J = (n_1', n_2', \dots, n_{r'-1}')$ and $\lambda_J = \boxtimes_{i=1}^{r'-1} \lambda_i$. The representation

$$\mathrm{ind}_{P_{(n-n_r, n_r)}(m)}^{K_n} \{\rho \boxtimes \tau_r\}$$

is contained in

$$\mathrm{ind}_{P_{(n-n_r, n_r)} \cap K_n}^{K_n} \{\rho \boxtimes \tau_r\}. \tag{11}$$

The representation (11) is contained in the representation

$$\mathrm{res}_{K_n}\, i_{P_{(n-n_r, n_r)}}^{G_n} \{i_{P_J}^{G_{n-n_r}}(\lambda_J) \boxtimes \sigma_r\}.$$

Since $[M_{I'}, \sigma_{I'}] \neq [M_J, \lambda_J]$ there exist an inertial class $[G_p, \sigma]$ occurring in the multi-set

$$\{[G_{n_1}, \sigma_1], [G_{n_2}, \sigma_2], \dots, [G_{n_{r-1}}, \sigma_{r-1}]\}$$

with a multiplicity not equal to its multiplicity in the multi-set

$$\{[G_{n'_1}, \lambda_1], [G_{n'_2}, \lambda_2], \dots, [G_{n'_{r'-1}}, \lambda_{r'-1}]\}.$$

Hence the classes $[M_I, \sigma_I]$ and $[M_J \times G_{n_r}, \lambda_J \boxtimes \sigma_r]$ represent two distinct Bernstein components for the group G_n. □

4. Proof of the main theorem

Proof of theorem 3.2. We prove the theorem by using induction on the positive integer n, the rank of G_n. The theorem is true for $n = 1$ since $U_m(\tau_I)$ is zero. We assume that the theorem is true for all positive integers less than $n + 1$. We will show the theorem for the positive integer $n + 1$. Let $s = [M_I, \sigma_I]$ be a level-zero inertial class. We assume that the partition $I = (n_1, n_2, \dots, n_r)$ of $n + 1$ satisfies the hypothesis $n_i \leq n_j$ for all $1 \leq i \leq j \leq r$. If $r = 1$ we have $U_m(\tau_I) = 0$ and the theorem holds by default. We now assume that $r > 1$ and let $I' = (n_1, n_2, \dots, n_{r-1})$.

We now break the proof into two cases. The first case is $n_r = 1$ and the second case is $n_r > 1$.

The case where $n_r = 1$. In this case $n_i = 1$ for $1 \leq i \leq r$ and $P_I = B_n$ where B_n is the Borel subgroup of GL_n. We denote by T_n and U_n the maximal torus and the unipotent radical respectively. We also use the notation $B_n(m)$ for the subgroup $P_I(m)$ and χ_{I_n} for τ_I since $I = (1, 1, \dots, 1)$ is a tuple of length n. The proof is by induction on the integer n, the rank of T_n. The statement is immediate for $n = 1$ and for $n = 2$ we refer to [BM02, A.2.4] for a proof. (We will require the proof for later use and we will recall it at that stage.) So we prove the theorem for $n \geq 3$. Suppose the theorem is true for some positive integer $n \geq 2$. By the definition of $U_m(\chi_{I_{n+1}})$ we have

$$\mathrm{ind}_{B_{n+1}(m)}^{K_{n+1}}(\chi_{I_{n+1}}) \simeq U_m(\chi_{I_{n+1}}) \oplus \mathrm{ind}_{B_{n+1}(1)}^{K_{n+1}}(\chi_{I_{n+1}}).$$

We have the isomorphism

$$\mathrm{ind}_{B_{n+1}(m)}^{K_{n+1}}(\chi_{I_{n+1}}) \simeq \mathrm{ind}_{P_{(n,1)}(m)}^{K_{n+1}}\{\mathrm{ind}_{B_n(m)}^{K_n}(\chi_{I_n}) \boxtimes \chi_{n+1})\}.$$

We also have the decomposition

$$\mathrm{ind}_{P_{(n,1)}(m)}^{K_{n+1}}\{\mathrm{ind}_{B_n(m)}^{K_n}(\chi_{I_n}) \boxtimes \chi_{n+1}\} \simeq$$
$$\mathrm{ind}_{P_{(n,1)}(m)}^{K_{n+1}}\{U_m(\chi_{I_n}) \boxtimes \chi_n\} \oplus \mathrm{ind}_{P_{(n,1)}(m)}^{K_{n+1}}\{\mathrm{ind}_{B_n(1)}^{K_n}(\chi_{I_n}) \boxtimes \chi_{n+1}\}.$$

By the inductive hypothesis and Lemma 3.10 irreducible sub-representations of

$$\mathrm{ind}_{P_{(n,1)}(m)}^{K_{n+1}}\{U_m(\chi_{I_n}) \boxtimes \chi_{n+1}\}$$

are atypical representations. We now consider the irreducible factors of the representation

$$\mathrm{ind}_{P_{(n,1)}(m)}^{K_{n+1}}\{\mathrm{ind}_{B_n(1)}^{K_n}(\chi_{I_n}) \boxtimes \chi_{n+1}\}. \tag{12}$$

