TATE COHOMOLOGY OF WHITTAKER LATTICES AND BASE CHANGE OF GENERIC REPRESENTATIONS OF GL_n

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Abstract. Let p and l be two distinct odd primes, and let $n \geq 2$ be a positive integer. Let E be a finite Galois extension of degree l of a p-adic field F. Let q be the cardinality of the residue field of F. Let π_F be an integral l-adic generic representation of $\mathrm{GL}_n(F)$, and let π_E be the base change of π_F. Let $J_l(\pi_F)$ (resp. $J_l(\pi_E)$) be the unique generic component of the mod-l reduction $r_l(\pi_F)$ (resp. $r_l(\pi_E)$). Assuming that l does not divide $|\mathrm{GL}_{n-1}(\mathbb{F}_q)|$, we prove that the Frobenius twist of $J_l(\pi_F)$ is the unique generic subquotient of the Tate cohomology group $\widehat{H}^0(\mathrm{Gal}(E/F), J_l(\pi_E))$–considered as a representation of $\mathrm{GL}_n(F)$.

1. Introduction

Let l be a prime number, and let F be a number field. Let $\mathbf{G}$ be a reductive algebraic group defined over F, and let σ be an automorphism of order l of $\mathbf{G}$. D.Treumann and A.Venkatesh have constructed a functorial lift of a mod-l automorphic form for $\mathbf{G}^\sigma$ to a mod-l automorphic form for $\mathbf{G}$ (see [TV16]). They conjectured that the mod-l local functoriality at ramified places must be realised in Tate cohomology, and they defined the notion of linkage (see [TV16, Section 6.3] for more details). Among many applications of this set up, we focus on local base change lifting from $\mathbf{G}^\sigma = \mathrm{GL}_n/F$ to $\mathbf{G} = \mathrm{Res}_{E/F}\mathrm{GL}_n/E$, where E/F is a Galois extension of p-adic fields with $[E : F] = l$. Truemann and Venkatesh's conjecture on linkage in Tate cohomology is verified for local base change of depth-zero cuspidal representations by N.Ronchetti, and a precise conjecture in the context of local base change of l-adic higher depth cuspidal representations was formulated in [Ron16, Conjecture 2]. In this article, using Whittaker models and Rankin-Selberg zeta functions, we prove this conjecture for GL_n under the assumption that l does not divide the pro-order of $\mathrm{GL}_{n-1}(F)$ whenever $n > 2$. In fact, when the prime l does not divide the pro-order of $\mathrm{GL}_{n-1}(F)$, we prove a much stronger theorem that the Frobenius twist of a mod-l generic representation of $\mathrm{GL}_n(F)$ occurs as a sub-quotient of the zeroth Tate cohomology of its base change lifting to $\mathrm{GL}_n(E)$ (see Theorem 6.7 for the precise result).

Let us introduce some notations to state the results of this article. From now, we assume that F is a finite extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$. Let E be a finite Galois extension of F with $[E : F] = l$, where l and p are distinct odd primes. Let π_F be an integral l-adic generic representation of $\mathrm{GL}_n(F)$. The mod-l-reduction of π_F has a unique generic component and it is denoted by $J_l(\pi_F)$ (see [Vig01, Section 1.8.4]). The base change lift of π_F to $\mathrm{GL}_n(E)$ is denoted by π_E (for the definition, see subsection (4.2)). Note that π_E is also an integral l-adic generic representation of $\mathrm{GL}_n(E)$. Moreover, there is an isomorphism $T : \pi_E \xrightarrow{\sim} \pi_E^\gamma$, where π_E^γ is the twist of π_E by a generator γ of $\mathrm{Gal}(E/F)$. Then the unique generic component $J_l(\pi_E)$ of the mod-l reduction $r_l(\pi_E)$ is also stable under the action of $\mathrm{Gal}(E/F)$–induced by T. In this article, Tate cohomology groups are always with respect to the action of $\mathrm{Gal}(E/F)$. We prove the following theorem:

Theorem 1.1. *Let F be a finite extension of $\mathbb{Q}_p$, and let E be a finite Galois extension of F with $[E : F] = l$, where p and l are distinct odd primes such that that l does not divide $|\mathrm{GL}_{n-1}(\mathbb{F}_q)|$. Let π_F be an integral l-adic generic representation of $\mathrm{GL}_n(F)$, and let π_E be the base change lifting of π_F to $\mathrm{GL}_n(E)$. Then, the Frobenius twist of $J_l(\pi_F)$ occurs a subquotient of the zeroth Tate cohomology group $\widehat{H}^0(J_l(\pi_E))$, considered as a representation of $\mathrm{GL}_n(F)$.*

We note some immediate remarks on the hypothesis in Theorem 1.1. As a consequence of Proposition 6.3 in Section 6, the Frobenius twist of $J_l(\pi_F)$, defined as $J_l(\pi_F) \otimes_{\mathrm{Frob}} \overline{\mathbb{F}}_l$, where Frob is the Frobenius automorphism of $\overline{\mathbb{F}}_l$, is in fact the unique generic sub-quotient of the Tate cohomology group $\widehat{H}^0(J_l(\pi_E))$. We use Kirillov and Whittaker models of generic representations to prove our main result. The hypothesis that l does not divide the pro-order of $\mathrm{GL}_{n-1}(F)$ is required in the proof of a vanishing result of Rankin–Selberg integrals on $\mathrm{GL}_{n-1}(F)$ (the analogue of [JPSS81, Lemma 3.5] or [BH03, 6.2.1]). This condition on l may be removed using γ-factors defined over local Artinian $\overline{\mathbb{F}}_l$-algebras as defined in the work of G.Moss and N.Matringe in [MM22]. However, the right notion of base change over local Artinian $\overline{\mathbb{F}}_l$-algebras is not clear to the authors and hence, we use the mild hypothesis that l does not divide $|\mathrm{GL}_{n-1}(\mathbb{F}_q)|$. If π_F and π_E are both cuspidal, then using the Kirillov model for cuspidal representations, one observes that the Tate cohomology group $\widehat{H}^0(r_l(\pi_E))$ is an irreducible $\mathrm{GL}_n(F)$ representation, and the above theorem says that this Tate cohomology space is isomorphic to the Frobenius twist of mod-l reduction of π_F (Corollary 6.9) when l does not divide the pro-order of $\mathrm{GL}_{n-1}(F)$. This is conjectured by Ronchetti in [Ron16, Conjecture 2].

Let $\mathcal{K}$ be the maximal unramified extension of $\mathbb{Q}_l$ in an algebraic closure $\overline{\mathbb{Q}}_l$ of $\mathbb{Q}_l$. Let Λ be the ring of integers of $\mathcal{K}$. We also prove an integral version of Theorem 1.1. To be precise, say π_E is an integral generic $\mathcal{K}$-representation of $\mathrm{GL}_n(E)$–which is absolutely irreducible (i.e., $\pi_E \otimes_{\mathcal{K}} \overline{\mathbb{Q}}_l$ is irreducible), such that $\pi_E \otimes_{\mathcal{K}} \overline{\mathbb{Q}}_l$ is the base change lift of an l-adic integral generic representation π_F of $\mathrm{GL}_n(F)$. We show that the Frobenius twist of $J_l(\pi_F)$ occurs as the unique generic subquotient of the zeroth Tate cohomology group $\widehat{H}^0(\mathbb{W}_\Lambda(\pi_E,\psi_E))$ (Corollary 6.8), where $\mathbb{W}_\Lambda(\pi_E,\psi_E)$ is the space of all Λ-valued functions in the Whittaker model of π_E, with respect to a $\mathrm{Gal}(E/F)$-equivariant character ψ_E. A priori, Vignéras showed that $\mathbb{W}_\Lambda(\pi_E,\psi_E)$ is a $\mathrm{GL}_n(E)$-invariant Λ-lattice in $\mathbb{W}(\pi_E,\psi_E)$.

When l does not divide the pro-order of $\mathrm{GL}_n(F)$, we obtain a much precise version of Theorem 1.1. We can show that the first Tate cohomology group of any $\mathrm{Gal}(E/F)$ invariant lattice $\mathcal{L}$ in a generic representation π_E as in Theorem 1.1 is trivial. Moreover, we show that the zeroth Tate cohomology group of the mod-l reduction $r_l(\pi_E)$ is an irreducible representation of $\mathrm{GL}_n(F)$ (see Theorem 8.3 and Corollary 8.4). Our method can also be extended to some non-generic representations as well. Especially for those irreducible representations of $\mathrm{GL}_n(E)$ which remains irreducible when restricted to the mirabolic subgroup, denoted by $P_n(E)$. This class of representations are exactly the Zelevinsky sub-representations. Assume that σ_E is an l-adic cuspidal representation obtained as a base change lifting of σ_F to $\mathrm{GL}_n(E)$. Let Δ be a segment (see Section 2.7.2) on the cuspidal line of π_E (defined in Theorem 1.1). We apply Theorem 1.1 to compute the Tate cohomology of mod-l Zelevinsky sub-representations $Z(\overline{\Delta})$ (see Theorem 7.3), where $\overline{\Delta}$ is the segment on the cuspidal line of $r_l(\sigma_E)$.

When F is a local function field, the above theorem follows from the work of T.Feng [Fen24]. T.Feng uses the constructions of V. Lafforgue and A. Genestier-V.Lafforgue [GL17]. Assuming that l and p do not divide n, Ronchetti proved the above result for depth-zero cuspidal representations using the compact induction model. Our methods are very different from the work of N.Ronchetti and the work of T.Feng. We rely on Rankin–Selberg integrals and Whittaker models. We do not require the explicit construction of cuspidal representations. We use various properties of local ϵ and γ-factors both in l-adic and mod-l situations associated with the representations of the p-adic group and the Weil group. The machinery of local ϵ and γ-factors of both l-adic and mod-l representations of $\mathrm{GL}_n(F)$ is made available by the seminal works of D.Helm, G.Moss, N.Matringe and R.Kurinczuk (see [HM18], [Mos16], [KM21], [KM17]).

The case where π_E is a cuspidal representation of $\mathrm{GL}_2(E)$ is considered in Theorem 6.5, the general case is proved in Theorem 6.7 using induction on n. The reader might quickly follow the proof of Theorem 6.5 before going to the general case. We sketch the proof of Theorem 1.1. The theorem is proved, inductively on n, using the Kirillov model. Let $\psi_F : F \to \overline{\mathbb{Q}}_l^\times$ be a non-trivial additive character and let ψ_E be the character $\psi_F \circ \mathrm{Tr}_{E/F}$, where $\mathrm{Tr}_{E/F} : E \to F$ is the trace function. Let (π_F, V) be an integral generic l-adic representation of $\mathrm{GL}_n(F)$. In particular, V is a $\overline{\mathbb{Q}}_l$-vector space. Let $N_n(F)$ be the group of unipotent upper triangular matrices in $\mathrm{GL}_n(F)$. Let $\Theta_F : N_n(F) \to \overline{\mathbb{Q}}_l^\times$ be a non-degenerate character corresponding to ψ_F. We denote by $\mathbb{W}(J_l(\pi_F),\overline{\psi}_F)$ the Whittaker model of the unique generic sub–quotient, denoted by $J_l(\pi_F)$, of the mod-l reduction of π_F. Here, $\overline{\psi}_F$ is the mod-l reduction of ψ_F. Let π_E be the base change lift of π_F to $\mathrm{GL}_n(E)$. Similar notations for π_E are followed where $\overline{\psi}_F$ is replaced with $\overline{\psi}_E$. It is easy to note that (Lemma 2.4) $\mathbb{W}(J_l(\pi_E),\overline{\psi}_E)$ is stable under the action of $\mathrm{Gal}(E/F)$ on the space $\mathrm{Ind}_{N_n(E)}^{\mathrm{GL}_n(E)}\overline{\Theta}_E$. Let $\mathbb{K}(J_l(\pi_F),\overline{\psi}_F)$ be the Kirillov model of $J_l(\pi_F)$. Using the result [MM22, Theorem 4.2], we get that the restriction to $P_n(F)$ map from $\mathbb{W}(J_l(\pi_F),\overline{\psi}_F)$ to $\mathbb{K}(J_l(\pi_F),\overline{\psi}_F)$ is a bijection.

The Kirillov model $\mathbb{K}(J_l(\pi_E),\overline{\psi}_E)$ contains the space of all smooth and compactly supported functions $\mathrm{ind}_{N_n(E)}^{P_n(E)}\overline{\Theta}_E$. Recall that the Tate cohomology group $\widehat{H}^0(\mathbb{K}(J_l(\pi_E),\overline{\psi}_E))$ (For definition, see Section 5) is an

$\overline{\mathbb{F}}_l$-representation of $P_n(F)$. Let Φ be the following map obtained by restriction of functions to $P_n(F)$:

$$\Phi : \widehat{H}^0(\mathbb{K}(J_l(\pi_E), \overline{\psi}_E)) \longrightarrow \mathrm{Ind}_{N_n(F)}^{P_n(F)} \overline{\Theta}_F^l.$$

Using compactly supported functions, one can show that the inverse image of $\mathbb{K}(J_l(\pi_F)^{(l)}, \overline{\psi}_F^l)$ under the map Φ is non-zero, and it is denoted by $\mathcal{M}(\pi_F, \psi_F)$. Here, $J_l(\pi_F)^{(l)}$ is the Frobenius twist of $J_l(\pi_F)$. To prove the main theorem, we show that the space $\mathcal{M}(\pi_F, \psi_F)$ is stable under $\mathrm{GL}_n(F)$ and the restriction of Φ to the space $\mathcal{M}(\pi_F, \psi_F)$ is $\mathrm{GL}_n(F)$ equivariant. This is just equivalent to showing that

$$I\big(X, \Phi(J_l(\pi_E)(w_n)W), \sigma(w_{n-1})W'\big) = I\big(X, J_l(\pi_F)^{(l)}(w_n)\Phi(W), \sigma(w_{n-1})W'\big), \tag{1.1}$$

for all $W \in \mathcal{M}(\pi_F, \psi_F)$ and $W' \in \mathbb{W}(\sigma, \overline{\psi}_F^{-l})$, where w_{n-1}, w_n are defined in subsection (2.2); and σ is an l-modular generic representation of $\mathrm{GL}_{n-1}(F)$. Here, $I(X, W, W')$ is a mod-l Rankin–Selberg zeta functions written as a formal power series in the variable X instead of the traditional q^{-s} ([KM17, Section 3]). We transfer the local Rankin–Selberg zeta functions $I\big(X, \Phi(J_l(\pi_E)(w_n)W), \sigma(w_{n-1})W'\big)$ made from integrals on $\mathrm{GL}_{n-1}(F)/N_{n-1}(F)$ to Rankin–Selberg zeta functions defined by integrals on $\mathrm{GL}_{n-1}(E)/N_{n-1}(E)$. Then, using local Rankin–Selberg functional equation, we show that the equality in (1.1) is equivalent to certain identities of mod-l local γ-factors, such as (6.13).

We briefly explain the contents of this article. In Section 2, we recall various notations, conventions on integral representations, Whittaker models and Kirillov models. In Section 3, we collect various results on local constants both in mod-l and l-adic settings. In Section 4, we put some well known results from l-adic local Langlands correspondence. In Section 5, we recall and set up some initial results on Tate cohomology of smooth integral representations as well as mod-l representations. In Section 6, we begin with a few observations on compatibility of Jacquet and twisted Jacquet functors with Tate cohomology. Then we prove our main result Theorem 6.7. In Sections 7 and 8, in the banal case, we completely compute the Tate cohomology of the representations $Z(\Delta)$ and $L(\Delta)$ using Theorem 6.7.

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2. Preliminaries

2.1. Let K be a non-Archimedean local field and let $\mathfrak{o}_K$ be the ring of integers of K. Let $\mathfrak{p}_K$ be the maximal ideal of $\mathfrak{o}_K$ and let ϖ_K be a uniformizer of K. Let q_K be the cardinality of the residue field $k_K = \mathfrak{o}_K/\mathfrak{p}_K$. Let $\upsilon_K : K^\times \to \mathbb{Z}$ be the normalized valuation. We denote by ν_K the normalized absolute value of K corresponding to υ_K. Let l and p be two distinct odd primes. Let F be a finite extension of $\mathbb{Q}_p$, and let E be a finite Galois extension of F with $[E : F] = l$. We denote the group $\mathrm{Gal}(E/F)$ by Γ.

2.2. For any ring A, let $M_{r\times s}(A)$ be the A-algebra of all $r \times s$ matrices with entries from A. Let $GL_n(K) \subseteq M_{n\times n}(K)$ be the group of all invertible $n \times n$ matrices. We denote by $G_n(K)$ the group $\mathrm{GL}_n(K)$ and $G_n(K)$ is equipped with locally compact topology induced from the local field K. For $r \in \mathbb{Z}$, let

$$G_n^r(K) = \{g \in G_n(K) : \upsilon_K(\det(g)) = r\}.$$

We set $P_n(K)$, the mirabolic subgroup, defined as the group:

$$\left\{ \begin{pmatrix} A & M \\ 0 & 1 \end{pmatrix} : A \in G_{n-1}(K), M \in M_{(n-1)\times 1}(K) \right\}.$$

Let $B_n(K)$ be the group of all invertible upper triangular matrices in $M_{n\times n}(K)$, and let $N_n(K)$ be its unipotent radical. We denote by w_n the following matrix of $G_n(K)$:

$$w_n = \begin{pmatrix} 0 & & & 1 \\ & & 1 & \\ & \cdot & & \\ \cdot & & & \\ 1 & & & 0 \end{pmatrix}.$$

Let X_K denote the coset space $N_{n-1}(K) \setminus G_{n-1}(K)$. For $r \in \mathbb{Z}$, we denote the coset space $\{N_{n-1}(K)g : g \in G_{n-1}^r(K)\}$ by X_K^r.

2.3. Fix an algebraic closure $\overline{\mathbb{Q}}_l$ of the field $\mathbb{Q}_l$. Let $\overline{\mathbb{Z}}_l$ be the integral closure of $\mathbb{Z}_l$ in $\overline{\mathbb{Q}}_l$ and let $\mathfrak{P}_l$ be the unique maximal ideal of $\overline{\mathbb{Z}}_l$. We have $\overline{\mathbb{Z}}_l/\mathfrak{P}_l \simeq \overline{\mathbb{F}}_l$. We fix a square root of q_F in $\overline{\mathbb{Q}}_l$, and it is denoted by $q_F^{1/2}$. The choice of $q_F^{1/2}$ is required for transferring the complex local Langlands correspondence to a local l-adic Langlands correspondence (see [BH06, Chapter 8]). Let $\mathcal{K}$ denote the maximal unramified extension of $\mathbb{Q}_l$ in $\overline{\mathbb{Q}}_l$, and let Λ be the ring of integers of $\mathcal{K}$. Let $W(\overline{\mathbb{F}}_l)$ be the ring of Witt vectors of $\overline{\mathbb{F}}_l$. The prime number l is called *banal* for $G_n(K)$ if l does not divide $|\mathrm{GL}_n(k_K)|$.

2.4. **Smooth representations and Integral representations.** Let G be a locally compact and totally disconnected group. A representation (π, V) is said to be smooth if, for every vector $v \in V$, the G-stabilizer of v is an open subgroup of G. All the representations are assumed to be smooth and the representation spaces are vector spaces over R, where $R = \overline{\mathbb{Q}}_l$ or $\overline{\mathbb{F}}_l$. A representation (π, V) is called l-adic when $R = \overline{\mathbb{Q}}_l$ and (π, V) is called l-modular when $R = \overline{\mathbb{F}}_l$. We denote by $\mathrm{Irr}(G, R)$, the set of all irreducible smooth R-representations of G. Let $C_c^\infty(G, R)$ denote the set of all locally constant and compactly supported functions on G taking values in a ring R.

Let (π, V) be an l-adic representation of G. A lattice in V is a free $\overline{\mathbb{Z}}_l$-module $\mathcal{L}$ such that $\mathcal{L} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l = V$. The representation (π, V) is said to be integral if it has finite length as a representation of G and there exists a G-invariant lattice $\mathcal{L}$ in V. A character is a smooth one-dimensional representation $\chi : G \longrightarrow R^\times$. For $G = G_n(K)$, a character $\chi : K^\times \to R^\times$ induces a character $\chi \circ \det : G_n(K) \to R^\times$. By abuse of notation, we denote the character $\chi \circ \det$ by χ. In particular, the normalized absolute value of K gives a character ν_K of $G_n(K)$. We say that a character $\chi : G \longrightarrow \overline{\mathbb{Q}}_l^\times$ is integral if it takes values in $\overline{\mathbb{Z}}_l$.

Let (π, V) be an integral l-adic representation of G. Choose a G-invariant lattice $\mathcal{L}$ in V. Then the group G acts on $\mathcal{L} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$, which is a vector space over $\overline{\mathbb{F}}_l$. This gives an l-modular representation, which depends on the choice of the G-invariant lattice $\mathcal{L}$. By [Vig96, II. 5.11.a and 5.11.b], the representation $\left(\pi, \mathcal{L} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l\right)$ is of finite length and its semisimplification is independent of the choice of the G-invariant lattice $\mathcal{L}$ in V. We denote by $r_l(\pi)$ the semisimplification of $\left(\pi, \mathcal{L} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l\right)$. The representation $r_l(\pi)$ is called the mod-l reduction of the l-adic representation π. We say that an l-modular representation σ lifts to an integral l-adic representation π if there exists a G-invariant lattice $\mathcal{L} \subseteq \pi$ such that $\mathcal{L} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l \simeq \sigma$.

2.5. **Parabolic induction.** Let H be a closed subgroup of G. Let Ind_H^G and ind_H^G be the smooth induction functor and compact induction functor respectively. We follow [BZ77] for the definitions.

Set $G = G_n(K)$, $P = P_n(K)$ and $N = N_n(K)$, where $G_n(K)$, $P_n(K)$ and $N_n(K)$ are defined in subsection (2.2). Let $\lambda = (n_1, n_2, ..., n_t)$ be an ordered partition of n. Let $Q_\lambda \subseteq G_n(K)$ be the group of matrices of the form

$$\begin{pmatrix} A_1 & * & * & * & * \\ & A_2 & * & * & * \\ & & . & * & * \\ & & & . & * \\ & & & & A_t \end{pmatrix},$$

where $A_i \in G_{n_i}(K)$, for all $1 \leq i \leq t$. Then $Q_\lambda = M_\lambda \ltimes U_\lambda$, where M_λ is the group of block diagonal matrices of the form

$$\begin{pmatrix} A_1 & & & & \\ & A_2 & & & \\ & & . & & \\ & & & . & \\ & & & & A_t \end{pmatrix}, \; A_i \in G_{n_i}(K),$$

for all $1 \leq i \leq t$ and U_λ is the unipotent radical of Q_λ consisting of matrices of the form

$$U_\lambda = \begin{pmatrix} I_{n_1} & * & * & * & * \\ & I_{n_2} & * & * & * \\ & & . & * & * \\ & & & . & * \\ & & & & I_{n_t} \end{pmatrix},$$

where I_{n_i} is the $n_i \times n_i$ identity matrix.

Let σ be an R-representation of M_λ. Then the representation σ is considered as a representation of Q_λ by inflation via the map $Q_\lambda \to Q_\lambda/U_\lambda \simeq M_\lambda$. The induced representation $\mathrm{Ind}_{Q_\lambda}^{G}(\sigma)$ is called the parabolic induction of σ. We denote the normalized parabolic induction of σ corresponding to the partition λ by $i_{Q_\lambda}^{G}(\sigma)$. For details, see [BZ77]. Let $\lambda = (n_1, ..., n_s)$ be a partition of n and let σ_i be R-representation of G_{n_i} for each i. We denote the parabolic induction $i_{Q_\lambda}^{G}(\sigma_1 \otimes \cdots \otimes \sigma_s)$ by the product symbol $\sigma_1 \times \cdots \times \sigma_s$.

2.5.1. Let λ be an ordered partition of n. Let σ be an integral l-adic representation of M_λ and let $\mathcal{L}$ be a G-invariant lattice in σ. Then by [Vig96, I. 9.3], the space $i_{Q_\lambda}^{G}(\mathcal{L})$, consisting of functions in $i_{Q_\lambda}^{G}(\sigma)$ taking values in $\mathcal{L}$, is a G-invariant lattice in $i_{Q_\lambda}^{G}(\sigma)$. Moreover, we have

$$i_{Q_\lambda}^{G}(\mathcal{L} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l) \simeq i_{Q_\lambda}^{G}(\mathcal{L}) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l.$$

Hence parabolic induction commutes with reduction modulo l that is,

$$r_l(i_{Q_\lambda}^{G}(\sigma)) \simeq [i_{Q_\lambda}^{G}(r_l(\sigma))],$$

where the square bracket denotes the semisimplification of $i_{Q_\lambda}^{G}(r_l(\sigma))$.

2.6. **Cuspidal and Supercuspidal representation.** Keeping the notation as in (2.5). Let π be an irreducible R-representation of G. Then π is called a cuspidal representation if for all proper subgroups $Q_\lambda = M_\lambda \ltimes U_\lambda$ of G and for all irreducible R-representations σ of M_λ, we have

$$\mathrm{Hom}_G(\pi, i_Q^G(\sigma)) = 0.$$

The representation π is called supercuspidal if for all proper subgroups $Q_\lambda = M_\lambda \ltimes U_\lambda$ of G and for all irreducible R-representations σ of M_λ, the representation (π, V) is not a subquotient of $i_Q^G(\sigma)$.

Remark 2.1. Let k be an algebraically closed field, and let π be a smooth k-representation of G. If the characteristic of k is 0, then π is cuspidal if and only if π is supercuspidal. But when characteristic of k is $l > 0$, there are cuspidal representations of G which are not supercuspidal. For details, see [Vig96, Section 2.5, Chapter 2].

2.7. **Generic representation.** Let $\psi_K : K \to R^\times$ be a non-trivial additive character of K. Let Θ_K be the character of $N_n(K)$, defined by

$$\Theta_K(x_{ij}) := \psi_K(\sum_{i=1}^{n-1} x_{i,i+1}).$$

Let (π, V) be an irreducible R-representation of $G_n(K)$. Then recall that

$$\dim_R\big(\mathrm{Hom}_{N_n(K)}(\pi, \Theta_K)\big) \leq 1.$$

For the proof, see [BZ76] when $R = \overline{\mathbb{Q}}_l$ and see [Vig96] when $R = \overline{\mathbb{F}}_l$. An irreducible R-representation (π, V) of $G_n(K)$ is called *generic* if

$$\dim_R\big(\mathrm{Hom}_{N_n(K)}(\pi, \Theta_K)\big) = 1.$$

2.7.1. *Whittaker Model.* Let (π, V) be a generic R-representation of $G_n(K)$. By Frobenius reciprocity, the representation π is embedded in the space $\mathrm{Ind}_{N_n(K)}^{G_n(K)}(\Theta_K)$. Let $\mathcal{W}_\pi$ be a non-zero linear functional in the space $\mathrm{Hom}_{N_n(K)}(\pi, \Theta_K)$. Let $\mathbb{W}(\pi, \psi_K) \subset \mathrm{Ind}_{N_n(K)}^{G_n(K)}(\Theta_K)$ be the space consisting of functions W_v, $v \in V$, where

$$W_v(g) := \mathcal{W}_\pi\big(\pi(g)v\big),$$

for all $g \in G_n(K)$. Then the map $v \mapsto W_v$ induces an isomorphism from (π, V) to $\mathbb{W}(\pi, \psi_K)$.

2.7.2. *Segments.* In this subsection, we recall the notion of segments and its associated representations. For details, see [Zel80] for $R = \overline{\mathbb{Q}}_l$ and [KM17], [MS14] for $R = \overline{\mathbb{F}}_l$.

Let $r, t \in \mathbb{Z}$ with $r \leq t$. A segment is a sequence $\Delta = (\nu_K^r\sigma, \nu_K^{r+1}\sigma, ..., \nu_K^t\sigma)$, with σ a cuspidal R-representation of $G_n(K)$. The length of Δ is defined to be $t - r + 1$. In [MS14, Definition 7.5] the authors, using Bushnell-Kutzko's simple types and the Hecke algebras associated with them, defined a certain quotient of the parabolically induced representation

$$\tau = \nu_K^r\sigma \times \nu_K^{r+1}\sigma \times \cdots \times \nu_K^t\sigma,$$

denoted by $\mathcal{L}(\Delta)$. The normalised Jacquet module of $\mathcal{L}(\Delta)$ with respect to the opposite of the parabolic subgroup $P_{(n,\cdots,n)}$ is equal to

$$\nu_K^r\sigma \otimes \nu_K^{r+1}\sigma \otimes \cdots \otimes \nu_K^t\sigma.$$

Moreover, there is a unique generic sub-quotient of τ, denoted by $\mathrm{St}(\sigma, [r,t])$ and it is called the generalised Steinberg representation associated to Δ. We denote by $\mathrm{St}(\sigma, k)$ the representation $\mathrm{St}(\sigma, [0, k-1])$, for $k \geq 1$.

2.7.3. Let σ be a cuspidal R-representation of $G_n(K)$. The set $\{\nu_K^r\sigma : r \in \mathbb{Z}\}$ is called the *cuspidal line* of σ and the cardinality of this set is denoted by $o(\sigma)$. Recall that [MS14, Section 5.2] defines a positive integer $e(\sigma)$ as follows:

$$e(\sigma) = \begin{cases} +\infty & \text{if } R = \overline{\mathbb{Q}}_l; \\ o(\sigma) & \text{if } R = \overline{\mathbb{F}}_l \text{ and } o(\sigma) > 1; \\ l & \text{if } R = \overline{\mathbb{F}}_l \text{ and } o(\sigma) = 1. \end{cases} \tag{2.1}$$

Then for a segment $\Delta = (\nu_K^r\sigma, \dots, \nu_K^t\sigma)$, with $r \leq t$, the representation $\mathcal{L}(\Delta)$ is equal to $\mathrm{St}(\sigma, [r,t])$ if and only if the length of the segment Δ is less than $e(\sigma)$([MS14, Remarque 8.14]). In this case, the segment Δ is called a generic segment. Note that every segment is generic for $R = \overline{\mathbb{Q}}_l$.

2.7.4. Two segments Δ_1 and Δ_2 are said to be linked if $\Delta_1 \not\subseteq \Delta_2$, $\Delta_2 \not\subseteq \Delta_1$ and $\Delta_1 \cup \Delta_2$ is a segment. The following theorem is proved by [MS14, Theorem 9.10] for $R = \overline{\mathbb{F}}_l$ and [Zel80, Theorem 9.7] for $R = \overline{\mathbb{Q}}_l$.

Theorem 2.2. *Let $\pi = \mathcal{L}(\Delta_1) \times \cdots \times \mathcal{L}(\Delta_t)$ be an R-representation of $G_n(K)$, where each Δ_j is a generic segment. Then π is irreducible if and only if the segments Δ_i and Δ_j are not linked for all i, j with $i \neq j$.*

An R-representation of the form $\mathcal{L}(\Delta_1) \times \cdots \times \mathcal{L}(\Delta_t)$, where each Δ_i is generic, is called a representation of *Whittaker type*. In [Zel80, Theorem 9.7] and [Vig98, Proposition V.3], it is shown that

Theorem 2.3. *An R-representation π of $G_n(K)$ is generic if and only if π is an irreducible R-representation of Whittaker type.*

2.7.5. In this subsection, we fix a standard lift of an l-modular generic representation of $G_n(K)$. First, recall that any l-modular cuspidal representation of $G_m(K)$ can be lifted to an l-adic cuspidal representation of $G_m(K)$ (see [Vig96, Chapter 3, 4.25]). Let ρ be an l-modular cuspidal representation of $G_m(K)$ and let $\Delta = (\rho, \overline{\nu}_K\rho, \dots, \overline{\nu}_K^{r-1}\rho)$ be a segment associated with ρ, where $\overline{\nu}_K$ is the mod-l reduction of ν_K. Let σ be a cuspidal lift of ρ. Then the segment $D = (\sigma, \nu_K\sigma, \dots, \nu_K^{r-1}\sigma)$ is called a standard lift of Δ. When $\mathcal{L}(\Delta) = \mathrm{St}(\rho, r)$, then the mod-$l$ representation $\mathcal{L}(\Delta)$ lifts to $\mathcal{L}(D) = \mathrm{St}(\sigma, r)$ ([KM17, Proposition 2.16]).

Let π be a generic l-modular representation of $G_n(K)$. Then π is of the form $\mathcal{L}(\Delta_1) \times \cdots \times \mathcal{L}(\Delta_t)$, where each Δ_i is a generic segment. For each $1 \leq i \leq t$, let D_i be a standard lift of Δ_i. Then the l-adic representation $\tau = \mathcal{L}(D_1) \times \cdots \times \mathcal{L}(D_t)$ is generic, and π lifts to τ ([KM17, Remark 2.31]). The representation τ is called a *standard lift* of π.

2.7.6. Let (π, V) be an integral, l-adic, generic representation of $G_n(K)$. We fix a non-trivial additive character $\psi_K : K \to \Lambda^\times$. By abuse of notation, the composition $K \xrightarrow{\psi_K} \Lambda^\times \hookrightarrow \overline{\mathbb{Q}}_l^\times$ is also denoted by ψ_K. Consider the space $\mathbb{W}^0(\pi, \psi_K)$ consisting of $W \in \mathbb{W}(\pi, \psi_K)$, taking values in $\overline{\mathbb{Z}}_l$. It follows from [Vig04, Theorem 2] that the $\overline{\mathbb{Z}}_l$-module $\mathbb{W}^0(\pi, \psi_K)$ is a $G_n(K)$-invariant lattice in $\mathbb{W}(\pi, \psi_K)$. The lattice $\mathbb{W}^0(\pi, \psi_K)$ is also called the *integral Whittaker model* or *Whittaker lattice*. Let τ be an l-modular generic representation of $G_n(K)$, and let π be an l-adic generic representation of $G_n(K)$. Then, the representation π is called a Whittaker lift of τ if there exists a lattice $\mathcal{L} \subseteq \mathbb{W}^0(\pi, \psi_K)$ such that

$$\mathcal{L} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l \simeq \mathbb{W}(\tau, \overline{\psi}_K),$$

where $\overline{\psi}_K$ is the reduction mod-l of ψ_K. Note that any standard lift of an l-modular generic representation π is a Whittaker lift (see [KM17, Theorem 2.26]). Let (π, V) be an l-adic integral generic representation defined over the field $\mathcal{K}$. Let V' be a $\mathcal{K}$-structure in V. There exists a $\Lambda[G_n(K)]$-lattice in V' (see [EH14, Corollary 4.4.4]). Thus, we get that the set of Λ-valued functions, denoted by $\mathbb{W}_\Lambda(\pi, \psi_K)$, in $\mathbb{W}^0(\pi, \psi_K)$ is a $G_n(K)$-stable lattice.

2.7.7. Now we follow the notations as in (2.1). Choose a generator γ of Γ. Let π be an R-representation of $G_n(E)$. The group $\Gamma = \mathrm{Gal}(E/F)$ acts on $G_n(E)$ component-wise i.e., for $\gamma \in \Gamma, g = (a_{ij})_{i,j=1}^n \in G_n(E)$, we set

$$\gamma.g := (\gamma(a_{ij}))_{i,j=1}^n.$$

Let π^γ be the representation of $G_n(E)$ on V, defined by

$$\pi^\gamma(g) := \pi(\gamma.g), \text{ for all } g \in G_n(E).$$

We say that the representation π of $G_n(E)$ is Γ-equivariant if the representations π and π^γ are isomorphic. We now prove a lemma concerning the Γ invariance of the Whittaker model of a Γ-equivariant representation π of $G_n(E)$. Let ψ_F and ψ_E be the non-trivial additive characters of F and E respectively such that $\psi_E = \psi_F \circ \mathrm{Tr}_{E/F}$ where, $\mathrm{Tr}_{E/F}$ is the trace map of the extension E/F. Let Θ_F and Θ_E be the characters of $N_n(F)$ and $N_n(E)$ respectively, as defined in (2.7). Now consider the action of Γ on the space $\mathrm{Ind}_{N_n(E)}^{G_n(E)}(\Theta_E)$, given by

$$(\gamma.f)(g) := f(\gamma^{-1}g),$$

for all $\gamma \in \Gamma$, $g \in G_n(E)$ and $f \in \mathrm{Ind}_{N_n(E)}^{G_n(E)}(\Theta_E)$.