We use induction on the integer m to show that the representation

$$\mathrm{ind}_{P_{(n,1)}(1)}^{K_{n+1}}\{\mathrm{ind}_{B_n(1)}^{K_n}(\chi_{I_n}) \boxtimes \chi_{n+1}\} \simeq \mathrm{ind}_{B_{n+1}(1)}^{K_{n+1}}(\chi_{I_{n+1}})$$

has a complement say $U_{1,m}(\chi_{In+1})$ in the representation (12) whose irreducible sub-representations are all atypical representations. This shows that irreducible sub-representations of $U_m(\chi_{I_{n+1}})$ are atypical. To reduce the notations we denote by $P(m)$ the subgroup $P_{(n,1)}(m)$. We note that $P_{n,1}(m) = P_{(n,1)}(1, m)$ and now applying the decomposition (8) to the parabolic subgroup $P_{(n,1)}$ we get that

$$\mathrm{ind}_{P(m+1)}^{P(m)}(\mathrm{id}) = \mathrm{id} \oplus \mathrm{ind}_{Z(\eta)}^{P(m)}(U_\eta)$$

where η is any non-trivial character of the group $K_{n+1}(m)U_{n,1}(\mathcal{O}_F)$ which is trivial on $K_{n+1}(m)U_{n,1}(\mathcal{O}_F) \cap P(m+1)$ and $K_{n+1}(m)$ is the principal congruence subgroup of level m (in the present situation we just have one orbit consisting of non-trivial characters.) Let $\theta_{(n,1)}$ be the isomorphism as in equation (7). We choose η such that $\theta_{(n,1)}(\eta) = [1, 0 \dots, 0]$.

With the above choice for the character η we have

$$\begin{aligned}&\mathrm{ind}_{P(m+1)}^{K_{n+1}}\{\mathrm{ind}_{B_n(1)}^{K_n}(\chi_{I_n})\boxtimes\chi_{n+1}\}\\ &\simeq \mathrm{ind}_{P(m)}^{K_{n+1}}\{\mathrm{ind}_{B_n(1)}^{K_n}(\chi_{I_n})\boxtimes\chi_{n+1}\}\oplus\mathrm{ind}_{Z(\eta)}^{K_{n+1}}\{U_\eta\otimes\mathrm{res}_{Z(\eta)\cap M_{(n,1)}}\{\mathrm{ind}_{B_n(1)}^{K_n}(\chi_{I_n})\boxtimes\chi_{n+1}\}\}.\end{aligned}$$

Since the representation $\mathrm{ind}_{B_n(1)}^{K_n}(\chi_{I_n})\boxtimes\chi_{n+1}$ is trivial on $K_n(1)\times K_1(1)$ we get that

$$\mathrm{res}_{Z(\eta)\cap M_{(n,1)}}\{\mathrm{ind}_{B_n(1)}^{K_n}(\chi_{I_n})\boxtimes\chi_{n+1}\}$$

is isomorphic to the inflation of the representation

$$\mathrm{res}_{\overline{Z(\eta)\cap M_{(n,1)}}}\{\mathrm{ind}_{B_n(k_F)}^{\mathrm{GL}_n(k_F)}(\chi_{I_n})\boxtimes\chi_{n+1}\}$$

where $\overline{Z(\eta)\cap M_{(n,1)}}$ is the mod-$\mathfrak{P}_F$ reduction of the group $Z(\eta)\cap M_{(n,1)}$. The group $\overline{Z(\eta)\cap M_{(n,1)}}$ is contained in the following subgroup

$$\left\{\begin{pmatrix}A & B & 0\\ 0 & d & 0\\ 0 & 0 & d\end{pmatrix} | A\in\mathrm{GL}_{n-1}(k_F), B\in M_{(n-1)\times 1}(k_F)\text{ and } d\in k_F^\times\right\}. \tag{13}$$

Let Mir_k be the mirabolic group

$$\left\{\begin{pmatrix}A & B\\ 0 & 1\end{pmatrix} | A\in\mathrm{GL}_{k-1}(k_F), B\in M_{(k-1)\times 1}(k_F)\right\}.$$

Now we have to understand the restriction

$$\mathrm{res}_{P_{(n-1,1)}}\mathrm{ind}_{B_n(k_F)}^{\mathrm{GL}_n(k_F)}(\chi_{I_n}).$$

We begin with understanding the restriction

$$\mathrm{res}_{\mathrm{Mir}_{n-1}}\mathrm{ind}_{B_n(k_F)}^{\mathrm{GL}_n(k_F)}(\chi_{I_n}).$$

We use the theory of derivatives (originally for G_n due to Bernstein and Zelevinsky (see [BZ76])) to describe this restriction in a way sufficient for our application. We refer to [Zel81, Chapter 3, §13] for details of these constructions.

In the case of finite fields from Clifford theory one can define four exact functors and we recall the formalism here. The precise definitions are not required for our purpose except for one functor Ψ^+ which will be recalled latter:

$$\mathcal{M}(\mathrm{Mir}_{k-1}) \underset{\Phi^-}{\overset{\Phi^+}{\rightleftarrows}} \mathcal{M}(\mathrm{Mir}_k) \underset{\Psi^+}{\overset{\Psi^-}{\rightleftarrows}} \mathcal{M}(\mathrm{GL}_{k-1}(k_F))$$

The key results we use from Zelevinsky are summarised below (see [Zel81, Chapter 3, §13]).

Theorem 4.1 (Zelevinsky)**.** *The functors Ψ^+ and Φ^- are left adjoint to Ψ^- and Φ^+ respectively. The compositions $\Phi^-\Phi^+$ and $\Psi^-\Psi^+$ are naturally equivalent to identity. Moreover $\Phi^+\Psi^-$ and $\Phi^-\Psi^+$ are zero. The diagram*

$$0\to\Phi^+\Phi^-\to\mathrm{id}\to\Psi^+\Psi^-\to 0$$

obtained from these properties is exact.

Using this theorem and following Bernstein-Zelevinsky one can define a filtration Fil on a finite dimensional representation τ of Mir_n, for all $n>1$. The filtration Fil is given by

$$0\subset\tau_n\subset\ldots\subset\tau_3\subset\tau_2\subset\tau_1=\tau$$

where $\tau_k=(\Phi^+)^{k-1}(\Phi^-)^{k-1}$ and $\tau_k/\tau_{k+1}=(\Phi^+)^{k-1}\Psi^+\Psi^-(\Phi^-)^{k-1}(\tau)$ for all $k\geq 1$. The representation $\tau^{(k)}:=\Psi^-(\Phi^-)^{k-1}(\tau)$ for all $k\geq 0$ of $\mathrm{GL}_{n-k}(k_F)$ is called the k^{th}-derivative of τ and by convention $\tau^{(0)}:=\tau$.

Let R_n be the Grothendieck group of $\mathrm{GL}_n(k_F)$ for all $n\geq 1$ and set $R_0=\mathbb{Z}$. Zelevinsky defined a ring structure on the group $R=\oplus_{n\geq 0}R_n$ by setting parabolic induction as the product rule. Recall that the ring R has a $\mathbb{Z}$-linear map D defined by setting $D(\pi)=\sum_{k\geq 0}(\pi|_{\mathrm{Mir}_n})^{(k)}$ for all π in R_n. It follows from [Zel81, Chapter 3, §13] that D is an endomorphism of the ring R. If π is a supercuspidal representation of $\mathrm{GL}_n(k_F)$ then by Gelfand-Kazhdan theory it follows that $\pi^{(n)}=1$, $\pi^{(0)}=\pi$ and all other derivatives are zero (see [Zel81, Chapter 3, §13]). Let $1_R\in R_0$ be the identity element of R.

In our present situation we have

$$D(\mathrm{ind}_{B_n(1)}^{K_n}(\chi_{I_n}))=\prod_{i=1}^{n}D(\chi_i)=\prod_{i=1}^{n}(\chi_i+1_R).$$

Let X_{n-k} be the term of degree $(n-k)$ in the expansion of the above product. (It is an actual representation of $\mathrm{GL}_{n-k}(k_F)$, since the coefficients of the above expansion are positive.) Then we have

$$\mathrm{res}_{\mathrm{Mir}_{n-1}}\mathrm{ind}_{B_n(k_F)}^{\mathrm{GL}_n(k_F)}(\chi_{I_n})\simeq\bigoplus_{k\geq 1}^{n}(\Phi^+)^{k-1}\Psi^+(X_{n-k}).$$

Observe that $P_{(n-1,1)}=\mathrm{Mir}_{(n-1)}\,k_F^\times$ (here $k_F^\times$ is the centre of $\mathrm{GL}_n(k_F)$) and $\mathrm{Mir}_{(n-1)}\cap k_F^\times=\mathrm{id}$. The representation

$$\rho:=(\Phi^+)^{k-1}\Psi^+(X_{n-k})$$

extends to a representation of $P_{(n-1,1)}$ by setting $\rho(a)=\chi(a)$ for all $a\in k_F^\times$ where χ is the central character of the representation

$$\mathrm{ind}_{B_n(k_F)}^{\mathrm{GL}_n(k_F)}(\boxtimes_{i=1}^{n}\chi_i).$$

Since the central character will play some role, we denote the extended representation by

$$ext\{(\Phi^+)^{k-1}\Psi^+(X_{n-k})\}.$$

By inflation and (later restriction) we extend the $P_{(n-1,1)}(k_F)\times k_F^\times$-representation

$$ext\{(\Phi^+)^{k-1}\Psi^+(X_{n-k})\}\boxtimes\chi_{n+1}$$

to a representation of $Z(\eta)\cap M_{(n,1)}$. We continue to use the notation

$$ext\{(\Phi^+)^{k-1}\Psi^+(X_{n-k})\}\boxtimes\chi_{n+1}$$

considered as a representation of $Z(\eta)\cap M_{(n,1)}$. We now have

$$\begin{aligned}\mathrm{ind}_{P(m+1)}^{K_{n+1}}(\chi_{I_n})\simeq{}&\mathrm{ind}_{P(m)}^{K_{n+1}}(\chi_{I_n})\oplus\\&\bigoplus_{k\geq 1}^{n}\mathrm{ind}_{Z(\eta)}^{K_{n+1}}\{ext\{(\Phi^+)^{k-1}\Psi^+(X_{n-k})\}\boxtimes\chi_{n+1}\}.\end{aligned}$$

We will show that any irreducible sub-representation of

$$\mathrm{ind}_{Z(\eta)}^{K_{n+1}}(ext\{(\Phi^+)^{k-1}\Psi^+(X_{n-k})\}\boxtimes\chi_{n+1})\}$$

is atypical for the component $[T_n,\chi_{I_n}]$.