Lemma 2.4. *Let (π, V) be a generic R-representation of $G_n(E)$ such that (π, V) is Γ-equivariant. Then the Whittaker model $\mathbb{W}(\pi, \psi_E)$ of π is invariant under the action of Γ.*

Proof. Let $\mathcal{W}_\pi$ be a Whittaker functional on the representation π. For $v \in V$, we have

$$\mathcal{W}_\pi(\pi^\gamma(x)v) = \Theta_E(\gamma(x))\mathcal{W}_\pi(v) = (\psi_F \circ \mathrm{Tr}_{E/F})\Big(\sum_{i=1}^{n-1} \gamma(x_{i,i+1})\Big)\mathcal{W}_\pi(v) = \Theta_E(x)\mathcal{W}_\pi(v),$$

for all $x \in N_n(E)$. Thus, $\mathcal{W}_\pi$ is also a Whittaker functional for the representation (π^γ, V). Let $W_v \in \mathbb{W}(\pi, \psi_E)$. Then

$$(\gamma^{-1}.W_v)(g) = \mathcal{W}_\pi(\pi^\gamma(g)v).$$

From the uniqueness of the Whittaker model, we have $\gamma^{-1}.W_v \in \mathbb{W}(\pi, \psi_E)$. Hence the lemma. □

2.8. **Kirillov Model.** Let π be a generic R-representation of $G_n(K)$. Following the notations as in the subsections (2.5) and (2.7), consider the space $\mathbb{K}(\pi, \psi_K)$ of all elements W restricted to $P_n(K)$, where W varies over $\mathbb{W}(\pi, \psi_K)$. Then the space $\mathbb{K}(\pi, \psi_K)$ is $P_n(K)$-invariant. By Frobenius reciprocity, there is a non-zero (unique upto a scalar) linear map $A_\pi : V \longrightarrow \mathrm{Ind}_{N_n(K)}^{P_n(K)}(\Theta_K)$, which is injective and compatible with the action of $P_n(K)$. In fact,

$$A_\pi(V) = \mathbb{K}(\pi, \psi_K) \simeq \mathbb{W}(\pi, \psi_K) \simeq \pi.$$

Moreover, $\mathcal{K}(\psi_K) = \mathrm{ind}_{N_n(K)}^{P_n(K)}(\Theta_K) \subseteq \mathbb{K}(\pi, \psi_K)$ and the equality holds if π is cuspidal. The space of all $\overline{\mathbb{Z}}_l$-valued functions in $\mathbb{K}(\pi, \psi_K)$ (resp. $\mathcal{K}(\psi_K)$) is denoted by $\mathbb{K}^0(\pi, \psi_K)$ (resp. $\mathcal{K}^0(\psi_K)$).

2.8.1. We now recall the Kirillov model for $n = 2$ and some of its properties. For details, see [BH06]. Up to isomorphism, any irreducible representation of $P_2(K)$, which is not a character, is isomorphic to

$$J_\psi := \mathrm{ind}_{N_2(K)}^{P_2(K)}(\psi), \tag{2.2}$$

for some non-trivial smooth additive character ψ of K, viewed as character of $N_2(K)$ via standard isomorphism $N_2(K) \simeq K$. Two different non-trivial characters of $N_2(K)$ induce isomorphic representations of $P_2(K)$. The space (2.2) is identified with the space of locally constant compactly supported functions on $K^\times$, to be denoted by $C_c^\infty(K^\times, \overline{\mathbb{Q}}_l)$. The action of $P_2(K)$ on the space $C_c^\infty(K^\times, \overline{\mathbb{Q}}_l)$ is given by

$$\left[J_\psi \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} f\right](y) = f(ay),$$
$$\left[J_\psi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} f\right](y) = \psi(xy)f(y),$$

for $a, y \in K^\times$ and $x \in K$. For any cuspidal representation of (π, V) of $G_2(K)$, we get a model for the representation (π, V) on the space $C_c^\infty(K^\times, \overline{\mathbb{Q}}_l)$. The action of the group $G_2(K)$ on $C_c^\infty(K^\times, \overline{\mathbb{Q}}_l)$ is denoted by $\mathbb{K}_\psi^\pi$; by definition the restriction of $\mathbb{K}_\psi^\pi$ to $P_2(K)$ is isomorphic to J_ψ. The operator $\mathbb{K}_\psi^\pi(w)$ completely describes the action of $G_2(K)$ on $C_c^\infty(K^\times, \overline{\mathbb{Q}}_l)$, where

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Here, we follow the exposition in [BH06, Section 37.3]. Let χ be a smooth character of $K^\times$ and let k be an integer. Define a function $\xi\{\chi, k\}$ in $C_c^\infty(K^\times, \overline{\mathbb{Q}}_l)$ by setting $\xi\{\chi, k\}(x) = \chi(x)$, for $\nu_K(x) = k$ and zero otherwise. Recall that ν_K is a discrete valuation on $K^\times$. Then we have :

$$\mathbb{K}_\psi^\pi(w)\xi\{\chi, k\} = \epsilon(\chi^{-1}\pi, \psi)\xi\{\chi^{-1}\varpi_\pi, -n(\chi^{-1}\pi, \psi) - k\}, \tag{2.3}$$

where ϖ_π is the central character of π. Here $\epsilon(\pi, \psi)$ is the Godement–Jacquet local ϵ-factor associated with a cuspidal representation π and some additive character ψ of F.

3. Review of Local Constants and Weil-Deligne representations

3.1. Keeping the notation as in Section 2, we briefly discuss about the Weil group and its Weil-Deligne representations. For a reference, see [BH06, Chapter 7] and [Del73, Chapter 4].

We choose a separable algebraic closure $\overline{K}$ of K. Let Ω_K be the absolute Galois group $\mathrm{Gal}(\overline{K}/K)$ and Let $\mathcal{I}_K$ be the inertia subgroup of Ω_K. Let $\mathcal{W}_K$ denote the Weil group of K. Fix a geometric Frobenius element Frob in $\mathcal{W}_K$. Then we have

$$\mathcal{W}_K = \mathcal{I}_K \rtimes \mathrm{Frob}^{\mathbb{Z}}.$$

There is a natural Krull topology on the absolute Galois group Ω_K and the inertia group $\mathcal{I}_K$, as a subgroup of Ω_K, is equipped with the subspace topology. Let the fundamental system of neighbourhoods of the Weil group $\mathcal{W}_K$ be such that each neighbourhood of the identity $\mathcal{W}_K$ contains an open subgroup of $\mathcal{I}_K$. Then under this topology, the Weil group $\mathcal{W}_K$ becomes a locally compact and totally disconnected group. If K_1/K is a finite extension with $K_1 \subseteq \overline{K}$, then the Weil group $\mathcal{W}_{K_1}$ is considered as a subgroup of $\mathcal{W}_K$.

An R-representation ρ of $\mathcal{W}_K$ is called unramified if ρ is trivial on I_K. Let ν be the unramified character of $\mathcal{W}_K$ which satisfies $\nu(\mathrm{Frob}) = q_K^{-1}$. We now define semisimple Weil-Deligne representations of $\mathcal{W}_K$.

3.2. **Semisimple Weil-Deligne representation.** A Weil-Deligne representation of $\mathcal{W}_K$ is a pair (ρ, U), where ρ is a finite dimensional R-representation of $\mathcal{W}_K$ and U is a nilpotent endomorphism of the vector space underlying ρ and intertwining the actions of $\nu\rho$ and ρ. A Weil-Deligne representation (ρ, U) of $\mathcal{W}_K$ is called semisimple if ρ is semisimple as a representation of $\mathcal{W}_K$. Note that any semisimple representation ρ of $\mathcal{W}_K$ is considered as a semisimple Weil-Deligne representation of the form $(\rho, 0)$. For two Weil-Deligne representations (ρ, U) and (ρ', U') of $\mathcal{W}_K$, let

$$\mathrm{Hom}_D((\rho, U), (\rho', U')) = \{f \in \mathrm{Hom}_{\mathcal{W}_K}(\rho, \rho') : f \circ U = U' \circ f\},$$

We say that (ρ, U) and (ρ', U') are isomorphic if there exists a map $f \in \mathrm{Hom}_D((\rho, U), (\rho', U'))$ such that f is bijective. Let $\mathcal{G}_{ss}^n(K)$ be the set of all n-dimensional semisimple Weil-Deligne representations of the Weil group $\mathcal{W}_K$.

3.3. **Local Constants of Weil-Deligne representation.** Keep the notations as in sections (3.1) and (3.2). In this subsection, we consider the local constants for l-adic Weil-Deligne representations of $\mathcal{W}_K$.

3.3.1. *L-factors.* Let (ρ, U) be an l-adic semisimple Weil-Deligne representation of $\mathcal{W}_K$. Then the L-factor corresponding to (ρ, U) is the following rational function in X:

$$L(X, (\rho, U)) = \det((\mathrm{id} - X\rho(\mathrm{Frob}))|_{\ker(U)^{\mathcal{I}_K}})^{-1}.$$

3.3.2. *Local ϵ-factors and γ-factors.* Let $\psi_K : K \to \overline{\mathbb{Q}}_l^\times$ be a non-trivial additive character and choose a self-dual additive Haar measure on K with respect to ψ_K. Let ρ be an l-adic representation of $\mathcal{W}_K$. The epsilon factor $\epsilon(X, \rho, \psi_K)$ of ρ, relative to ψ_K is defined in [Del73]. Let K'/K be a finite extension inside $\overline{K}$. Let $\psi_{K'}$ denotes the character of K', where $\psi_{K'} = \psi_K \circ \mathrm{Tr}_{K'/K}$. Then the epsilon factor satisfies the following properties :

(1) If ρ_1 and ρ_2 are two l-adic representations of $\mathcal{W}_K$, then
$$\epsilon(X, \rho_1 \oplus \rho_2, \psi_K) = \epsilon(X, \rho_1, \psi_K)\epsilon(X, \rho_2, \psi_K).$$
(2) ρ is an l-adic representation of $\mathcal{W}_{K'}$, then
$$\frac{\epsilon\big(X, \mathrm{ind}_{\mathcal{W}_{K'}}^{\mathcal{W}_K}(\rho), \psi_K\big)}{\epsilon(X, \rho, \psi_{K'})} = \left\{\frac{\epsilon\big(X, \mathrm{ind}_{\mathcal{W}_{K'}}^{\mathcal{W}_K}(1_{K'}), \psi_K\big)}{\epsilon(X, 1_{K'}, \psi_{K'})}\right\}^{\dim(\rho)}, \tag{3.1}$$
where $1_{K'}$ denotes the trivial character of $\mathcal{W}_K$.
(3) If ρ is an l-adic representation of $\mathcal{W}_K$, then
$$\epsilon\big(X, \rho, \psi_K\big)\epsilon\big(q_K^{-1}X^{-1}, \rho^\vee, \psi_K\big) = \det\big(\rho(-1)\big), \tag{3.2}$$
where $\rho^\vee$ denotes the dual of the representation ρ.
(4) For an l-adic representation ρ of $\mathcal{W}_K$, there exists an integer $n(\rho, \psi_K)$ for which
$$\epsilon(X, \rho, \psi_K) = (q_K^{\frac{1}{2}}X)^{n(\rho,\psi_K)}\epsilon(\rho, \psi_K).$$

Now for an l-adic semisimple Weil-Deligne representation (ρ, U), the ϵ-factor is defined as
$$\epsilon\big(X, (\rho, U), \psi_K\big) = \epsilon(X, \rho, \psi_K)\frac{L(q_K^{-1}X^{-1}, \rho^\vee)}{L(X, \rho)}\frac{L(X, (\rho, U))}{L(q_K^{-1}X^{-1}, (\rho, U)^\vee)},$$
where $(\rho, U)^\vee = (\rho^\vee, -U^\vee)$. Set
$$\gamma(X, (\rho, U), \psi_K) = \epsilon(X, (\rho, U), \psi_K)\frac{L(X, (\rho, U))}{L(q_K^{-1}X^{-1}, (\rho, U)^\vee)}.$$
The element $\gamma(X, (\rho, U), \psi_K)$ is called the γ-factor of the Weil-Deligne representation (ρ, U).

Now we state a result [KM21, Proposition 5.11] which concerns the fact that the γ-factors are compatible with reduction modulo l. For $P \in \overline{\mathbb{Z}}_l[X]$, we denote by $r_l(P) \in \overline{\mathbb{F}}_l[X]$ the polynomial obtained by reduction mod-l to the coefficients of P. For $Q \in \overline{\mathbb{Z}}_l[X]$, such that $r_l(Q) \neq 0$, we set $r_l(P/Q) = r_l(P)/r_l(Q)$.

Proposition 3.1. *Let ρ be an integral l-adic semisimple representation of $\mathcal{W}_K$. Then*
$$r_l\big(\gamma(X, \rho, \psi_K)\big) = \gamma\big(X, r_l(\rho), \overline{\psi}_K\big),$$
where $\overline{\psi}_K$ is the reduction mod-l of ψ_K.

We end this subsection with a lemma which will be needed later in the proof of Theorem (1.1).

Lemma 3.2. *Let E/F be a cyclic Galois extension of prime degree l and assume $l \neq 2$. Let ρ be an l-adic representation of $\mathcal{W}_E$ of even dimension. Then*
$$\epsilon(X, \rho, \psi_E) = \epsilon\big(X, \mathrm{ind}_{\mathcal{W}_E}^{\mathcal{W}_F}(\rho), \psi_F\big).$$

Proof. Let $\mathcal{C}_{E/F}(\psi_F) = \dfrac{\epsilon\big(X, \mathrm{ind}_{\mathcal{W}_E}^{\mathcal{W}_F}(1_E), \psi_F\big)}{\epsilon(X, 1_E, \psi_E)}$, where 1_E denotes the trivial character of $\mathcal{W}_E$. Then $\mathcal{C}_{E/F}(\psi_F)$ is independent of X (see [BH06, Corollary 30.4, Chapter 7]). Using the equality (3.1), we get
$$\frac{\epsilon\big(X, \mathrm{ind}_{\mathcal{W}_E}^{\mathcal{W}_F}(\rho), \psi_F\big)}{\epsilon(X, \rho, \psi_E)} = (\mathcal{C}_{E/F}(\psi_F))^{\dim\rho}.$$

In view of the functional equation (3.2), we have
$$\mathcal{C}_{E/F}(\psi_F)^2 = \xi_{E/F}(-1),$$

where $\xi_{E/F} = \det\big(\mathrm{ind}_{\mathcal{W}_E}^{\mathcal{W}_F}(1_E)\big)$, a quadratic character of $\mathcal{W}_F$. Note that $\xi_{E/F}$ is a character of the quotient group $\mathcal{W}_F/\mathcal{W}_E$–which is a cyclic group of order l, and this implies that $\xi_{E/F}^l = 1$. Since l is odd, we get that $\xi_{E/F} = 1_F$, the trivial character of $\mathcal{W}_F$. Hence the lemma. □

3.4. **Local constants of p-adic representations.** Following the notations as in Section (2.7), we now define the L-factors and γ-factors for irreducible R-representations of $G_n(K)$. For details, see [KM17]. Let π be an R-representation of Whittaker type of $G_n(K)$ and let π' be an R-representation of Whittaker type of $G_{n-1}(K)$. Let $W \in \mathbb{W}(\pi, \psi_K)$ and $W' \in \mathbb{W}(\pi', \psi_K^{-1})$. The function $W\begin{pmatrix} g & 0 \\ 0 & 1\end{pmatrix} W'(g)$ is compactly supported on Y_K^r [KM17, Proposition 3.3]. Then the following integral

$$c_r^K(W, W') = \int_{Y_K^r} W\begin{pmatrix} g & 0 \\ 0 & 1\end{pmatrix} W'(g)\, dg,$$

is well defined for all $r \in \mathbb{Z}$, and vanishes for $r << 0$. In this paper, we deal with base change where two different p-adic fields are involved. So to avoid confusion, we use the notation $c_r^K(W, W')$ instead of the notation $c_r(W, W')$ used in [KM17, Proposition 3.3] for these integrals on Y_K^r. Now consider the functions $\widetilde{W}$ and $\widetilde{W'}$, defined as

$$\widetilde{W}(x) = W\big(w_n(x^t)^{-1}\big)$$

and

$$\widetilde{W'}(g) = W'\big(w_{n-1}(g^t)^{-1}\big),$$

for all $x \in G_n(K)$, $g \in G_{n-1}(K)$. Then making a change of variables, we have the following relation:

$$c_r^K(\widetilde{W}, \widetilde{W'}) = c_{-r}^K\big(\pi(w_n)W, \pi'(w_{n-1})W'\big). \tag{3.3}$$

Let $I(X, W, W')$ be the following power series:

$$I(X, W, W') = \sum_{r\in\mathbb{Z}} c_k^K(W, W') q_K^{r/2} X^r \in R((X)). \tag{3.4}$$

Note that $I(X, W, W')$ is a rational function in X (see [KM17, Theorem 3.5]).

3.4.1. *L -factors.* Let π and π' be two R-representations of Whittaker type of $G_n(K)$ and $G_{n-1}(K)$ respectively. Then the R-submodule spanned by $I(X, W, W')$ as W varies in $\mathbb{W}(\pi, \psi_K)$ and W' varies in $\mathbb{W}(\pi', \psi_K^{-1})$, is a fractional ideal of $R[X, X^{-1}]$ and it has a unique generator which is an Euler factor denoted by $L(X, \pi, \pi')$. The generator $L(X, \pi, \pi')$ called the L-factor associated to π, π' and ψ_K.

Remark 3.3. *If π and π' are l-adic representations of Whittaker type of $G_n(K)$ and $G_{n-1}(K)$ respectively, then $1/L(X, \pi, \pi') \in \overline{\mathbb{Z}}_l[X]$.*

We conclude this section with a theorem [KM17, Theorem 4.3,] which describes L-factors of cuspidal representations.

Theorem 3.4. Let π_1 and π_2 be two cuspidal R-representations of $G_n(K)$ and $G_m(K)$ respectively. Then $L(X, \pi_1, \pi_2)$ is equal to 1, except in the following case : π_1 is banal in the sense of [MS14] and $\pi_2 \simeq \chi\pi_1^\vee$ for some unramified character χ of $K^\times$.

In the proof of Theorem(1.1), we only consider the case when $m = n - 1$, and by the above theorem the L-factor $L(X, \pi_1, \pi_2)$ associated with the cuspidal R-representations π_1 and π_2 is equal to 1.

3.4.2. *Functional Equations and Local γ-factors.* Let π and π' be two R-representations of Whittaker type of $G_n(K)$ and $G_{n-1}(K)$ respectively. Then there is an invertible element $\epsilon(X, \pi, \pi', \psi_K)$ in $R[X, X^{-1}]$ such that for all $W \in \mathbb{W}(\pi, \psi_K)$, $W' \in \mathbb{W}(\pi', \psi_K^{-1})$, we have the following functional equation :

$$\frac{I(q_K^{-1}X^{-1}, \widetilde{W}, \widetilde{W'})}{L(q_K^{-1}X^{-1}, \widetilde{\pi}, \widetilde{\pi'})} = \omega_{\pi'}(-1)^{n-2}\epsilon(X, \pi, \pi', \psi_K)\frac{I(X, W, W')}{L(X, \pi, \pi')},$$

where $\widetilde{W}$ is defined as in (3.4) and $\omega_{\pi'}$ denotes the central character of the representation π'. We call $\epsilon(X, \pi, \pi', \psi_K)$ the local ϵ-factor associated to π, π' and ψ_K. Moreover, if π and π' be l-adic representations of Whittaker type of $G_n(K)$ and $G_{n-1}(K)$ respectively, then the factor $\epsilon(X, \pi, \pi', \psi_K)$ is of the form cX^k for a unit $c \in \overline{\mathbb{Z}}_l$. In particular, there exists an integer $n(\pi, \pi', \psi_K)$ such that

$$\epsilon(X, \pi, \pi', \psi_K) = (q_K^{\frac{1}{2}} X)^{n(\pi,\pi',\psi_K)} \epsilon(\pi, \pi', \psi_K). \tag{3.5}$$

Now the local γ-factor associated with π, π' and ψ is defined as:

$$\gamma(X, \pi, \pi', \psi_K) = \epsilon(X, \pi, \pi', \psi_K) \frac{L(q_K^{-1} X^{-1}, \widetilde{\pi}, \widetilde{\pi'})}{L(X, \pi, \pi')}.$$

3.4.3. *Compatibility with reduction modulo l.* Let τ and τ' be two l-modular representations of Whittaker type of $G_n(K)$ and $G_{n-1}(K)$ respectively. Let π and π' be the respective Whittaker lifts of τ and τ'. Then

$$L(X, \tau, \tau') | r_l\big(L(X, \pi, \pi')\big)$$

and

$$r_l\big((\gamma(X, \pi, \pi', \psi_K)\big) = \gamma(X, \tau, \tau', \overline{\psi}_K).$$

For details, see [KM17, Section 3.3].

3.4.4. *Generic part of mod-l reduction.* Let π be an integral l-adic generic representation of $G_n(K)$. The mod-l-reduction of π, denoted by $r_l(\pi)$, has a unique generic component and it is denoted by $J_l(\pi)$ (see [Vig01, Section 1.8.4]). Let σ be an integral l-adic generic representation of $G_{n-1}(K)$. Now, the functional equation for the pair $(J_l(\pi), J_l(\sigma))$ gives

$$I(q_K^{-1} X^{-1}, \widetilde{W}, \widetilde{W'}) = \varpi_\sigma(-1)^{n-2} \gamma(X, J_l(\pi), J_l(\sigma), \overline{\psi}_K) I(X, W, W'),$$

for all $W \in \mathbb{W}(J_l(\pi), \overline{\psi}_K)$ and $W' \in \mathbb{W}(J_l(\sigma), \overline{\psi}_K^{-1})$. Let us consider the following commutative diagram:

$$\begin{array}{ccc}
\mathbb{W}^0(\pi, \psi_K) & \xrightarrow{\Lambda_\pi} & \mathrm{Ind}_{N_n(K)}^{G_n(K)} \overline{\Theta}_K \\
\downarrow{\scriptstyle \mathrm{Res}_{P_n(K)}} & & \downarrow{\scriptstyle \mathrm{Res}_{P_n(K)}} \\
\mathbb{K}^0(\pi, \psi_K) & \xrightarrow[\lambda_\pi]{} & \mathrm{Ind}_{N_n(K)}^{P_n(K)} \overline{\Theta}_K
\end{array}$$

Note that the restriction to $P_n(K)$ map on $W^0(\pi, \psi_K)$ is an isomorphism. Here Λ_π and λ_π are the pointwise mod-l reduction maps. Since $\mathcal{K}^0(\psi_K)$ is contained in $\mathbb{K}^0(\pi, \psi_K)$ and λ_π maps $\mathcal{K}^0(\psi_K)$ onto $\mathcal{K}(\overline{\psi}_K)$, the $P_n(K)$-equivariant map λ_π is non-zero. It then follows from the commutativity of the above diagram that Λ_π is non-zero. Since $J_l(\pi)$ is the unique generic subquotient of $r_l(\pi)$, the image of Λ_π contains the Whittaker space $\mathbb{W}(J_l(\pi), \overline{\psi}_K)$. Similarly, the image of Λ_σ contains $\mathbb{W}(J(\sigma), \overline{\psi}_K)$. Let U (resp. U') be an element of $\mathbb{W}^0(\pi, \psi_K)$ (resp. $\mathbb{W}^0(\sigma, \psi_K)$) such that $\Lambda_\pi(U) = W$ (resp. $\Lambda_\tau(U') = W'$). From the functional equation for the pair (π, σ), we get the following relation

$$I(q_K^{-1} X^{-1}, \widetilde{U}, \widetilde{U'}) = \varpi_\sigma(-1)^{n-2} \gamma(X, \pi, \sigma, \psi_K) I(X, U, U').$$

After reducing the above equality modulo-l, we have

$$I(q_K^{-1} X^{-1}, \widetilde{W}, \widetilde{W'}) = \varpi_\sigma(-1)^{n-2} r_l(\gamma(X, \pi, \sigma, \psi_K)) I(X, W, W'),$$

Thus, we get that

$$r_l(\gamma(X, \pi, \sigma, \psi_K)) = \gamma(X, J_l(\pi), J_l(\sigma), \overline{\psi}_K). \tag{3.6}$$

4. Local Langlands Correspondence

4.1. **The l-adic local Langlands correspondence.** In this subsection, we recall the l-adic local Langlands correspondence. Keep the notation as in section (2). Let ψ_K be a non-trivial additive character of K. Recall that local Langlands correspondence over $\overline{\mathbb{Q}}_l$ is the bijection

$$\Pi_K : \mathrm{Irr}\big(GL_n(K), \overline{\mathbb{Q}}_l\big) \longrightarrow \mathcal{G}^n_{ss}(K)$$

such that

$$\gamma(X, \sigma \times \sigma', \psi_K) = \gamma(X, \Pi_K(\sigma) \otimes \Pi_K(\sigma'), \psi_K)$$

and

$$L(X, \sigma \times \sigma') = L(X, \Pi_K(\sigma) \otimes \Pi_K(\sigma')),$$

for all $\sigma \in \mathrm{Irr}(G_n(K), \overline{\mathbb{Q}}_l)$, $\sigma' \in \mathrm{Irr}(G_m(K), \overline{\mathbb{Q}}_l)$. Moreover, the set of all cuspidal l-adic representations of $GL_n(K)$ is mapped onto the set n-dimensional irreducible l-adic representations of the Weyl group $\mathcal{W}_K$ via the bijection Π_K (see [HT01], [Hen00] or [Sch13]). Note that the classical local Langlands correspondence is a bijection between $\mathrm{Irr}(\mathrm{GL}_n(K), \mathbb{C})$ and the isomorphism classes of n-dimensional, complex semisimple Weil–Deligne representations. To get a correspondence over $\overline{\mathbb{Q}}_l$, one twists the original correspondence by the character $\nu^{(1-n)/2}$. For details see [Clo90, Conjecture 4.4, Section 4.2], [Hen01, Section 7] and for $n = 2$ see [BH06, Theorem 35.1].

4.2. **Local base Change for the extension E/F.** Now we recall local base change for a cyclic extension of a p-adic field. The base change operation on irreducible smooth representations of $\mathrm{GL}_n(F)$ over complex vector spaces is characterised by certain character identities (see [AC89, Chapter 3]). Let us recall the relation between l-adic local Langlands correspondence and local base change for GL_n. Let π_F be an l-adic irreducible smooth representation of $\mathrm{GL}_n(F)$. Let (ρ_F, U) be a semisimple Weil-Deligne representation such that $\Pi_F(\pi_F) = \rho_F$, where Π_F is the l-adic local Langlands correspondence as described in the previous section. Let π_E be the l-adic irreducible representation of $GL_n(E)$ such that

$$\mathrm{res}_{\mathcal{W}_E}\big(\Pi_F(\pi_F)\big) \simeq \Pi_E(\pi_E).$$

The representation π_E is the base change of π_F. Note that in this case $\pi_E \simeq \pi_E^\gamma$, for all $\gamma \in \Gamma$.

4.2.1. *Base change for $L(\Delta)$.* Let k be a positive integer. Let $\Delta = \{\tau_F, \tau_F\nu_F, ..., \tau_F\nu_F^{k-1}\}$ be a segment, where τ_F is an l-adic cuspidal representation of $G_m(F)$. Consider the generic representation $\mathcal{L}(\Delta)$ of $G_{km}(F)$. Then

$$\Pi_F\big(\mathcal{L}(\Delta)\big) = \Pi_F(\tau_F) \otimes \mathrm{Sp}_F(k),$$

where $\mathrm{Sp}_F(k)$ is the semisimple Weil-Deligne representation of $\mathcal{W}_F$, defined as in [BH06, Section 31, Example 31.1]. If l does not divide m, then there exists a cuspidal representation τ_E of $G_m(E)$ such that τ_E is a base change of τ_F that is,

$$\mathrm{res}_{\mathcal{W}_E}\big(\Pi_F(\tau_F)\big) = \Pi_E(\tau_E).$$

Then we have

$$\mathrm{res}_{\mathcal{W}_E}\big(\Pi_F(\mathcal{L}(\Delta))\big) = \Pi_E(\tau_E) \otimes \mathrm{Sp}_E(k) = \Pi_E\big(\mathcal{L}(D)\big),$$

where D is the segment $\{\tau_E, \tau_E\nu_E, ..., \tau_E\nu_E^{k-1}\}$. Hence, it follows that the generic representation $\mathcal{L}(D)$ of $G_{km}(E)$ is a base change of $\mathcal{L}(\Delta)$. Next, we prove a lemma about base change lifting and integrality.

Lemma 4.1. *Let π_F be an irreducible l-adic representation of $G_n(F)$, and let π_E be the base change lifting of π_F to $G_n(E)$. Then π_F is integral if and only if the base change lifting π_E is integral.*

Proof. Let π be an irreducible l-adic smooth representation of $G_n(K)$. Let $\mathrm{scs}(\pi)$ be the supercuspidal support of π (see [Vig98, III.3.] for the definition). The representation π is integral if and only if $\mathrm{scs}(\pi)$ is integral (see [Vig01, Section 1.4] and for general reductive groups see [DHKM24, Corollary 1.6] for a reference). The representation $\mathrm{scs}(\pi)$ is integral if and only if the central character of $\mathrm{scs}(\pi)$ is integral.

Assume that π_F is integral. Let (ρ_F, U) be the l-adic, semisimple Weil-Deligne representation of $\mathcal{W}_F$ associated with π_F under the local Langlands correspondence (LLC) Π_F. Under LLC, the representation $\mathrm{scs}(\pi_F)$ corresponds to the $\mathcal{W}_F$-representation ρ_F. Since the determinant character of each irreducible component of ρ_F is integral, we get that ρ_F is integral. This implies that the restriction $\mathrm{res}_{\mathcal{W}_E}(\rho_F)$ is integral. Under LLC, the supercuspidal support of π_E corresponds to the restriction $\mathrm{res}_{\mathcal{W}_E}(\rho_F)$. Thus, the supercuspidal support of π_E is integral and we get that π_E is integral.

Conversely assume that π_E is integral. Let (ρ_E, U_E) be the l-adic, semisimple Weil-Deligne representation of $\mathcal{W}_E$ associated with π_E under LLC. Since the supercuspidal support of π_E is integral, the $\mathcal{W}_E$-representation ρ_E (which is $\mathrm{res}_{\mathcal{W}_E}(\rho_F)$) is integral. Let $\mathcal{L}_E$ be a $\mathcal{W}_E$-stable lattice in ρ_E. Then,

$$\sum_{x \in \mathcal{W}_F/\mathcal{W}_E} \rho_F(x)\mathcal{L}_E.$$

is a $\mathcal{W}_F$-stable lattice in ρ_F. Thus, the representation ρ_F is also integral, which implies that the supercuspidal support $\mathrm{scs}(\pi_F)$ is integral. Hence, π_F is integral. □

5. Tate Cohomology

In this section, we recall Tate cohomology and some useful results on Γ-equivariant l-sheaves of Λ-modules on an l-space X equipped with an action of Γ. We refer to [TV16, Section 3] for details.

5.1. Fix a generator γ of Γ. Let M be a $\Lambda[\Gamma]$-module, and let T_γ be the automorphism of M defined by

$$T_\gamma(m) = \gamma.m, \text{ for } \gamma \in \Gamma, m \in M.$$

Let $N_\gamma = \mathrm{id} + T_\gamma + T_{\gamma^2} + ... + T_{\gamma^{l-1}}$ be the norm operator. The Tate cohomology groups $\widehat{H}^0(M)$ and $\widehat{H}^1(M)$ are defined as :

$$\widehat{H}^0(M) = \frac{\ker(\mathrm{id} - T_\gamma)}{\mathrm{Im}(N_\gamma)},\ \widehat{H}^1(M) = \frac{\ker(N_\gamma)}{\mathrm{Im}(\mathrm{id} - T_\gamma)}.$$

5.2. **Tate Cohomology of sheaves on l-spaces.** Let X be an l-space equipped with an action of a finite group $\langle\gamma\rangle$ of order l. Let $\mathcal{F}$ be an l-sheaf of Λ modules on X. Write $\Gamma_c(X;\mathcal{F})$ for the space of compactly supported sections of $\mathcal{F}$. In particular, if $\mathcal{F}$ is the constant sheaf with stalk Λ, then $\Gamma_c(X;\mathcal{F}) = C_c^\infty(X,\Lambda)$. The assignment $\mathcal{F} \mapsto \Gamma_c(X;\mathcal{F})$ is a covariant exact functor. If $\mathcal{F}$ is γ-equivariant, then γ can be regarded as a map of sheaves $\mathcal{F}|_{X^\gamma} \to \mathcal{F}|_{X^\gamma}$ and the Tate cohomology is defined as

$$\widehat{H}^0(\mathcal{F}|_{X^\gamma}) := \ker(1-\gamma)/\mathrm{Im}(N),$$
$$\widehat{H}^1(\mathcal{F}|_{X^\gamma}) := \ker(N)/\mathrm{Im}(1-\gamma).$$

A compactly supported section of $\mathcal{F}$ can be restricted to a compactly supported section of $\mathcal{F}|_{X^\gamma}$. The following result is often useful in calculating Tate cohomology groups.

Proposition 5.1 (Treumann-Venkatesh, [TV16])**.** *The restriction map induces an isomorphism of the following spaces:*

$$\widehat{H}^i(\Gamma_c(X;\mathcal{F})) \longrightarrow \Gamma_c(X^\gamma;\widehat{H}^i(\mathcal{F})) \textit{ for } i = 0,1.$$

5.2.1. The above proposition is very useful to compute the Tate cohomology groups of compactly induced representations. For instance, the following argument is used at many places in the paper. Let E/F be a Galois extension of degree l with Galois group Γ. Let $\psi_F : F \to \overline{\mathbb{Z}}_l^\times$ be an additive character and let ψ_E be the character $\psi_F \circ \mathrm{Tr}_{E/F}$, where $\mathrm{Tr}_{E/F}$ is the trace function. Let Θ_E and Θ_F be the non-degenerate characters of $N_n(E)$ and $N_n(F)$ associated with ψ_F and ψ_E respectively (see Subsection 2.7). We will use the notations $\overline{\psi}_E$, $\overline{\psi}_F$, $\overline{\Theta}_E$, $\overline{\Theta}_F$ for the respective mod-l reductions. Recall the notation $\mathcal{K}(\psi_K)$ for the compact induction $\mathrm{ind}_{N_n(K)}^{P_n(K)} \psi_K$. Note that the Galois group Γ acts on the representation $\mathcal{K}(\overline{\psi}_E)$, by setting

$$(\gamma f)(x) = f(\gamma^{-1}(x)),\ \gamma \in \Gamma,\ x \in P_n(E),\ f \in \mathcal{K}(\overline{\psi}_E).$$

The restriction to $P_n(F)$ map:

$$\mathcal{K}(\overline{\psi}_E) \to \mathcal{K}(\overline{\psi}_F^l),\ f \mapsto \mathrm{res}_{P_n(F)} f$$

factorizes through

$$\widehat{H}^0(\mathcal{K}(\overline{\psi}_E)) \to \mathcal{K}(\overline{\psi}_F^l). \tag{5.1}$$

We set $Y_K = P_n(K)/N_n(K)$. Note that the pointed set $H^1(\Gamma, N_n(E))$ is trivial and hence $Y_E^\Gamma = Y_F$. Applying Proposition 5.1 for the case where $X = Y_E$ and $\mathcal{F}$ equals the sheaf associated with the induced representation $\mathrm{ind}_{N_n(E)}^{P_n(E)} \overline{\Theta}_E$, we get that the map (5.1) is an isomorphism. Proposition 5.1 also shows that $\widehat{H}^1(\mathcal{K}^0(\psi_E))$ is trivial. Here, $\mathcal{K}^0(\psi_E)$ is the space of $\overline{\mathbb{Z}}_l$-valued functions in the l-adic representation $\mathrm{ind}_{N_n(E)}^{P_n(E)} \psi_E$.

5.3. **Comparison of integrals of smooth functions.** The group $\Gamma = \langle\gamma\rangle$ acts on the space $X_E = G_{n-1}(E)/N_{n-1}(E)$ and hence its action on the space $C_c^\infty(X_E, \overline{\mathbb{F}}_l)$ is given by the following equality:

$$(\gamma.\phi)(x) := \phi(\gamma^{-1}x), \text{ for all } x \in X_E, \text{ and } \phi \in C_c^\infty(X_E, \overline{\mathbb{F}}_l).$$

Let $C_c^\infty(X_E, \overline{\mathbb{F}}_l)^\Gamma$ be the space of all Γ-invariant functions in $C_c^\infty(X_E, \overline{\mathbb{F}}_l)$. We end this section with a proposition comparing the integrals on the spaces X_E and X_F.

Proposition 5.2. *Let $d\mu_E$ and $d\mu_F$ be Haar measures on X_E and X_F respectively. Then, there exists a non-zero scalar $c \in \overline{\mathbb{F}}_l$ such that*

$$\int_{X_E} \phi\, d\mu_E = c \int_{X_F} \phi\, d\mu_F,$$

for all $\phi \in C_c^\infty(X_E, \overline{\mathbb{F}}_l)^\Gamma$.