We first consider the case when $k\geq 2$. The representation X_{n-k} is a direct sum of the representations:

$$\mathrm{ind}_{B_{n-k}(k_F)}^{\mathrm{GL}_{n-k}(k_F)}(\chi_{i_1}\boxtimes\chi_{i_2}\boxtimes\cdots\boxtimes\chi_{i_{n-k}}).$$

The above term also occurs in the expansion

$$\prod_{j=1}^{n-k}(1_R+\chi_{i_j})(1_R+\lambda)$$

where λ is a cuspidal representation of $\mathrm{GL}_k(k_F)$. To shorten the notation we use the symbol $\times$ for the multiplication in the ring R. We get that the representation

$$(\Phi^+)^{k-1}\Psi^+(\chi_{i_1}\times\chi_{i_2}\times\ldots\times\chi_{i_{n-k}})$$

occurs in the representation

$$\mathrm{res}_{\mathrm{Mir}_{n-1}}(\chi_{i_1}\times\chi_{i_2}\times\ldots\times\chi_{i_{n-k}}\times\lambda).$$

Note that the mod-$\mathfrak{P}_F$ reduction of the group $Z(\eta)\cap M_{(n,1)}$ is contained in the subgroup of the form (13) and recall that the $(n-1)^{th}$ diagonal entry of any element in (13) is the same as its n^{th} diagonal entry. Let

η_1 and η_2 be the central characters of $\chi_1 \times \chi_2 \times \ldots \times \chi_n$ and $(\chi_{i_1} \times \chi_{i_2} \times \ldots \times \chi_{i_{n-k}}) \times \lambda$ respectively. Let χ'_{n+1} be the character $\chi_{n+1}\eta_2^{-1}\eta_1$ and now the representation

$$\mathrm{res}_{Z(\eta)\cap M_{(n,1)}}\{ext((\Phi^+)^{k-1}\Psi^+(\chi_{i_1} \times \chi_{i_2} \times \ldots \times \chi_{i_{n-k}}))\} \boxtimes \chi_{n+1}$$

occurs in the representation

$$\mathrm{res}_{Z(\eta)\cap M_{(n,1)}}(\chi_{i_1} \times \chi_{i_2} \times \ldots \times \chi_{i_{n-k}}) \times \lambda) \boxtimes \chi'_{n+1}.$$

Hence an irreducible sub-representation of

$$\mathrm{ind}_{Z(\eta)}^{K_{n+1}}\{(ext\{(\Phi^+)^{k-1}\Psi^+(X_{n-k})\} \boxtimes \chi_{n+1}) \otimes U_\eta\} \tag{14}$$

occurs as a sub-representation of

$$\mathrm{ind}_{Z(\eta)}^{K_{n+1}}\{\{(\chi_{i_1} \boxtimes \chi_{i_2} \boxtimes \cdots \boxtimes \chi_{i_{n-k}} \boxtimes \lambda \boxtimes \chi'_{n+1})\} \otimes U_\eta\}. \tag{15}$$

The above representation occurs as a sub-representation of

$$\mathrm{ind}_{P_{(1,1,\ldots,k,1)}\cap K_{n+1}}^{K_{n+1}}\{\chi_{i_1} \boxtimes \chi_{i_2} \boxtimes \cdots \boxtimes \chi_{i_{n-k}} \boxtimes \lambda \boxtimes \chi'_{n+1}\}. \tag{16}$$

Hence the sub-representation of (14) are not typical representations.

Now we are left with the term

$$\mathrm{ind}_{Z(\eta)}^{K_{n+1}}\{(ext\{\Psi^+(X_{n-1})\} \boxtimes \chi_{n+1}) \otimes U_\eta\}. \tag{17}$$

If we repeat the same strategy as for $k \geq 2$ then λ is one-dimensional so the representations (16) and $\chi_1 \times \chi_2 \times \ldots \chi_{n+1}$ may not have distinct inertial support. In order to tackle the terms of the above representation we use a different technique. We now recall the definition of the representation U_η, the functor Ψ^+ and some facts due to Casselman regarding the restriction of an irreducible smooth representation to the maximal compact subgroup $\mathrm{GL}_2(\mathcal{O}_F)$.

The representation U_η is a character on the group $Z(\eta)$. From (13) any element of the group $Z(\eta)$ is of the form

$$\begin{pmatrix} A & B & X' \\ \varpi_F C & d & y \\ \varpi_F^m X & \varpi_F^m y' & e \end{pmatrix} \tag{18}$$

where $A \in \mathrm{GL}_{n-1}(\mathcal{O}_F)$; $(X'), X^{tr}, B, C^{tr} \in M_{(n-1)\times 1}(\mathcal{O}_F)$; $e, d \in \mathcal{O}_F^\times$; $y, y' \in \mathcal{O}_F$ and $d \equiv e \pmod{\mathfrak{P}_F}$. The character U_η is given by

$$\begin{pmatrix} A & B & X' \\ \varpi_F C & d & y \\ \varpi_F^m X & \varpi_F^m y' & e \end{pmatrix} \mapsto \eta(\varpi_F^m y').$$

The functor

$$\Psi^+ : \mathcal{M}(\mathrm{GL}_{k-1}(k_F)) \to \mathcal{M}(\mathrm{Mir}_k)$$

is the inflation functor via the quotient map of Mir_k modulo the unipotent radical of Mir_k.