Proof. Since $N_{n-1}(E)$ is stable under the action of Γ on $G_{n-1}(E)$, we have the following long exact sequence of non-abelian cohomology [Ser, Chapter VII, Appendix]:

$$0 \longrightarrow N_{n-1}(E)^\Gamma \longrightarrow G_{n-1}(E)^\Gamma \longrightarrow X_E^\Gamma \longrightarrow H^1\big(\Gamma; N_{n-1}(E)\big) \longrightarrow H^1\big(\Gamma; G_{n-1}(E)\big).$$

Since $H^1\big(\Gamma; N_{n-1}(E)\big) = 0$, we get from the above exact sequence that

$$X_E^\Gamma \simeq X_F.$$

Since X_F is closed in X_E, we have the following exact sequence of Γ-modules :

$$0 \longrightarrow C_c^\infty\big(X_E \setminus X_F, \overline{\mathbb{F}}_l\big) \longrightarrow C_c^\infty(X_E, \overline{\mathbb{F}}_l) \longrightarrow C_c^\infty(X_F, \overline{\mathbb{F}}_l) \longrightarrow 0. \tag{5.2}$$

Now, the action of Γ on $X_E \setminus X_F$ is free, and it follows from Proposition 5.1 that

$$H^1\big(\Gamma, C_c^\infty(X_E \setminus X_F, \overline{\mathbb{F}}_l)\big) = 0. \tag{5.3}$$

Using (5.2) and (5.3), we get the following exact sequence :

$$0 \longrightarrow C_c^\infty\big(X_E \setminus X_F, \overline{\mathbb{F}}_l\big)^\Gamma \longrightarrow C_c^\infty(X_E, \overline{\mathbb{F}}_l)^\Gamma \longrightarrow C_c^\infty(X_F, \overline{\mathbb{F}}_l) \longrightarrow 0.$$

Again the free action of Γ on $X_E \setminus X_F$ gives a fundamental domain U such that $X_E \setminus X_F = \bigsqcup_{i=0}^{l-1} \gamma^i U$, and we have

$$\int_{X_E \setminus X_F} \phi\, d\mu_E = l \sum_{i=0}^{l-1} \int_U \phi\, d\mu_E = 0,$$

for all $\phi \in C_c^\infty\big(X_E \setminus X_F, \overline{\mathbb{F}}_l\big)^\Gamma$. Therefore the linear functional $d\mu_E$ induces a $G_{n-1}(F)$-invariant linear functional on $C_c^\infty(X_F, \overline{\mathbb{F}}_l)$ and we have

$$\int_{X_E} \phi\, d\mu_E = c \int_{X_F} \phi\, d\mu_F,$$

for some scalar c. Now we will show that $c \neq 0$. By [Vig96, Chapter 1, Section 2.8], we have a surjective map $\Psi : C_c^\infty(G_n(E), \overline{\mathbb{F}}_l) \longrightarrow C_c^\infty(X_E, \overline{\mathbb{F}}_l)$, defined by

$$\Psi(f)(g) := \int_{N_n(E)} f(ng)\, dn,$$

for all $f \in C_c^\infty(G_n(E), \overline{\mathbb{F}}_l)$, where dn is a Haar measure on $N_n(E)$. Then there exists a Γ-invariant compact open subgroup $I \subseteq G_n(E)$ such that $\Psi(1_I) \neq 0$, where 1_I denotes the characteristic function on I. So the Haar measure $d\mu_E$ is non-zero on the space $C_c^\infty(X_E, \overline{\mathbb{F}}_l)^\Gamma$ and this implies that $c \neq 0$. Hence the proposition follows. □

Remark 5.3. Keep the notations and hypothesis in Proposition 5.2. From now, the Haar measures $d\mu_E$ and $d\mu_F$ on X_E and X_F respectively, are chosen so as to make $c = 1$. Then we have

$$\int_{X_E} \phi\, d\mu_E = \int_{X_F} \phi\, d\mu_F.$$

Moreover, if e is the ramification index of the extension E over F, then for all $r \notin \{te : t \in \mathbb{Z}\}$, we have

$$\int_{(X_E^r)^\Gamma} \phi \, d\mu_F = 0$$

and for all $r \in \{te : t \in \mathbb{Z}\}$, we have

$$\int_{(X_E^r)^\Gamma} \phi \, d\mu_F = \int_{X_F^{\frac{r}{e}}} \phi \, d\mu_F.$$

5.4. **Finiteness of Tate cohomology.** In this part, we prove some results on finiteness of the Tate cohomology of finite length representations of GL_n. First, we introduce some notations. Let $U_n(K)$ be the subgroups of $G_n(K)$, given by

$$U_n(K) = \left\{ \begin{pmatrix} 1 & C \\ 0 & 1 \end{pmatrix} : C \in K^{n-1} \right\}$$

respectively. Note that $U_n(K)$ is contained in the mirabolic subgroup $P_n(K)$. We use the short notation $Z_{K,n}$ to denote the coset space $P_n(K)/P_{n-1}(K)U_n(K)$. Let $\mathrm{Rep}_R(G)$ denote the category of smooth R-representations of a locally profinite group G, where R denotes either $\mathcal{K}$ or $\overline{\mathbb{Q}}_l$ or $\overline{\mathbb{F}}_l$. Then we have four fundamental functors:

$$\Psi^- : \mathrm{Rep}_R(P_n) \to \mathrm{Rep}_R(G_{n-1}), \Psi^+ : \mathrm{Rep}_R(G_{n-1}) \to \mathrm{Rep}_R(P_n)$$
$$\Phi^- : \mathrm{Rep}_R(P_n) \to \mathrm{Rep}_R(P_{n-1}), \Phi^+ : \mathrm{Rep}_R(P_{n-1}) \to \mathrm{Rep}_R(P_n).$$

For the definitions of the functors $\Phi^\pm$ and $\Psi^\pm$, see [BZ77, Section 3] for $R = \overline{\mathbb{Q}}_l$, and [Vig96, Chapter III, Section 1] for $R = \overline{\mathbb{F}}_l$.

5.4.1. Let τ be a smooth R-representation of $P_n(K)$. The m-th derivative of τ, denoted by $\tau^{(m)}$, is defined as the representation $\Psi^-(\Phi^-)^{m-1}(\tau)$ of $G_{n-m}(K)$. There is a functorial filtration on τ, given by

$$0 \subseteq \tau_n \subseteq \tau_{n-1} \subseteq \cdots \subseteq \tau_2 \subseteq \tau_1 = \tau,$$

where $\tau_m = (\Phi^+)^{m-1}(\Phi^-)^{m-1}(\tau)$ and $\tau_m/\tau_{m+1} = (\Phi^+)^{m-1}(\Psi^+)(\tau^{(m)})$. We have the following easy lemma.

Lemma 5.4. *Let ρ be a finite length representation of $G_t(K)$, where $1 \leq t < n$. Then*

$$(\Phi^+)^{n-t-1}(\Psi^+)(\rho)$$

is also of finite length as a representation of $P_n(K)$.

Proof. This is an immediate consequence of [Vig96, Chapter III, Subsection 1.5] and the exactness of the functor $(\Phi^+)^{n-t-1}(\Psi^+)$. ☐

5.4.2. Let E be a finite Galois extension of a p-adic field F with $[E : F] = l$, where l and p are distinct primes. Fix a non-trivial additive character $\psi_F : F \to \Lambda^\times$. By abuse of notation, the composition

$$F \xrightarrow{\psi_F} \Lambda^\times \hookrightarrow \overline{\mathbb{Q}}_l^\times$$

is also denoted by ψ_F. Let ψ_E be the character of E, defined by the composition $\psi_F \circ \mathrm{Tr}_{E/F}$, where $\mathrm{Tr}_{E/F}$ denotes the trace map of the extension E/F. The mod-l reductions of ψ_F and ψ_E are denoted by $\overline{\psi}_F$ and $\overline{\psi}_E$, respectively. Then, we have the following finiteness result of the Tate cohomology groups.

Proposition 5.5. *Let Π be a finite length l-modular representation of $G_n(E)$ with an isomorphism $T : \Pi \to \Pi^\gamma$ and $T^l = \mathrm{id}$. Then, the Tate cohomology $\widehat{H}^i(\Pi)$, with respect to the operator T, is a finite length representation of $G_n(F)$.*

Proof. We prove the proposition using induction on the integer n. The case $n = 1$ is clear. So, we assume that the proposition is true for all finite length l-modular representations of $G_t(E) \rtimes \Gamma$ and for all $t < n$. Now, we consider Π as a representation of the mirabolic subgroup $P_n(E)$. Since $\overline{\psi}_E(\gamma(x)) = \overline{\psi}_E(x)$, for $x \in E$, we get the isomorphism

$$\Phi^-(\Pi^\gamma) \simeq \Phi^-(\Pi)^\gamma, \tag{5.4}$$

as representation of $P_{n-1}(E)$. Similarly, for any smooth l-modular representation τ of $P_{n-1}(E)$, we have $P_n(E)$-equivariant isomorphism

$$\Phi^+(\tau^\gamma) \simeq \Phi^+(\tau)^\gamma. \tag{5.5}$$

Using (5.4) and (5.5), and the isomorphism T, we get an isomorphism between the representations Π_m and Π_m^γ, and also between the representations $\Pi^{(m)}$ and $(\Pi^{(m)})^\gamma$, for all $m \leq n$.

Recall that $Z_{E,m}$ denote the coset space $P_m(E)/P_{m-1}(E)U_m(E)$. Since $P_{m-1}(E)U_m(E)$ is a Γ-stable subgroup of $P_m(E)$, we have the following long exact sequence of non-abelian cohomology ([Ser, Appendix, Proposition 1]):

$$0 \longrightarrow P_{m-1}(F)U_m(F) \longrightarrow P_m(F) \longrightarrow Z_{E,m}^\Gamma \longrightarrow H^1(\Gamma, P_{m-1}(E)U_m(E)) \longrightarrow H^1(\Gamma, P_m(E)). \tag{5.6}$$

Consider the short exact sequence of non-abelian Γ-modules

$$0 \longrightarrow U_m(E) \longrightarrow P_{m-1}(E)U_m(E) \longrightarrow P_{m-1}(E) \longrightarrow 0. \tag{5.7}$$

From Hilbert's theorem 90, we get that $H^1(\Gamma, U_m(E))$ and $\widehat{H}^1(\Gamma, P_{m-1}(E))$ are trivial. Then, from the long exact sequence of non-abelian cohomology corresponding to (5.7), we have $H^1(\Gamma, P_{m-1}(E)U_m(E)) = 0$. Hence, the long exact sequence (5.6) gives the equality $Z_{E,m}^\Gamma = Z_{F,m}$. Now, using Proposition 5.1 repeatedly $(m-1)$-times, we get the $P_n(F)$-equivariant isomorphism

$$\widehat{H}^i(\Pi_m/\Pi_{m+1}) \simeq (\Phi^+)^{m-1}(\Psi^+)\big(\widehat{H}^i(\Pi^{(m)})\big).$$

Using Leibniz formula for derivatives ([Vig96, Lemma 1.10, Chapter 3]), we get that $\Pi^{(m)}$ is a finite length representation of $G_{n-m}(E)$. By induction hypothesis, the $G_{n-m}(F)$-representation $\widehat{H}^i(\Pi^{(m)})$ is of finite length, for all $m < n$. In view of Lemma 5.4, it follows from the above isomorphism that the $P_n(F)$-representation $\widehat{H}^i(\Pi_m/\Pi_{m+1})$ is of finite length for all $m < n$.

Now, for each $m \in \{1, 2, \dots, n-1\}$, we consider the short exact sequence of $P_n(E)$-representations

$$0 \longrightarrow \Pi_{m+1} \longrightarrow \Pi_m \longrightarrow \Pi_m/\Pi_{m+1} \longrightarrow 0.$$

Since Γ is cyclic, the corresponding long exact sequence of Tate cohomology gives the following diagram:

$$\begin{array}{ccccc}
 & \widehat{H}^0(\Pi_{m+1}) & \longrightarrow & \widehat{H}^0(\Pi_m) & \\
\nearrow & & & & \searrow \\
\widehat{H}^1(\Pi_m/\Pi_{m+1}) & & & & \widehat{H}^0(\Pi_m/\Pi_{m+1}) \\
\nwarrow & & & & \swarrow \\
 & \widehat{H}^1(\Pi_m) & \longleftarrow & \widehat{H}^1(\Pi_{m+1}) &
\end{array}$$

We denote the above exact sequence by $S(m)$. Now, consider the largest integer r for which Π_r is non-zero. Using induction hypothesis, the Tate cohomology groups $\widehat{H}^i(\Pi_r)$ is of finite length as a representation of $P_n(F)$. Now, using the exact sequence $S(r-1)$ and the finiteness of $\widehat{H}^i(\Pi_{r-1}/\Pi_r)$, we get that $\widehat{H}^i(\Pi_{r-1})$ is a finite length representation of $P_n(F)$. Again, using the finiteness of both the representations $\widehat{H}^i(\Pi_{r-1})$ and $\widehat{H}^i(\Pi_{r-2}/\Pi_{r-1})$, it follows from the exact sequence $S(r-2)$ that $\widehat{H}^i(\Pi_{r-2})$ is of finite length. Thus, inductively, we get that $\widehat{H}^i(\Pi)$ is of finite length as a representation of $P_n(F)$ and hence of $G_n(F)$. This completes the proof. □

As a corollary, we have

Corollary 5.6. *Let (Π, V) be an integral $\mathcal{K}$-representation of $G_n(E)$ with an isomorphism $T : (\Pi, V) \to (\Pi^\gamma, V)$ and $T^l = \mathrm{id}$. Then, for any $G_n(E) \rtimes \Gamma$-invariant Λ-lattice $\mathcal{L}$ in V (Here, Γ acts on V via T), the Tate cohomology groups $\widehat{H}^i(\mathcal{L})$, $i \in \{0, 1\}$, are of finite length as representations of $G_n(F)$.*

Proof. Recall that $\mathcal{L}$ is a free Λ-module and $\mathcal{L}/l\mathcal{L}$ is a finite length $G_n(E)$-representation (see [Vig96, II.5.11.a] for finiteness). Consider the short exact sequence of $G_n(E) \rtimes \Gamma$-modules

$$0 \longrightarrow \mathcal{L} \xrightarrow{\text{mult.}l} \mathcal{L} \longrightarrow \mathcal{L}/l\mathcal{L} \longrightarrow 0.$$

By Proposition 5.5, the Tate cohomology group $\widehat{H}^i(\mathcal{L}/l\mathcal{L})$ has finite length as representations of $G_n(F)$. From the long exact sequence of Tate cohomology corresponding to the above short exact sequence, we have

$$0 \to \widehat{H}^0(\mathcal{L}) \to \widehat{H}^0(\mathcal{L}/l\mathcal{L}) \to \widehat{H}^1(\mathcal{L}) \to 0.$$

and

$$0 \to \widehat{H}^1(\mathcal{L}) \to \widehat{H}^1(\mathcal{L}/l\mathcal{L}) \to \widehat{H}^0(\mathcal{L}) \to 0.$$

Thus, we get that each $\widehat{H}^i(\mathcal{L})$ is of finite length. □

5.5. **Frobenius Twist.** Let G be a locally profinite group (i.e., locally compact and totally disconnected). Let (σ, V) be an l-modular representation of G. Consider the vector space $V^{(l)}$, where the underlying additive group structure of $V^{(l)}$ is same as that of V but the scalar action $*$ on $V^{(l)}$ is given by

$$c * v = c^{\frac{1}{l}} v, \text{ for all } c \in \overline{\mathbb{F}}_l,\ v \in V.$$

Then the action of G on V induces a representation $\sigma^{(l)}$ of G on $V^{(l)}$. The representation $(\sigma^{(l)}, V^{(l)})$ is called the Frobenius twist of the representation (σ, V).

We end this subsection with a lemma which will be used in the main result.

Lemma 5.7. *Let ψ be a non-trivial l-modular additive character of F and let Θ be the non-degenerate character of $N_n(F)$ corresponding to ψ. If (π, V_π) and (σ, V_σ) are two l-modular generic representations of $G_n(F)$ and $G_{n-1}(F)$ respectively, then*

$$\gamma(X, \pi, \sigma, \psi)^l = \gamma(X^l, \pi^{(l)}, \sigma^{(l)}, \psi^l).$$

Proof. Let W_π be a Whittaker functional on the representation π. Then the composite map

$$V_\pi \xrightarrow{W_\pi} \overline{\mathbb{F}}_l \xrightarrow{x \mapsto x^l} \overline{\mathbb{F}}_l,$$

denoted by $W_{\pi^{(l)}}$, is a Whittaker functional (with respect to $\psi^l : N_n(F) \to \overline{\mathbb{F}}_l^\times$) on the representation $\pi^{(l)}$, as we have:

$$W_{\pi^{(l)}}(c.v) = W_\pi((c^{\frac{1}{l}} v))^l = c W_{\pi^{(l)}}(v)$$

and

$$W_{\pi^{(l)}}(\pi^{(l)}(n)v) = (\Theta(n) W_\pi(v))^l = \Theta^l(n) W_{\pi^{(l)}}(v),$$

for all $v \in V_\pi$, $c \in \overline{\mathbb{F}}_l$ and all $n \in N_n(F)$.

So the Whittaker model $\mathbb{W}(\pi^{(l)}, \psi^l)$ consists of the functions W_v^l, where W_v varies in $\mathbb{W}(\pi, \psi)$. Similarly the Whittaker model $\mathbb{W}(\sigma^{(l)}, \psi^l)$ of $\sigma^{(l)}$ consists of the functions U_v^l, where U_v varies in $\mathbb{W}(\sigma, \psi)$. Then by the Rankin-Selberg functional equation in the subsection (3.4.2), we have

$$\sum_{r \in \mathbb{Z}} c_r^F(\widetilde{W_v}, \widetilde{U_v})^l q_F^{-lr/2} X^{-lr} = \omega_\sigma(-1)^{n-2} \gamma(X, \pi, \sigma, \psi)^l \sum_{r \in \mathbb{Z}} c_r^F(W_v, U_v)^l q_F^{lr/2} X^{lr} \tag{5.8}$$

and

$$\sum_{r \in \mathbb{Z}} c_r^F(\widetilde{W_v^l}, \widetilde{U_v^l}) q_F^{-r/2} X^{-r} = \omega_{\sigma^{(l)}}(-1)^{n-2} \gamma(X, \pi^{(l)}, \sigma^{(l)}, \psi^l) \sum_{r \in \mathbb{Z}} c_r^F(W_v^l, U_v^l) q_F^{r/2} X^r. \tag{5.9}$$

Replace X by X^l to the equation (5.9), we have

$$\sum_{r \in \mathbb{Z}} c_r^F(\widetilde{W_v^l}, \widetilde{U_v^l}) q_F^{-r/2} X^{-lr} = \omega_{\sigma^{(l)}}(-1)^{n-2} \gamma(X^l, \pi^{(l)}, \sigma^{(l)}, \psi^l) \sum_{r \in \mathbb{Z}} c_r^F(W_v^l, U_v^l) q_F^{r/2} X^{lr}. \tag{5.10}$$

Then from the equations (5.8) and (5.10), we get

$$\gamma(X, \pi, \sigma, \psi)^l = \gamma(X^l, \pi^{(l)}, \sigma^{(l)}, \psi^l).$$

□

6. Tate Cohomology of Whittaker Lattice

As before, we fix a non-trivial additive character $\psi_F : F \to \Lambda^\times$. Let ψ_E be the composition $\psi_F \circ \mathrm{Tr}_{E/F}$, where $\mathrm{Tr}_{E/F}$ is the trace map of the extension E/F. The mod-l reductions of ψ_F and ψ_E are denoted by $\overline{\psi}_F$ and $\overline{\psi}_E$ respectively. Let Θ_F and Θ_E be the characters of $N_n(F)$ and $N_n(E)$ respectively, as defined in (2.7). Similarly, we denote by $\overline{\Theta}_F$ and $\overline{\Theta}_E$ the mod-l reductions of Θ_F and Θ_E respectively.

6.1. Let (π, V) be a generic R-representation of $G_n(E)$, where $R = \overline{\mathbb{Q}}_l$ or $\overline{\mathbb{F}}_l$, such that π is isomorphic to π^γ, for all $\gamma \in \Gamma$. Let $\mathbb{W}(\pi, \psi_E)$ be the Whittaker model of π. For $W \in \mathbb{W}(\pi, \psi_E)$, we recall that $\gamma.W$ is a function given by

$$(\gamma.W)(g) = W(\gamma^{-1}(g)),$$

for all $g \in G_n(E)$. Note that $\gamma.W \in \mathbb{W}(\pi, \psi_E)$ (see Lemma 2.4). Thus, we define

$$T_\gamma : \mathbb{W}(\pi, \psi_E) \to \mathbb{W}(\pi, \psi_E)$$

by setting $T_\gamma(W) = \gamma.W$, for all $W \in \mathbb{W}(\pi, \psi_E)$. The map T_γ gives an isomorphism between (π^γ, V) and (π, V) as we have

$$T_\gamma(\pi(g)W)(h) = \pi(g)W(\gamma^{-1}(h)) = W(\gamma^{-1}(h)g)$$

and

$$[\pi^\gamma(g)T_\gamma(W)](h) = T_\gamma(W)(h\gamma(g)) = W(\gamma^{-1}(h)g),$$

for all $g, h \in G_n(E)$.

6.2. **Jacquet-functors and Tate cohomology.** We begin with a few elementary results on the compatibility of Jacquet (twisted Jacquet) functors with Tate cohomology. Let $(\pi, \mathcal{L})$ be a smooth $\Lambda[G_n(E) \rtimes \Gamma]$-module. Let $\lambda = (n_1, n_2, \dots, n_r)$ be a partition of n and let $P_\lambda = M_\lambda N_\lambda$ be a parabolic subgroup of G_n with N_λ its unipotent radical and M_λ is a standard Levi subgroup. Let $\mathcal{L}(N_\lambda(E))$ be the space spanned by the set of vectors

$$\big\{v - \pi(n)v : v \in \mathcal{L}, n \in N_\lambda(E)\big\}.$$

Note that the space $\mathcal{L}(N_\lambda(E))$ is stable under the action of Γ.

Lemma 6.1. *The image of the natural map $\widehat{H}^0(\mathcal{L}(N_\lambda(E))) \to \widehat{H}^0(\mathcal{L})$ is equal to $\widehat{H}^0(\mathcal{L})(N_\lambda(F))$.*

Proof. Let ϕ be the natural map $\widehat{H}^0(\mathcal{L}(N_\lambda(E))) \to \widehat{H}^0(\mathcal{L})$. Let $v \in \mathrm{img}(\phi)$ and let $\tilde{v}$ be a lift of v in $\mathcal{L}(N_\lambda(E))^\Gamma$. Then there exists a compact open subgroup $\mathcal{N}$ of $N_\lambda(E)$ such that

$$\int_{\mathcal{N}} \pi(n)\tilde{v}\, dn = 0. \tag{6.1}$$

Since π is smooth, there exists a compact open subgroup $\mathcal{N}'$ of $\mathcal{N}$ of finite index such that

$$\int_{\mathcal{N}} \pi(n)\tilde{v}\, dn = \sum_{n \in \mathcal{N}/\mathcal{N}'} \pi(n)\tilde{v}\, dn.$$

Since $N_\lambda(E)$ has a filtration of Γ-stable compact open subgroups, we may assume that $\mathcal{N}$ is Γ-stable. If X denotes the coset space $\mathcal{N}/\mathcal{N}'$, then we have

$$\sum_{x \in X} \pi(x)\tilde{v} = \sum_{y \in X^\Gamma} \pi(y)\tilde{v} \ + \sum_{z \in X \setminus X^\Gamma} \pi(z)\tilde{v}. \tag{6.2}$$

Since the Γ-action on $X \setminus X^\Gamma$ is free, there exists a subset U such that $X \setminus X^\Gamma$ is the disjoint union of $\gamma^i U$, $1 \leq i \leq l$. As $\tilde{v}$ is Γ-invariant, we have

$$\sum_{z \in X \setminus X^\Gamma} \pi(z)\tilde{v} = \sum_{i=1}^{l} \sum_{u \in U} \pi(\gamma^i(u))\tilde{v} = N\big(\sum_{u \in U} \pi(u)\tilde{v}\big),$$

where $N = 1 + \gamma + \dots + \gamma^{l-1}$. This shows that

$$\sum_{z \in X \setminus X^\Gamma} \pi(z)v = 0$$

in $\widehat{H}^0(\mathcal{L})$. Therefore, it follows from (6.1) and (6.2) that

$$\sum_{y \in X^\Gamma} \pi(y)v = 0.$$

This implies that v belongs to $\widehat{H}^0(\mathcal{L})(N_\lambda(F))$. Conversely, let w be an element of $\widehat{H}^0(\mathcal{L})(N_\lambda(F))$. Let $\mathcal{N}_F$ be the compact open subgroup of $N_\lambda(F)$ with

$$\int_{\mathcal{N}_F} \pi(n)w\,dn = 0.$$

Choose a compact open subgroup $\mathcal{N}'_F$ of $\mathcal{N}_F$ of finite index such that

$$\int_{\mathcal{N}_F} \pi(n)w\,dn = \sum_{n\in\mathcal{N}_F/\mathcal{N}'_F} \pi(n)w\,dn.$$

Let $\tilde{w}$ be a lift of w in $\mathcal{L}^\Gamma$. Consider the element

$$\tilde{w}_1 = \tilde{w} - \frac{1}{|\mathcal{N}_F/\mathcal{N}'_F|}\sum_{n\in\mathcal{N}_F/\mathcal{N}'_F} \pi(n)\tilde{w}.$$

Then $\tilde{w}_1$ belongs to $\mathcal{L}(N_\lambda(E))^\Gamma$ and $\phi(\tilde{w}_1) = w$. Moreover, we have

$$\int_{\mathcal{N}_F} \pi(n)\tilde{w}_1\,dn = \sum_{n\in\mathcal{N}_F/\mathcal{N}'_F} \pi(n)\tilde{w}_1 = 0.$$

This completes the proof. $\square$

Lemma 6.2. *Let (π, V) be a smooth l-modular representation of $G_n(E)\rtimes\Gamma$ such that $\widehat{H}^0(V)_{N_\lambda(F)}$ is non-zero and $\widehat{H}^0(V_{N_\lambda(E)})$ is irreducible for the action of $M_\lambda(F)$. Then, the $M_\lambda(F)$-representation $\widehat{H}^0(V)_{N_\lambda(F)}$ is isomorphic to $\widehat{H}^0(V_{N_\lambda(E)})$.*

Proof. The long exact sequence of Tate cohomology groups associated with the exact sequence

$$0 \longrightarrow V(N_\lambda(E)) \longrightarrow V \longrightarrow V_{N_\lambda(E)} \longrightarrow 0,$$

is equal to :

$$\widehat{H}^0(V(N_\lambda(E))) \xrightarrow{\phi} \widehat{H}^0(V) \to \widehat{H}^0(V_{N_\lambda(E)}) \to \widehat{H}^1(V(N_\lambda(E))) \to \widehat{H}^1(V) \to 0.$$

Using Lemma 6.1, we get that $\phi\big(\widehat{H}^0(V(N_\lambda(E)))\big)$ is equal to $\widehat{H}^0(V)(N_\lambda(F))$, and therefore the $N_\lambda(F)$-coinvariants $\widehat{H}^0(V)_{N_\lambda(F)}$ is isomorphic to a subrepresentation of $\widehat{H}^0(V_{N_\lambda(E)})$. Since $\widehat{H}^0(V)_{N_\lambda(F)}$ is non-zero, the lemma follows from the irreducibility of $\widehat{H}^0(V_{N_\lambda(E)})$. $\square$

Using similar ideas, we can prove that zeroth Tate cohomology of a generic representation has a unique generic subquotient. For any integral l-adic generic representation (π, V) of $G_n(K)$, defined over $\mathcal{K}$, we recall that $\mathbb{W}_\Lambda(\pi, \psi_K)$ denotes the set of functions in $\mathbb{W}(\pi_E, \psi_E)$ with values in Λ. The Λ-module $\mathbb{W}_\Lambda(\pi, \psi_K)$ gives a Λ structure of π (see 2.7.6).

Proposition 6.3. *Let π_E be an l-modular generic representation (or an integral l-adic generic representation defined over $\mathcal{K}$) of $G_n(E)$. Assume that π_E is stable under the action of Γ. Then there exists a unique generic subquotient of the $G_n(F)$ representation $\widehat{H}^0(\mathbb{W}(\pi_E, \psi_E))$ (resp. $\widehat{H}^0(\mathbb{W}_\Lambda(\pi_E, \psi_E))$).*

Proof. Let $\mathcal{W}$ be the Whittaker space $\mathbb{W}(\pi_E, \overline{\psi}_E)$ (resp. $\mathbb{W}_\Lambda(\pi_E, \psi_E)$). Let $\mathcal{W}(N_n(E), \Theta_E)$ be the $\overline{\mathbb{F}}_l$ (resp. Λ)-span of the vectors of the form $\pi_E(n)v - \overline{\Theta}_E(n)v$ (resp. $\pi_E(n)v - \Theta_E(n)v$), for all $v \in \mathcal{W}$ and $n \in N_n(E)$. We have the following exact sequence :

$$0 \to \mathcal{W}(N_n(E), \Theta_E) \to \mathcal{W} \to \mathcal{W}_{N_n(E),\Theta_E} \to 0.$$

The space $\mathcal{W}_{N_n(E),\Theta_E}$ is a one dimensional vector space over $\overline{\mathbb{F}}_l$ (resp. a free Λ-module of rank one). The long exact sequence in the Tate cohomology gives us

$$\widehat{H}^0(\mathcal{W}(N_n(E), \Theta_E)) \xrightarrow{f} \widehat{H}^0(\mathcal{W}) \xrightarrow{g} \widehat{H}^0(\mathcal{W}_{N_n(E),\Theta_E}) \to \widehat{H}^1(\mathcal{W}(N_n(E), \Theta_E)) \to \widehat{H}^1(\mathcal{W}).$$

Using arguments of Lemma 6.1, the image of the morphism f is equal to $\widehat{H}^0(\mathcal{W})(N_n(F), \Theta_F^l)$. The Tate cohomology of the Kirillov model $\mathbb{K}(\pi_E, \psi_E)$ contains $\mathcal{K}(\psi_F^l)$ as $P_n(F)$ subrepresentation (see 6.5.1). Since

Γ acts trivially on $\mathcal{W}_{N_n(E),\Theta_E}$, the Tate cohomology space $\widehat{H}^0(\mathcal{W}_{N_n(E),\Theta_E})$ is a one dimensional vector space over $\overline{\mathbb{F}}_l$. Hence, the map g induces the isomorphism:

$$\widehat{H}^0(\mathcal{W})_{N_n(F),\Theta_F^l} \simeq \widehat{H}^0(\mathcal{W}_{N_n(E),\Theta_E}).$$

Now, it follows from the exactness of the Jacquet functor and Proposition 5.5, that $\widehat{H}^0(\mathcal{W})$ admits a unique generic subquotient. □

Remark 6.4. The above lemmas will be used to compute the Tate cohomology of the base change of mod-l the Zelevinsky sub-representation $Z(\Delta)$. The Jacquet functor of $Z(\Delta)$ with respect to the parabolic subgroup of type $(n/k, n/k, \dots, n/k)$, where k is the length of the segment Δ, is an l-modular cuspidal representation and the hypothesis in Lemma 6.2 are applicable. The precise definitions will be recalled in the next section.

6.3. The GL_2 case.

Theorem 6.5. *Let F be a finite extension of $\mathbb{Q}_p$, and let E be a finite Galois extension of F with $[E:F]=l$. Assume that l and p are distinct odd primes. Let π_F be an integral l-adic cuspidal representations of $G_2(F)$ and let π_E be the representation of $G_2(E)$ such that π_E is the base change of π_F. Then*

$$\widehat{H}^0(r_l(\pi_E)) \simeq r_l(\pi_F)^{(l)}.$$

Proof. First note that the base change lift π_E is cuspidal, and hence the mod-l reduction $r_l(\pi_E)$ is also cuspidal. Let ψ_E and ψ_F be defined as in subsection (2.7.7). Let $\big(\mathbb{K}_{\overline{\psi}_E}^{r_l(\pi_E)}, C_c^\infty(E^\times, \overline{\mathbb{F}}_l)\big)$ be a Kirillov model of $r_l(\pi_E)$. Recall that the group Γ acts on $C_c^\infty(E^\times, \overline{\mathbb{F}}_l)$. We denote by $\widehat{H}^0(r_l(\pi_E))$ the cohomology group $\widehat{H}^0\big(C_c^\infty(E^\times, \overline{\mathbb{F}}_l)\big)$. Then, using Proposition 5.1, we have

$$\widehat{H}^0(r_l(\pi_E)) \simeq C_c^\infty(F^\times, \overline{\mathbb{F}}_l).$$

The space $\widehat{H}^0(r_l(\pi_E))$ is isomorphic to $\mathrm{ind}_{N_2(F)}^{P_2(F)}(\overline{\psi}_F^l)$ as a representation of $P_2(F)$, where $\overline{\psi}_F$ is the mod-l reduction of ψ_F; and the induced action of the operator $\mathbb{K}_{\overline{\psi}_E}^{r_l(\pi_E)}(w)$ on $\widehat{H}^0(r_l(\pi_E))$ is denoted by $\overline{\mathbb{K}_{\overline{\psi}_E}^{r_l(\pi_E)}(w)}$. The theorem now follows from the following claim.

Claim 1. $\overline{\mathbb{K}_{\overline{\psi}_E}^{r_l(\pi_E)}(w)}(f) = \mathbb{K}_{\overline{\psi}_F^l}^{r_l(\pi_F)^{(l)}}(w)(f)$, for all $f \in C_c^\infty(F^\times, \overline{\mathbb{F}}_l)$.

Now, for a function $f \in C_c^\infty(F^\times, \overline{\mathbb{F}}_l)$, any covering of $\mathrm{supp}(f)$ by open subsets of $F^\times$ has a finite refinement of pairwise disjoint open compact subgroups of $F^\times$. So we may assume that $\mathrm{supp}(f) \subseteq \varpi^r x U_F^1$, where $r \in \mathbb{Z}$, ϖ_F is an uniformizer of F and x is a unit in $(\mathfrak{o}_F/\mathfrak{p}_F)^\times$ embedded in $F^\times$. Therefore it is sufficient to prove the claim for functions $f \in C_c^\infty(F^\times, \overline{\mathbb{F}}_l)$ with $\mathrm{supp}(f) \subseteq U_F^1$, and we have

$$f = c_{\chi_F} \sum_{\chi_F \in \widehat{U_F^1}} \xi\{\chi_F, 0\},$$

where $c_{\chi_F} \in \overline{\mathbb{F}}_l$ and $\widehat{U_F^1}$ is the set of smooth characters of

$$F^\times = \langle \varpi_F \rangle \times k_F^\times \times U_F^1$$

which are trivial on $k_F^\times$ and ϖ_F. We now prove the claim for the function $\xi\{\chi_F, 0\}$ for $\chi_F \in \widehat{U_F^1}$. There exists a character $\chi_0 \in \widehat{U_F^1}$ such that $\chi_0^l = \chi_F$. Let $\widetilde{\chi}_0$ be the l-adic lift of the character χ_0. Define the characters χ_E and $\widetilde{\chi}_E$ of $E^\times$ as follows

$$\chi_E(x) = \chi_0(\mathrm{Nr}_{E/F}(x))$$

and

$$\widetilde{\chi}_E(x) = \widetilde{\chi}_0(\mathrm{Nr}_{E/F}(x)),$$

for $x \in E^\times$. Here, $\mathrm{Nr}_{E/F} : E^\times \to F^\times$ denotes the norm map. Note that $\widetilde{\chi}_E$ and χ_E extends the character χ_F. We have the following relations :

$$\overline{\mathbb{K}_{\overline{\psi}_E}^{r_l(\pi_E)}(w)}\big(\xi\{\chi_F, 0\}\big) = \epsilon(\chi_E^{-1} r_l(\pi_E), \overline{\psi}_E)\xi\left\{\chi_F, \frac{-n(\chi_E^{-1} r_l(\pi_E), \overline{\psi}_E)}{e}\right\} \tag{6.3}$$

and

$$\mathbb{K}^{r_l(\pi_F)^{(l)}}_{\overline{\psi}^l_F}(w)\big(\xi\{\chi_F,0\}\big)=\epsilon(\chi_F^{-1}r_l(\pi_F)^{(l)},\overline{\psi}^l_F)\xi\{\chi_F,-n(\chi_F^{-1}r_l(\pi_F)^{(l)},\overline{\psi}^l_F)\},\tag{6.4}$$

where e denotes the ramification index of the extension E/F. Next, we aim to prove the following identity:

$$\epsilon(\chi_E^{-1}r_l(\pi_E),\overline{\psi}_E)=\epsilon(X,\chi_F^{-1}r_l(\pi_F)^{(l)},\overline{\psi}^l_F).$$

It follows from Theorem 3.4 that the ϵ-factor is same as the γ-factor in both l-adic and mod-l cases. Now, using the identity in [AC89, Proposition 6.9], we get

$$\epsilon(X,\widetilde{\chi}_E^{-1}\pi_E,\psi_E)=\prod_{\eta}\epsilon(X,\widetilde{\chi}_0^{-1}\pi_F\otimes\eta,\psi_F),\tag{6.5}$$

where η runs over all the characters of the group $F^\times/\mathrm{Nr}_{E/F}(E^\times)$– which is isomorphic to $\mathrm{Gal}(E/F)$ via local class field theory. Using the identity (3.6), we have

$$r_l\big(\epsilon(X,\widetilde{\chi}_E^{-1}\pi_E,\psi_E)\big)=\epsilon(X,\chi_E^{-1}r_l(\pi_E),\overline{\psi}_E)$$

and

$$r_l\big(\epsilon(X,\widetilde{\chi}_0^{-1}\pi_F\otimes\eta,\psi_F)\big)=\epsilon(X,\chi_0^{-1}r_l(\pi_F),\overline{\psi}_F),$$

for each character η. Using the above identities and taking mod-l reduction to (6.5), we get

$$\epsilon(X,\chi_E^{-1}r_l(\pi_E),\overline{\psi}_E)=\epsilon(X,\chi_0^{-1}r_l(\pi_F),\overline{\psi}_F)^l.$$

Using Lemma 5.7, we obtain the following identity

$$\epsilon(X,\chi_E^{-1}r_l(\pi_E),\overline{\psi}_E)=\epsilon(X^l,\chi_F^{-1}r_l(\pi_F)^{(l)},\overline{\psi}^l_F).$$

Now, using the identity (3.5) and comparing the degree of X from above relation, we get

$$\frac{n(\chi_E^{-1}r_l(\pi_E),\overline{\psi}_E)}{e}=n(\chi_F^{-1}r_l(\pi_F)^{(l)},\overline{\psi}^l_F)$$

and

$$\epsilon(\chi_E^{-1}r_l(\pi_E),\overline{\psi}_F)=\epsilon(\chi_F^{-1}r_l(\pi_F)^{(l)},\overline{\psi}^l_F).$$

Thus it follows from (6.3) and (6.4) that

$$\overline{\mathbb{K}^{r_l(\pi_E)}_{\overline{\psi}_E}(w)}\big(\xi\{\chi_F,0\}\big)=\mathbb{K}^{r_l(\pi_F)^{(l)}}_{\overline{\psi}^l_F}(w)\big(\xi\{\chi_F,0\}\big).$$

Hence we prove the claim, and the theorem follows. □

6.4. Our main result uses the following lemma which is the analogue of completeness of Whittaker models in the complex case.