Let (π, V_π) be an irreducible smooth representation of $\mathrm{GL}_2(F)$. We denote by $c(\pi)$ and ϖ_π the conductor (see [Cas73a, Theorem 1]) and central character of the representation π respectively. Let V^N be the space of all vectors fixed by the principal congruence subgroup of level N for all $N \geq 1$. For all $i > c(\varpi_\pi)$ we define the representation $U_i(\varpi_\pi)$ as the complement of the representation $\mathrm{ind}_{B_2(i-1)}^{\mathrm{GL}_2(\mathcal{O}_F)}(\varpi_\pi)$ in $\mathrm{ind}_{B_2(i)}^{\mathrm{GL}_2(\mathcal{O}_F)}(\varpi_\pi)$. For $i = c(\varpi_\pi)$ we set

$$U_i(\varpi_\pi) = \mathrm{ind}_{B_2(i)}^{\mathrm{GL}_2(\mathcal{O}_F)}(\varpi_\pi \boxtimes \mathrm{id}).$$

It follows from [Cas73b, Proposition 1] that the representation $U_i(\varpi_\pi)$ is an irreducible representation of $\mathrm{GL}_2(\mathcal{O}_F)$. From the result [Cas73b, Proposition 2] we get that $c(\pi) \geq c(\varpi_\pi)$. By [Cas73b, Theorem 1] we have

$$\mathrm{res}_{\mathrm{GL}_2(\mathcal{O}_F)} V_\pi = V^{(c(\pi)-1)} \oplus \bigoplus_{i \geq c(\pi)} U_i(\varpi_\pi). \tag{19}$$

We now describe the representation $U_i(\varpi_\pi)$ in our language. Let η be a non-trivial character of the group $K_2(m)U_{(1,1)}(\mathcal{O}_F)$ trivial modulo

$$K_2(m)U_{(1,1)}(\mathcal{O}_F) \cap B_2(m+1).$$

Let $Z(\eta)$ be the $B_2(m)$ stabilizer of η. Any element of the group $Z(\eta)$ is of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $a, d \in \mathcal{O}_F^\times$; $b \in \mathcal{O}_F$, $c \in \mathfrak{P}_F^m$ and $d \equiv a$ modulo $\mathfrak{P}_F$. We define a character U_η by setting

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \eta(c).$$

We then have

$$U_m(\varpi) \simeq \mathrm{ind}_{Z(\eta)}^{K_2}(U_\eta \otimes (\varpi \boxtimes \mathrm{id})).$$

Now let us resume the proof in the general case $n > 2$ the representation

$$\mathrm{ind}_{Z(\eta)}^{K_{n+1}}\{(ext\{\Psi^+(X_{n-1})\} \boxtimes \chi_{n+1}) \otimes U_\eta\}$$

is contained in the representation

$$\mathrm{ind}_{P_{(n-1,2)}(m)}^{K_{n+1}}(X_{n-1} \boxtimes U_m(\chi)) \tag{20}$$

where χ is given by $\prod_{i=1}^n \chi_i$ of $\mathcal{O}_F^\times$. This representation, by the theorem of Casselman (see the decomposition (19)) is contained in the representation

$$\mathrm{ind}_{P_{(n-1,2)} \cap K_{n+1}}^{K_{n+1}}(X'_{n-1} \boxtimes \sigma)$$

where σ is a supercuspidal representation of level-zero with central character χ (see the remark below for the existence) and X'_{n-1} is the $(n-1)^{\mathrm{th}}$ derivative of the representation

$$i_{B_n}^{G_n}(\chi_{I_n}).$$

Hence irreducible sub-representations of (17) are atypical. This completes the proof that irreducible sub-representations of

$$\mathrm{ind}_{Z(\eta)}^{K_{n+1}}\{U_\eta \otimes \mathrm{res}_{Z(\eta) \cap M_{(n,1)}}\{\mathrm{ind}_{B_n(1)}^{K_n}(\chi_{I_n}) \boxtimes \chi_{n+1}\}\}$$

are atypical. From the decomposition

$$\begin{aligned} &\mathrm{ind}_{P(m+1)}^{K_{n+1}}\{\mathrm{ind}_{B_n(1)}^{K_n}(\chi_{I_n}) \boxtimes \chi_{n+1}\} \\ &\simeq \mathrm{ind}_{P(m)}^{K_{n+1}}\{\mathrm{ind}_{B_n(1)}^{K_n}(\chi_{I_n}) \boxtimes \chi_{n+1}\} \\ &\qquad \oplus \mathrm{ind}_{Z(\eta)}^{K_{n+1}}\{U_\eta \otimes \mathrm{res}_{Z(\eta) \cap M_{(n,1)}}\{\mathrm{ind}_{B_n(1)}^{K_n}(\chi_{I_n}) \boxtimes \chi_{n+1}\}\}. \end{aligned}$$

we get the theorem for the case where $n_r = 1$.

Remark 4.2. *The existence of the cuspidal representation of* $\mathrm{GL}_2(k_F)$ *with a given central character can be deduced from the explicit formula for such representations, we refer to* [BH06, Theorem section 6.4]. *To be precise we begin with a quadratic extension* k *of* k_F *and* θ *a character of* $k^\times$ *such that* $\theta^q \neq \theta$ *where* $q = \#k_F$. *These characters are called regular characters and for any regular character one can define a supercuspidal representation* π_θ *and conversely all supercuspidal representations are of the form* π_θ *for some regular character* θ. *The central character of* π_θ *is given by* $\mathrm{res}_{k_F^\times}(\theta)$. *Now to get a supercuspidal representation with a central character* χ *we begin with a character* χ *on* $k_F^\times$, *there are* $\#k_F + 1$ *possible extensions to* $k^\times$. *The set of characters* θ *such that* $\theta^q = \theta$ *has cardinality* $\#k_F - 1$. *Hence there exists at least one supercuspidal representation with a given central character* χ. *This shows that irreducible sub-representations of* (20) *are not typical and this completes the proof of the theorem in this case.*

The case where $n_r > 1$. By transitivity of induction we have

$$\mathrm{ind}_{P_I(m)}^{P_I(1)}(\tau_I) \simeq \mathrm{ind}_{P_I(1,m)}^{P_I(1)}\{\mathrm{ind}_{P_I(m)}^{P_I(1,m)}(\tau_I)\}.$$

We note that $P_I(1,m) \cap U_{(n-n_r+1,n_r)}$ is equal to $P_I(m) \cap U_{(n-n_r+1,n_r)}$ and $P_I(1,m) \cap \bar{U}_{(n-n_r+1,n_r)}$ is equal to $P_I(m) \cap \bar{U}_{(n-n_r+1,n_r)}$ hence Lemma 2.6 gives the isomorphism

$$\mathrm{ind}_{P_I(1,m)}^{P_I(1)}\{\mathrm{ind}_{P_I(m)}^{P_I(1,m)}(\tau_I)\} \simeq \mathrm{ind}_{P_I(1,m)}^{P_I(1)}\{(\mathrm{ind}_{P_{I'}(m)}^{P_{I'}(1)}(\tau_{I'})) \boxtimes \tau_r)\}.$$

Splitting the representation $\mathrm{ind}_{P_{I'}(m)}^{P_{I'}(1)}(\tau_{I'})$ as $\tau_{I'} \oplus U_m^0(\tau_{I'})$ we get that

$$\mathrm{ind}_{P_I(1,m)}^{P_I(1)}\{(\mathrm{ind}_{P_{I'}(m)}^{P_{I'}(1)}(\tau_{I'})) \boxtimes \tau_r)\} \simeq \mathrm{ind}_{P_I(1,m)}^{P_I(1)}\{U_m^0(\tau_{I'}) \boxtimes \tau_r\} \oplus \mathrm{ind}_{P_I(1,m)}^{P_I(1)}(\tau_I).$$

From Frobenius reciprocity the representation τ_I occurs in $\mathrm{ind}_{P_I(1,m)}^{P_I(1)}(\tau_I)$ with multiplicity one. Let $U_{(1,m)}^0(\tau_I)$ be the complement of τ_I in $\mathrm{ind}_{P_I(1,m)}^{P_I(1)}(\tau_I)$. With this we conclude that

$$\mathrm{ind}_{P_I(m)}^{P_I(1)}(\tau_I) \simeq \mathrm{ind}_{P_I(1,m)}^{P_I(1)}\{U_m^0(\tau_{I'}) \boxtimes \tau_r\} \oplus U_{(1,m)}^0(\tau_I) \oplus \tau_I.$$

By definition $U_m(\tau_I) = \mathrm{ind}_{P_I(1)}^{K_{n+1}}(U_m^0(\tau_I))$ which shows that

$$U_m(\tau_I) \simeq \mathrm{ind}_{P_I(1,m)}^{K_{n+1}}\{U_m^0(\tau_{I'}) \boxtimes \tau_r\} \oplus \mathrm{ind}_{P_I(1)}^{K_{n+1}}(U_{(1,m)}^0(\tau_I)).$$

We observe that

$$P_I(1,m) \cap U_{(n-n_r+1,n_r)} = P_{(n-n_r+1,n_r)}(m) \cap U_{(n-n_r+1,n_r)}$$

and

$$P_I(1,m) \cap \bar{U}_{(n-n_r+1,n_r)} = P_{(n-n_r+1,n_r)}(m) \cap \bar{U}_{(n-n_r+1,n_r)}$$

hence Lemma 2.6 applied to the groups $J_2 = P_I(1,m)$ and $J_1 = P_{(n-n_r+1,n_r)}(m)$ and $\lambda = U_m^0(\tau_{I'}) \boxtimes \tau_r$ gives us the isomorphism

$$\mathrm{ind}_{P_I(1,m)}^{K_{n+1}}\{U_m^0(\tau_{I'}) \boxtimes \tau_r\} \simeq \mathrm{ind}_{P_{(n-n_r+1,n_r)}(m)}^{K_{n+1}}\{U_m(\tau_{I'}) \boxtimes \tau_r\}.$$