Lemma 6.6. *Assume that l does not divide $|G_n(k_K)|$ and let $\overline{\psi}_K$ be the mod-l reduction of ψ_K. Let $\overline{\Theta}_K$ be the non-degenerate character of $N_n(K)$ associated with $\overline{\psi}_K$ (see Section 2.7). Let $\phi\in\mathrm{ind}_{N_n(K)}^{G_n(K)}(\overline{\Theta}_K)$. If*

$$\int_{N_n(K)\backslash G_n(K)}\phi(t)W(t)\,dt=0,$$

for all $W\in\mathbb{W}(\sigma,\overline{\psi}_K^{-1})$ and for all generic representations σ of $G_n(K)$, then $\phi=0$.

Proof. Suppose ϕ is non-zero. Let $\mathrm{Rep}_{W(\overline{\mathbb{F}}_l)}(G_n(K))$ be the category of smooth $W(\overline{\mathbb{F}}_l)[G_n(K)]$-modules, and let $\mathcal{Z}_n$ be its center. Let W_n be the smooth $W(\overline{\mathbb{F}}_l)[G_n(K)]$-module $\mathrm{ind}_{N_n(K)}^{G_n(K)}(\Theta_K)$. Recall that for any primitive idempotent e in $\mathcal{Z}_n$, the space eW_n is a smooth co-Whittaker $e\mathcal{Z}_n[G_n(K)]$-module (see [Hel16b, Theorem 6.3]). According to [Mos21, Corollary 4.3], there exists a primitive idempotent e' of $\mathcal{Z}_n$ and an element $U\in\mathbb{W}(e'W_n\otimes_{W(\overline{\mathbb{F}}_l)}\overline{\mathbb{F}}_l,\overline{\psi}_K^{-1})$ such that the integral

$$\langle\phi,U\rangle:=\int_{N_n(K)\backslash G_n(K)}\phi(t)\otimes U(t)\,dt$$

is non-zero in $\overline{\mathbb{F}}_l\otimes_{W(\overline{\mathbb{F}}_l)}e'\mathcal{Z}_n$. As described in [Hel16a], the primitive idempotent e' corresponds to an inertial equivalence class of pairs (M,π), where M is a Levi subgroup of $G_n(K)$ and π is a supercuspidal $\overline{\mathbb{F}}_l$-representation of M. Let R' denote the ring $\overline{\mathbb{F}}_l\otimes_{W(\overline{\mathbb{F}}_l)}e'\mathcal{Z}_n$.

For the inertial equivalence class $[M,\pi]$, consider the subcategory $\mathrm{Rep}_{W(\overline{\mathbb{F}}_l)}(G_n(K))_{[M,\pi]}$, consisting of objects Π in $\mathrm{Rep}_{W(\overline{\mathbb{F}}_l)}(G_n(K))$ whose irreducible sub-quotients have mod-l inertial supercuspidal support $[M,\pi]$. Let $A_{[M,\pi]}$ denote the center of the subcategory $\mathrm{Rep}_{W(\overline{\mathbb{F}}_l)}(G_n(K))_{[M,\pi]}$. Since l does not divide $|G_n(k_K)|$, it follows from [Hel16a, Example 13.9] that

$$A_{[M,\pi]} = C_{[M,\pi]},$$

where $C_{[M,\pi]}$ is a $W(\overline{\mathbb{F}}_l)$-subalgebra of $A_{[M,\pi]}$, as defined in [Hel16a, Theorem 12.5]. Then, there is an isomorphism of $C_{[M,\pi]} \otimes_{W(\overline{\mathbb{F}}_l)} \overline{\mathbb{F}}_l$ with the reduced quotient of $A_{[M,\pi]} \otimes_{\Lambda} \overline{\mathbb{F}}_l$ (see [Hel16a, Corollary 12.13]), and hence we get that the $W(\overline{\mathbb{F}}_l)$-algebra R' is reduced. In particular, the element $\langle \phi, U\rangle$ is not nilpotent. Therefore, the basic open set $D(\langle \phi, U\rangle)$ is non-empty, and hence intersects the dense set of closed points of the affine $W(\overline{\mathbb{F}}_l)$-scheme associated with R'. This implies that there exists a map $f : R' \to \overline{\mathbb{F}}_l$ such that the image of $\langle \phi, U\rangle$ under f, which is equal to

$$\int_{N_n(K)\backslash G_n(K)} \phi(t) W_0(t)\, dt$$

for some $W_0 \in \mathbb{W}\big(e' W_n \otimes_{R',f} \overline{\mathbb{F}}_l, \overline{\psi}_K^{-1}\big)$, is non-zero in $\overline{\mathbb{F}}_l$. Note that $e' W_n \otimes_{R',f} \overline{\mathbb{F}}_l$, as $\overline{\mathbb{F}}_l$-representation, admits a generic quotient with same Whittaker space. Hence, the lemma. □

6.5. **The general case.** Let π_F be an integral generic l-adic representation of $G_n(F)$, and let π_E be the base change lifting of π_F to $G_n(E)$. We observe that the unique generic component $J_l(\pi_E)$ of the mod-l reduction of π_E is stable under the action of Γ. We will now prove the main theorem of our article.

Theorem 6.7. *Let F be a finite extension of $\mathbb{Q}_p$, and let E be a finite Galois extension of F with $[E:F] = l$, where p and l are distinct primes such that l does not divide $|G_{n-1}(k_F)|$. Let π_F be an integral l-adic generic representation of $G_n(F)$ with $J_l(\pi_F)$, the unique generic component of the mod-l reduction of π_F. Let π_E be the base change lift of π_F. Then, the representation $J_l(\pi_F)^{(l)}$ is the unique generic sub-quotient of $\widehat{H}^0(J_l(\pi_E))$.*

Proof. We begin with a summary of the proof. We prove the above theorem using induction on the integer n. The proof is divided into four parts. In the first part, we isolate a subspace $\mathcal{M}(\pi_F, \psi_F)$ of the Tate cohomology of the Kirillov model of $J_l(\pi_E)$ which will eventually give $J_l(\pi_F)^{(l)}$ as a quotient. In the second part, we will set up comparison of Zeta integrals on homogeneous spaces of F with those on homogeneous spaces of E. In the third part we reduce the theorem to an identity of local γ-factors. In the fourth-part we deal with these local γ-factor identities and we show that $\mathcal{M}(\pi_F, \psi_F)$ is stable under the action of $G_n(F)$. At the end of the fourth part, we get a natural onto map from $\mathcal{M}(\pi_F, \psi_F)$ to the mod-l Kirillov model $\mathbb{K}(J_l(\pi_F)^{(l)}, \overline{\psi}_F^l)$ as $G_n(F)$ representations.

6.5.1. Notations on Whittaker and Kirillov models are defined in the subsections (2.7.1) and (2.8). Consider the Whittaker model $\mathbb{W}(J_l(\pi_E), \overline{\psi}_E)$ of $J_l(\pi_E)$. The restriction map $W \mapsto \mathrm{res}_{P_n(E)}(W)$ is an isomorphism from $\mathbb{W}(J_l(\pi_E), \overline{\psi}_E)$ onto $\mathbb{K}(J_l(\pi_E), \overline{\psi}_E)$ (see [MM22, Theorem 4.2]). Recall that $\mathcal{K}(\overline{\psi}_E)$ denotes the compactly induced representation $\mathrm{ind}_{N_n(E)}^{P_n(E)} \overline{\Theta}_E$. Note that $\mathcal{K}(\overline{\psi}_E)$ is contained in $\mathbb{K}(J_l(\pi_E), \overline{\psi}_E)$. Let I_n be the following natural map:

$$I_n : \widehat{H}^0(\mathcal{K}(\overline{\psi}_E)) \longrightarrow \widehat{H}^0(\mathbb{K}(J_l(\pi_E), \overline{\psi}_E)).$$

Let $\Phi_n : \mathbb{K}(J_l(\pi_E), \overline{\psi}_E)^{\Gamma} \to \mathrm{Ind}_{N_n(F)}^{P_n(F)} \overline{\Theta}_F^l$ be the restriction to $P_n(F)$ map. Note that the map Φ_n factorizes through

$$\Phi_n : \widehat{H}^0(\mathbb{K}(J_l(\pi_E), \overline{\psi}_E)) \longrightarrow \mathrm{Ind}_{N_n(F)}^{P_n(F)} \overline{\Theta}_F^l.$$

The composition $\Phi_n \circ I_n$ is induced by the restriction to $P_n(F)$ map from $\mathcal{K}(\overline{\psi}_E)^{\Gamma}$ to $\mathcal{K}(\overline{\psi}_F^l)$, and hence, $\Phi_n \circ I_n$ is an isomorphism onto the space $\mathcal{K}(\overline{\psi}_F^l)$ by Proposition 5.1 (see Subsection 5.2.1). This implies that the image of Φ_n contains $\mathcal{K}(\overline{\psi}_F^l)$. Let $\mathcal{M}(\pi_F, \psi_F)$ be the space $\Phi_n^{-1}(\mathbb{K}(J_l(\pi_F)^{(l)}, \overline{\psi}_F^l))$. The space $\mathcal{M}(\pi_F, \psi_F)$ is a non-zero $P_n(F)$ sub-representation of $\widehat{H}^0(\mathbb{K}(J_l(\pi_E), \overline{\psi}_E))$, and the map

$$\Phi_n : \mathcal{M}(\pi_F, \psi_F) \longrightarrow \mathbb{K}(J_l(\pi_F)^{(l)}, \overline{\psi}_F^l)$$

is non-zero. Then, using induction on n, we will show that the space $\mathcal{M}(\pi_F, \psi_F)$ is stable under the action of $G_n(F)$ and the map Φ_n is $G_n(F)$-equivariant.

6.5.2. Let $\overline{J_l(\pi_E)}(w_n)$ be the induced action of $J_l(\pi_E)(w_n)$ on the space $\widehat{H}^0(\mathbb{K}(J_l(\pi_E),\overline{\psi}_E))$. Let V be an element in $\mathcal{M}(\pi_F,\psi_F)$. Then there exists $W\in\mathbb{W}(J_l(\pi_E),\overline{\psi}_E)^\Gamma$ such that W is mapped to V under the map

$$\mathbb{W}(J_l(\pi_E),\overline{\psi}_E)^\Gamma \longrightarrow \mathbb{K}(J_l(\pi_E),\overline{\psi}_E)^\Gamma \longrightarrow \widehat{H}^0\big(\mathbb{K}(J_l(\pi_E),\overline{\psi}_E)\big). \tag{6.6}$$

Let $\overline{\sigma}_F$ be an arbitrary l-modular generic representation of $G_{n-1}(F)$, and let σ_F be its l-adic lift. In this case, the generic mod-l representation $J_l(\sigma_F)$ is equal to $\overline{\sigma}_F$. Let σ_E be an l-adic generic representation of $G_{n-1}(E)$ obtained as a base change of σ_F. Note that the map

$$\tilde{\Phi}_{n-1}:\widehat{H}^0(\mathbb{W}(J_l(\sigma_E),\overline{\psi}_E^{-1}))\longrightarrow \mathrm{Ind}_{N_{n-1}(F)}^{G_{n-1}(F)}\overline{\Theta}_F^{-l}$$

is non-zero. Here, $\tilde{\Phi}_{n-1}$ is the restriction to $G_{n-1}(F)$ map on the space $\mathrm{Ind}_{N_{n-1}(E)}^{G_{n-1}(E)}\overline{\Theta}_E$. Assuming the induction hypothesis for $n-1$ and using the fact that the representation $\widehat{H}^0(\mathbb{W}(J_l(\sigma_E),\overline{\psi}_E^{-1}))$ has a unique generic subquotient (Proposition 6.3), the image of $\tilde{\Phi}_{n-1}$ contains $\mathbb{W}(\overline{\sigma}_F^{(l)},\overline{\psi}_F^{-l})$. Thus, for any $W'\in\mathbb{W}(\overline{\sigma}_F^{(l)},\overline{\psi}_F^{-l})$, there exists an element $\mathcal{S}\in\mathbb{W}(J_l(\sigma_E),\overline{\psi}_E^{-1})^\Gamma$ such that $\tilde{\Phi}_{n-1}(\mathcal{S})=W'$ and

$$\tilde{\Phi}_{n-1}\big(J_l(\sigma_E)(w_{n-1})S\big)=\overline{\sigma}_F^{(l)}(w_{n-1})W'.$$

Now the functional equation in (3.4.2) gives the following relation :

$$\sum_{r\in\mathbb{Z}}c_r^E\big(\widetilde{W},\widetilde{\mathcal{S}}\big)q_F^{-\frac{r}{2}f}X^{-fr}=\omega_{J_l(\sigma_E)}(-1)^{n-2}\gamma\big(X,J_l(\pi_E),J_l(\sigma_E),\psi_E\big)\sum_{r\in\mathbb{Z}}c_r^E(W,\mathcal{S})q_F^{\frac{r}{2}f}X^{fr}, \tag{6.7}$$

where f denotes the residue degree of the extension E/F. Note that $\omega_{\sigma_E}(-1)=\omega_{\sigma_F}(-1)$ as l is an odd prime. Applying Proposition 5.2, we get

$$\begin{aligned}\int_{(X_E^r)^\Gamma}W\begin{pmatrix}g&0\\0&1\end{pmatrix}\mathcal{S}(g)dg&=\int_{X_F^{\frac{r}{e}}}W\begin{pmatrix}g&0\\0&1\end{pmatrix}\mathcal{S}(g)dg,\\ \int_{(X_E^r)^\Gamma}\widetilde{W}\begin{pmatrix}g&0\\0&1\end{pmatrix}\widetilde{\mathcal{S}}(g)dg&=\int_{X_F^{\frac{r}{e}}}\widetilde{W}\begin{pmatrix}g&0\\0&1\end{pmatrix}\widetilde{\mathcal{S}}(g)dg,\end{aligned} \tag{6.8}$$

for each $r\in\mathbb{Z}$. Using the above equalities and Remark 5.3, the functional equation (6.7) becomes

$$\sum_{r\in\mathbb{Z}}c_r^F(\widetilde{W},\widetilde{\mathcal{S}})q_F^{-\frac{r}{2}}X^{-efr}=\omega_{J_l(\sigma_F)}(-1)^{n-2}\gamma(X,J_l(\pi_E),J_l(\sigma_E),\overline{\psi}_E)\sum_{r\in\mathbb{Z}}c_r^F(W,\mathcal{S})q_F^{\frac{r}{2}}X^{efr}.$$

Using the modification as in (3.3), the above equality becomes

$$\begin{aligned}&\sum_{r\in\mathbb{Z}}c_{-r}^F\big(\overline{J_l(\pi_E)}(w_n)W,\overline{\sigma}_F^{(l)}(w_{n-1})W'\big)q_F^{-\frac{r}{2}}X^{-lr}=\\ &\qquad\omega_{J_l(\sigma_F)}(-1)^{n-2}\gamma(X,J_l(\pi_E),J_l(\sigma_E),\overline{\psi}_E)\sum_{r\in\mathbb{Z}}c_r^F(W,W')q_F^{\frac{r}{2}}X^{lr}.\end{aligned} \tag{6.9}$$

6.5.3. For any $V\in\mathcal{M}(\pi_F,\psi_F)$, we show that

$$\Phi_n(\overline{J_l(\pi_E)}(w_n)V)=J_l(\pi_F)^{(l)}(w_n)\Phi_n(V). \tag{6.10}$$

Let U be an element of $\mathbb{W}(J_l(\pi_F)^{(l)},\overline{\psi}_F^l)$ such that $\mathrm{res}_{P_n(F)}(U)$ is equal to $\Phi_n(V)$. By Lemma 6.6, the assertion (6.10) is equivalent to the following equality :

$$\begin{aligned}&\sum_{r\in\mathbb{Z}}c_{-r}^F\big(\overline{J_l(\pi_E)}(w_n)W,\overline{\sigma}_F^{(l)}(w_{n-1})W'\big)q_F^{-r/2}X^{-r}=\\ &\qquad\sum_{r\in\mathbb{Z}}c_{-r}^F\big(J_l(\pi_F)^{(l)}(w_n)U,\overline{\sigma}_F^{(l)}(w_{n-1})W'\big)q_F^{-r/2}X^{-r},\end{aligned} \tag{6.11}$$

for all $W'\in\mathbb{W}(\overline{\sigma}_F,\overline{\psi}_F^{-l})$ and for all l-modular generic representations $\overline{\sigma}_F$ of $G_{n-1}(F)$. Now, consider an l-modular generic representation $\overline{\sigma}_F$ of $G_{n-1}(F)$ and take an l-adic lift of $\overline{\sigma}_F$, say σ_F (see subsection 2.7.5).