With this we are in a place to use the induction hypothesis through the isomorphism

$$U_m(\tau_I) \simeq \mathrm{ind}_{P_{(n-n_r+1,n_r)}(m)}^{K_{n+1}}\{U_m(\tau_{I'}) \boxtimes \tau_r\} \oplus \mathrm{ind}_{P_I(1)}^{K_{n+1}}(U_{(1,m)}^0(\tau_I)). \tag{21}$$

By induction hypothesis $\mathrm{GL}_{n-n_r+1}(\mathcal{O}_F)$-irreducible sub-representations of $U_m(\tau_{I'})$ are atypical for the component $[M_{I'}, \sigma_{I'}]$. Now Lemma 3.10 and the equation (21) reduce the proof of the theorem to showing that irreducible sub-representations of $\mathrm{ind}_{P_I(1)}^{K_{n+1}}(U_{(1,m)}^0(\tau_I))$ are atypical representations.

Proposition 4.3. *The irreducible sub-representations of*

$$\mathrm{ind}_{P_I(1)}^{K_{n+1}}(U_{(1,m)}^0(\tau_I))$$

are atypical for $m \geq 1$.

Proof. We observe that

$$\mathrm{ind}_{P_I(1,m+1)}^{P_I(1)}(\tau_I) \simeq \mathrm{ind}_{P_I(1,m)}^{P_I(1)}\{\mathrm{ind}_{P_I(1,m+1)}^{P_I(1,m)}(\tau_I)\}$$

and the decomposition (8) gives us the isomorphism

$$\mathrm{ind}_{P_I(1,m+1)}^{P_I(1)}(\tau_I) = \mathrm{ind}_{P_I(1,m)}^{P_I(1)}(\tau_I) \oplus \bigoplus_{\eta_k \neq \mathrm{id}} \mathrm{ind}_{P_I(1,m)}^{P_I(1)}\{(\mathrm{ind}_{Z(\eta_k)}^{P_I(1,m)}(U_{\eta_k}) \otimes \tau_I\}$$

which gives the equality

$$U_{(1,m+1)}^0(\tau_I) = U_{(1,m)}^0(\tau_I) \oplus \bigoplus_{\eta_k \neq \mathrm{id}} \mathrm{ind}_{P_I(1,m)}^{P_I(1)}\{(\mathrm{ind}_{Z(\eta_k)}^{P_I(1,m)}(U_{\eta_k})) \otimes \tau_I\}.$$

If we show that the irreducible sub-representations of

$$\mathrm{ind}_{P_I(1,m)}^{K_{n+1}}\{(\mathrm{ind}_{Z(\eta_k)}^{P_I(1,m)}(U_{\eta_k})) \otimes \tau_I\}$$

(for $\eta_k \neq \mathrm{id}$) are atypical for $[M_I, \sigma_I]$ then induction on the positive integer m completes the proof of the proposition in this case. To begin with we note that

$$\mathrm{ind}_{P_I(1,m)}^{K_{n+1}}\{\mathrm{ind}_{Z(\eta_k)}^{P_I(1,m)}(U_{\eta_k}) \otimes \tau_I\} \simeq \mathrm{ind}_{P_I(1,m)}^{K_{n+1}}\{\mathrm{ind}_{Z(\eta_k)}^{P_I(1,m)}(U_{\eta_k} \otimes \mathrm{res}_{Z(\eta_k) \cap M_I} \tau_I)\}.$$

The representation τ_I is trivial on $M_I \cap K_{n+1}(1)$. Hence $\mathrm{res}_{Z(\eta_k) \cap M_I} \tau_I$ is isomorphic to the inflation of the representation $\mathrm{res}_{\overline{Z(\eta_k) \cap M_I}} \tau_I$ where $\overline{Z(\eta_k) \cap M_I}$ is mod-$\mathfrak{P}_F$ reduction of $Z(\eta_k) \cap M_I$. Let $A = \theta_I(\eta_k)$ where θ_I is the map defined in equation (7). The mod-$\mathfrak{P}_F$ reduction $\overline{Z(\eta_k) \cap M_I}$ is contained in $Z_{P_{I'}(k_F) \times \mathrm{GL}_{n_r}(k_F)}(A)$.

If η_k is a nontrivial character then $A \neq 0$. Applying Lemma 3.8 to the parabolic corresponding to the partition $I = (n_1, n_2, \ldots, n_r)$ and the assumption that $n_r > 1$ shows that

$$Z_{P_{I'}(k_F)\times \mathrm{GL}_{n_r}(k_F)}(A) \cap M_I(k_F)$$

has trivial intersection with a unipotent subgroup U of $M_I(k_F)$. If A satisfies condition (1) of the Lemma 3.8 then let U_j be the unipotent radical of the opposite parabolic subgroup containing the image of p_j (see Lemma 3.8) and if A satisfies condition (2) of 3.8 then let U_r be any unipotent radical of a parabolic subgroup of $\mathrm{GL}_{n_r}(k_F)$. Now U can be chosen to be the group

$$\{(u_1, u_2, \ldots, u_r) \ \in M_I(k_F)\ |u_j \in U_j \text{ and } u_k = 1_k \ \forall\ k \neq j\}$$

if A satisfies condition (1) in Lemma 3.8 and U to be

$$\{(u_1, u_2, \ldots, u_r) \ \in M_I(k_F)\ |u_r \in U_r \text{ and } u_k = 1_k \ \forall\ k \neq r\}$$

if A satisfies the condition (2) in Lemma 3.8.

Now applying Lemma 3.9 to the reductive group $G = M_I$ and the representation $\tau = \tau_I$, we get that every irreducible subrepresentation Γ of

$$\mathrm{ind}_{P_I(1,m)}^{K_{n+1}}\{\mathrm{ind}_{Z(\eta_k)}^{P_I(1,m)}(U_{\eta_k} \otimes \mathrm{res}_{Z(\eta_k)\cap M_I}\ \tau_I)\}$$

occurs as a subrepresentation of some representation of the form

$$\mathrm{ind}_{P_I(1,m)}^{K_{n+1}}\{\mathrm{ind}_{Z(\eta_k)}^{P_I(1,m)}(U_{\eta_k} \otimes \mathrm{res}_{Z(\eta_k)\cap M_I}\ \tau_I')\}, \tag{22}$$

with τ_I' some non-cuspidal representation of $M_I(k_F)$. The representation 22 occurs as a subrepresentation of

$$\mathrm{ind}_{P_I\cap K_{n+1}}^{K_{n+1}}\ \tau_I',$$

which occurs as a subrepresentation of

$$\mathrm{res}_{K_{n+1}}\ i_{P_I}^{G_{n+1}}\sigma_I'$$

where σ_I' is a non-cuspidal representation such that ${\sigma_I'}^{K_{n+1}(1)\cap M_I} \simeq \tau_I'$. Hence the representation Γ is not a typical representation for the component $[M_I, \sigma_I]$.

This completes the proof of the proposition and also the proof of the theorem. □

□

References

[BH06] Colin J. Bushnell and Guy Henniart, *The local Langlands conjecture for* GL(2), Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 335, Springer-Verlag, Berlin, 2006. MR 2234120 (2007m:22013)

[BK93] Colin J. Bushnell and Philip C. Kutzko, *The admissible dual of* GL(N) *via compact open subgroups*, Annals of Mathematics Studies, vol. 129, Princeton University Press, Princeton, NJ, 1993. MR 1204652 (94h:22007)

[BK99] ———, *Semisimple types in* GL_n, Compositio Math. **119** (1999), no. 1, 53–97. MR 1711578 (2000i:20072)

[BM02] Christophe Breuil and Ariane Mézard, *Multiplicités modulaires et représentations de* $\mathrm{GL}_2(\mathbf{Z}_p)$ *et de* $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ *en* $l = p$, Duke Math. J. **115** (2002), no. 2, 205–310, With an appendix by Guy Henniart. MR 1944572 (2004i:11052)

[BZ76] I. N. Bernšteĭn and A. V. Zelevinskiĭ, *Representations of the group $GL(n, F)$, where F is a local non-Archimedean field*, Uspehi Mat. Nauk **31** (1976), no. 3(189), 5–70. MR 0425030 (54 #12988)

[Cas73a] William Casselman, *On some results of Atkin and Lehner*, Math. Ann. **201** (1973), 301–314. MR 0337789

[Cas73b] ———, *The restriction of a representation of* $\mathrm{GL}_2(k)$ *to* $\mathrm{GL}_2(\mathfrak{o})$, Math. Ann. **206** (1973), 311–318. MR 0338274 (49 #3040)

[EG14] Matthew Emerton and Toby Gee, *A geometric perspective on the Breuil-Mézard conjecture*, J. Inst. Math. Jussieu **13** (2014), no. 1, 183–223. MR 3134019

[Pas05] Vytautas Paskunas, *Unicity of types for supercuspidal representations of* GL_N, Proc. London Math. Soc. (3) **91** (2005), no. 3, 623–654. MR 2180458 (2007b:22018)

[Ren10] David Renard, *Représentations des groupes réductifs p-adiques*, Cours Spécialisés [Specialized Courses], vol. 17, Société Mathématique de France, Paris, 2010. MR 2567785 (2011d:22019)

[SZ99] P. Schneider and E.-W. Zink, *K-types for the tempered components of a p-adic general linear group*, J. Reine Angew. Math. **517** (1999), 161–208, With an appendix by Schneider and U. Stuhler. MR 1728541 (2001f:22029)

[Wil10] Conley William, *Inertial types and automorphic representations with prescribed ramification*, Phd Thesis, UCLA (2010).

[Zel81] Andrey V. Zelevinsky, *Representations of finite classical groups*, Lecture Notes in Mathematics, vol. 869, Springer-Verlag, Berlin-New York, 1981, A Hopf algebra approach. MR 643482

Santosh Nadimpalli,
School of Mathematics, Tata Institute of Fundamental Research, Mumbai, 400005.
nvrnsantosh@gmail.com, nsantosh@math.tifr.res.in