Note that $J_l(\sigma_F) = \overline{\sigma}_F$. Let σ_E be the l-adic generic representation of $G_{n-1}(E)$ obtained as a base change lift of σ_F. From the functional equation with its modifications as in (3.3), we have

$$\sum_{r\in\mathbb{Z}} c_{-r}^F\big(J_l(\pi_F)^{(l)}(w_n)U,\overline{\sigma}_F^{(l)}(w_{n-1})W'\big)q_F^{-\frac{r}{2}}X^{-lr} = \omega_{J_l(\sigma_F)}(-1)^{n-2}\gamma\big(X^l, J_l(\pi_F)^{(l)}, \overline{\sigma}_F^{(l)}, \overline{\psi}_F^l\big)\sum_{r\in\mathbb{Z}} c_r^F(U, W')q_F^{\frac{r}{2}}X^{lr},$$

where we replace the variable X by X^l. Note that $\mathrm{res}_{P_n(F)}(W)$ is equal to $\mathrm{res}_{P_n(F)}(U)$. Thus, comparing the above functional equation with (6.9), the relation (6.10) is now equivalent to the following equality:

$$\gamma(X, J_l(\pi_E), J_l(\sigma_E), \overline{\psi}_E) = \gamma\big(X^l, J_l(\pi_F)^{(l)}, \overline{\sigma}_F^{(l)}, \overline{\psi}_F^l\big).$$

6.5.4. Recall that

$$\gamma(X, \pi_E, \sigma_E, \psi_E) = \epsilon(X, \pi_E, \sigma_E, \psi_E)\frac{L(q_E^{-1}X^{-1}, \widetilde{\pi_E}, \widetilde{\sigma_E})}{L(X, \pi_E, \sigma_E)}.$$

Now using the identity in [AC89, Proposition 6.9], we have

$$L(X, \pi_E, \sigma_E) = \prod_\eta L(X, \pi_F, \sigma_F \otimes \eta)$$

and

$$\epsilon(X, \pi_E, \sigma_E, \psi_E) = \mathcal{C}_{E/F}(\psi_F)^{n(n-1)}\prod_\eta \epsilon(X, \pi_F, \sigma_F \otimes \eta, \psi_F),$$

where η runs over all the characters of the group $F^\times/\mathrm{Nr}_{E/F}(E^\times)$, which is isomorphic to $\mathrm{Gal}(E/F)$ via local class field theory. Here, $\mathcal{C}_{E/F}(\psi_F)$ is the Langlands constant, defined as in the proof of Lemma 3.2, and $\mathcal{C}_{E/F}(\psi_F)^2 = 1$. Then the above relations implies that

$$\gamma(X, \pi_E, \sigma_E, \psi_E) = \prod_\eta \gamma(X, \pi_F, \sigma_F \otimes \eta, \psi_F). \tag{6.12}$$

Now, using the identity (3.6), we have

$$r_l(\gamma(X, \pi_E, \sigma_E, \psi_E)) = \gamma(X, J_l(\pi_E), J_l(\sigma_E), \overline{\psi}_E)$$

and

$$r_l(\gamma(X, \pi_F, \sigma_F \otimes \eta, \psi_E)) = \gamma(X, J_l(\pi_F), \overline{\sigma}_F, \overline{\psi}_F),$$

for each character η. Taking mod-l reduction to the identity (6.12) and using these relations, we get

$$\gamma(X, J_l(\pi_E), J_l(\sigma_E), \overline{\psi}_E) = \gamma(X, J_l(\pi_F), \overline{\sigma}_F, \overline{\psi}_F)^l. \tag{6.13}$$

Finally, it follows from Lemma 5.7 that

$$\gamma(X, J_l(\pi_E), J_l(\sigma_E), \overline{\psi}_E) = \gamma(X^l, J_l(\pi_F)^{(l)}, \overline{\sigma}_F^{(l)}, \overline{\psi}_F^l).$$

The identity (6.10) shows that space $\mathcal{M}(\pi_F, \psi_F)$ is stable under the action of $G_n(F)$ and the map

$$\Phi_n : \mathcal{M}(\pi_F, \psi_F) \to \mathbb{K}(J_l(\pi_F)^{(l)}, \overline{\psi}_F^l)$$

is surjective. Using Proposition 6.3, the $G_n(F)$ representation $\widehat{H}^0(\mathbb{W}(J_l(\pi_E), \overline{\psi}_E))$ has a unique generic sub-quotient, which is necessarily equal to $J(\pi_F)^{(l)}$. This completes the proof. □

Now we deduce some corollaries of Theorem 6.7. We keep the same assumptions that E/F is a finite Galois extension p-adic fields with $[E : F] = l$, where l and p are distinct primes, and l does not divide $|G_{n-1}(k_F)|$.

Corollary 6.8. *Let π_E be an integral generic $\mathcal{K}$-representation of $G_n(E)$ which is absolutely irreducible. Assume that $\pi_E^\gamma \simeq \pi_E$, for all $\gamma \in \Gamma$. Let $\mathbb{W}_\Lambda(\pi_E, \psi_E)$ be the space of all Λ-valued functions in the Whittaker model of π_E. Let π_F be the integral generic $\overline{\mathbb{Q}}_l$-representation of $G_n(F)$ such that $\pi_E \otimes_\mathcal{K} \overline{\mathbb{Q}}_l$ is the base change lift of π_F. Then the Frobenius twist of $J_l(\pi_F)$ occurs as a unique generic subquotient of the zeroth Tate cohomology group $\widehat{H}^0(\mathbb{W}_\Lambda(\pi_E, \psi_E))$.*

Proof. The outline of the proof is same as Theorem 6.7. For the sake of completeness, we discuss some crucial steps. As one can observe, the previous and the present theorems are similar in spirit to local converse theorem for $(n, n-1)$, we precisely use Theorem 6.7 at the $(n-1)$ step.

First, note that the Λ-lattice $\mathbb{W}_\Lambda(\pi_E, \psi_E)$ is stable under the action of $G_n(E) \rtimes \Gamma$ (see [Vig04, Theorem 2] and Lemma 2.4). Consider the integral Kirillov model $\mathbb{K}_\Lambda(\pi_E, \psi_E)$. The restriction map $W \mapsto \mathrm{res}_{P_n(E)}(W)$ is then a bijection from $\mathbb{W}_\Lambda(\pi_E, \psi_E)$ onto $\mathbb{K}_\Lambda(\pi_E, \psi_E)$. Let Φ_n be the following $P_n(F)$-equivariant map, defined as the composition of restriction to $P_n(F)$ map and (pointwise) mod-l reduction map

$$\Phi_n : \mathbb{K}_\Lambda(\pi_E, \psi_E)^\Gamma \longrightarrow \mathbb{K}(J_l(\pi_F)^{(l)}, \overline{\psi}_F^l).$$

Then Φ_n is non-zero and it factorizes through the Tate cohomology space $\widehat{H}^0(\mathbb{K}_\Lambda(\pi_E, \psi_E))$. As before, we consider the non-zero space $\Phi_n^{-1}(\mathbb{K}(J_l(\pi_F)^{(l)}, \overline{\psi}_F^l))$ and denote it by $\mathcal{M}(\pi_F, \psi_F)$. To prove the above corollary, it is sufficient prove that

$$\Phi_n(\overline{\pi_E(w_n)}V) = J_l(\pi_F)^{(l)}(w_n)\Phi_n(V),$$

for all $V \in \mathcal{M}(\pi_F, \psi_F)$. It is enough to prove the following identity of Laurent series

$$\sum_{r\in\mathbb{Z}} c_r^F\left(\Phi_n(\overline{\pi_E(w_n)}V), W'\right) q_F^{r/2} X^r = \sum_{r\in\mathbb{Z}} c_r^F\left(J_l(\pi_F)^{(l)}(w_n)\Phi_n(V), W'\right) q_F^{r/2} X^r, \tag{6.14}$$

for all $W' \in \mathbb{W}(\overline{\sigma}_F^{(l)}, \overline{\psi}_F^{-l})$ and for all l-modular generic representations $\overline{\sigma}_F$ of $G_{n-1}(F)$. Take such mod-l generic representation $\overline{\sigma}_F$ of $G_{n-1}(F)$. Let σ_F be an l-adic lift of $\overline{\sigma}_F$ and let σ_E be the base change lift of σ_F to $G_{n-1}(E)$. Theorem 6.7 gives a $G_{n-1}(F)$-stable subspace $\mathcal{N}(\sigma_F, \psi_F)$ of the Tate cohomology group $\widehat{H}^0(\mathbb{W}(J_l(\sigma_E), \overline{\psi}_E))$ with the following $G_{n-1}(F)$-equivariant surjection

$$\Phi_{n-1} : \mathcal{N}(\sigma_F, \psi_F) \longrightarrow \mathbb{W}(\overline{\sigma}_F^{(l)}, \overline{\psi}_F^{-l}).$$

Now, lifting the function W' via Φ_{n-1} and the function V to the respective Γ-invariant Kirillov models and using the identities (6.8), the above relation (6.14) is then equivalent to the following identity of gamma factors :

$$r_l(\gamma(X, \pi_E, \sigma_E, \psi_E)) = \gamma(X^l, J_l(\pi_F)^{(l)}, \overline{\sigma}_F^{(l)}, \overline{\psi}_F^l).$$

This follows from the arguments of the subsection (6.5.4) in the proof of Theorem 6.7. ∎

Corollary 6.9. *Let $n \geq 3$, and let π_F be an integral l-adic cuspidal representation of $G_n(F)$ and let π_E be the base change lift of π_F. Then we have*

$$\widehat{H}^0(r_l(\pi_E)) \simeq r_l(\pi_F)^{(l)}.$$

Proof. Since l does not divide n, the representation π_E is cuspidal. As the Kirillov model $\mathbb{K}(r_l(\pi_E), \overline{\psi}_E)$ is equal to $\mathcal{K}(\overline{\psi}_E)$, we get that $\widehat{H}^0(\mathbb{K}(r_l(\pi_E), \overline{\psi}_E))$ is equal to $\mathcal{K}(\overline{\psi}_F^l)$. Thus, the action of $G_n(F)$ on $\widehat{H}^0(\mathbb{K}(r_l(\pi_E), \overline{\psi}_E))$ is irreducible, and the corollary follows from Theorem 6.7. ∎

7. Base change for $Z(\Delta)$

In this section, we study the Tate cohomology of the base change of the Zelevinsky subrepresentations of the form $Z(\Delta)$. In [Zel80], Zelevinsky uses the notation $\langle\Delta\rangle$ for $Z(\Delta)$. In this section, we continue with the assumptions in Corollary 6.9, i.e., $l \neq p$ and l does not divide $|G_{n-1}(\mathbb{F}_q)|$. Note that l does not divide n. Recall that q is the cardinality of the residue field of F. We will crucially use the fact that $Z(\Delta)$ remains irreducible under the restriction to P_n and it is characterised by this property.

7.1. Keeping the notations as in subsection (2.7.2), let $\Delta = \left\{\sigma, \sigma\nu_K, \dots, \sigma\nu_K^{r-1}\right\}$ be a segment, where K is a p-adic field and σ is a cuspidal l-adic representation of $G_m(K)$. We denote by $\ell(\Delta)$ the length of Δ, i.e., the integer r. The parabolic induction

$$\sigma \times \sigma\nu_K \times \cdots \sigma\nu_K^{r-1}$$

admits a unique irreducible subrepresentation, denoted by $Z(\Delta)$. Moreover, $Z(\Delta)$ can be characterised as those irreducible representation of $G_{rm}(K)$ that remain irreducible after restricting to $P_{rm}(K)$, and the restriction is isomorphic to $(\Phi^+)^{m-1} \circ \Psi^+(Z(\Delta^-))$, where $\Delta^- = \Delta \setminus \left\{\sigma\nu_K^{r-1}\right\}$. We refer to [BZ77, Section 3], [Vig96, Section 1.2, Chapter III] for the definitions of the functors $\Phi^\pm$ and $\Psi^\pm$, and for the definition of $Z(\Delta)$ and its restriction to $P_n(K)$, we refer to [Zel80, Section 3].

7.2. Let F be a finite extension of $\mathbb{Q}_p$, and let E be a finite Galois extension of F of prime degree l with $l \neq p$. Let Γ denote the Galois group $\mathrm{Gal}(E/F)$ with generator, say γ. Let σ_F and σ_E be the integral cuspidal l-adic representations of $G_m(F)$ and $G_m(E)$ respectively, such that σ_E is a base change lift of σ_F. Consider the segments

$$\Delta_F = \{\sigma_F, \sigma_F\nu_F, \dots, \sigma_F\nu_F^{k-1}\}$$
$$\Delta_E = \{\sigma_E, \sigma_E\nu_E, \dots, \sigma_E\nu_E^{k-1}\}.$$

Then we have the irreducible l-adic representations $Z(\Delta_F)$ and $Z(\Delta_E)$ of $G_n(F)$ and $G_n(E)$ respectively, where $n = km$. If we let σ'_F (resp. σ'_E) to be the representation $\sigma_F\nu_F^{k-1}$ (resp. $\sigma_E\nu_E^{k-1}$), then we have

$$\Pi_F(Z(\Delta_F)) = \Pi_F(\sigma'_F) \oplus \Pi_F(\sigma'_F\nu_F^{-1}) \oplus \cdots \oplus \Pi_F(\sigma_F)$$

and

$$\Pi_E(Z(\Delta_E)) = \Pi_E(\sigma'_E) \oplus \Pi_E(\sigma'_E\nu_E^{-1}) \oplus \cdots \oplus \Pi_F(\sigma_E),$$

where Π_F and Π_E are the local Langlands correspondences defined as in subsection (4.1). This shows that

$$\mathrm{Res}_{\mathcal{W}_E}\big(\Pi_F(Z(\Delta_F))\big) \simeq \Pi_E(Z(\Delta_E)).$$

Thus the representation $Z(\Delta_E)$ is the base change of $Z(\Delta_F)$.

7.3. Let $\mathcal{L}_0$ be a $G_m(E)$-invariant lattice in σ_E, and let $S_\gamma : \sigma_E \to \sigma_E^\gamma$ be an isomorphism with $S_\gamma^l = \mathrm{id}$ and $S_\gamma(\mathcal{L}_0) = \mathcal{L}_0$. Recall that the representation $\pi_E = \sigma_E \times \sigma_E\nu_E \times \cdots \times \sigma_E\nu_E^{k-1}$ admits a $G_n(E)$-invariant lattice, say $\mathcal{L}'$, which is induced via $\mathcal{L}_0$. Then $\mathcal{L} = \mathcal{L}' \cap Z(\Delta_E)$ is a $G_n(E)$-invariant lattice in $Z(\Delta_E)$. Now, the map S_γ induces an isomorphism $T_\gamma : Z(\Delta_E) \to Z(\Delta_E)^\gamma$ such that $T_\gamma^l = \mathrm{id}$ and T_γ stabilizes $\mathcal{L}$. Moreover, choosing a $G_m(E)$-invariant, S_γ-stable lattice in σ_E is equivalent to choosing a $G_n(E)$-invariant, T_γ-stable lattice in $Z(\Delta_E)$.

Remark 7.1. Since σ_E is cuspidal, the restriction $\sigma_E|_{P_m(E)}$ is isomorphic to the compact induction $\mathcal{K}(\psi_E)$ as $\overline{\mathbb{Q}}_l$ representations. So, the restriction of $\mathcal{L}_0$ to the subgroup $P_m(E)$ is isomorphic to the space of $\overline{\mathbb{Z}}_l$ valued functions in $\mathcal{K}(\psi_E)$. This implies that $\widehat{H}^1(\mathcal{L}_0) = 0$. For details, see [Ron16, Theorem 6].

From this, we now deduce the following result.

Proposition 7.2. *Let $\mathcal{L}$ be a lattice in $Z(\Delta_E)$ that is stable under the action of both $G_n(E)$ and T_γ. Then we have $\widehat{H}^1(\mathcal{L}) = 0$.*

Proof. We prove the above claim using induction on $\ell(\Delta_E)$. If the length of Δ_E is 1, then the proposition clearly follows from Remark 7.1. Recall that

$$Z(\Delta_E)|_{P_n(E)} \simeq (\Phi^+)^{m-1} \circ \Psi^+(Z(\Delta_E^-)).$$

Here, $Z(\Delta_E)^-$ is identified with the m-th derivative of $Z(\Delta_E)$. The m-th derivative is the composition of the following maps:

$$Z(\Delta_E) \xrightarrow{r_{N_{n-m,m}(E)}} Z(\Delta_E^-) \otimes \sigma_E \xrightarrow{\mathrm{id} \otimes r_{N_m(E),\Theta_E}} Z(\Delta_E^-),$$

where $r_{N_{n-m,m}(E)}$ and $\mathrm{id} \otimes r_{N_m(E),\Theta_E}$ are the natural quotient maps of corresponding Jacquet module and twisted Jacquet module. Note that the maps $r_{N_{n-m,m}(E)}$ and $r_{N_m(E),\Theta_E}$ preserve integral structures (see [Dat05, Proposition 1.4(i)] for $r_{N_{n-m,m}(E)}$, and [Vig04, Theorem III.2] for $r_{N_m(E),\Theta_E}$). Thus, we get that $\mathcal{L}^-$ defined as

$$\mathcal{L}^- = \Psi^- \circ (\Phi^-)^{m-1}(\mathcal{L})$$

is a lattice in $Z(\Delta_E^-)$. We have the following isomorphism of $P_n(E) \rtimes \Gamma$-modules

$$\mathcal{L}|_{P_n(E)} \simeq (\Phi^+)^{m-1} \circ \Psi^+(\mathcal{L}^-).$$

Applying Proposition 5.1 repeatedly $(m-1)$-times, we get that

$$\widehat{H}^1(\mathcal{L}) \simeq (\Phi^+)^{m-1} \circ \Psi^+\big(\widehat{H}^1(\mathcal{L}^-)\big). \tag{7.1}$$

When k is 2, we have $\Delta_E = \{\sigma_E, \sigma_E\nu_E\}$ and in this case, the representation $Z(\Delta_E^-)$ is equal to σ_E and $\mathcal{L}^-$ is a $G_m(E) \rtimes \Gamma$-stable lattice in σ_E. Then, it follows from Remark 7.1 and the isomorphism (7.1) that $\widehat{H}^1(\mathcal{L}) = 0$. Suppose the result is true for all $Z(\Delta)$'s where the length of Δ is strictly less than k. Recall that the length of Δ_E^- is $k-1$. By induction hypothesis, we have $\widehat{H}^1(\mathcal{L}^-)=0$. Then, using (7.1), we get that $\widehat{H}^1(\mathcal{L})=0$. □

7.4. We now recall the mod-l reduction of the representation $Z(\Delta_F)$. Let us introduce the following notations:

$$r_l(\Delta_F)=\{r_l(\sigma_F), r_l(\sigma_F)\overline{\nu}_F,\ldots,r_l(\sigma_F)\overline{\nu}_F^{k-1}\}$$

and

$$r_l(\Delta_F)^{(l)}=\{r_l(\sigma_F)^{(l)},(r_l(\sigma_F)\overline{\nu}_F)^{(l)},\ldots,(r_l(\sigma_F)\overline{\nu}_F^{k-1})^{(l)}\},$$

where $r_l(\sigma_F)$ is the mod-l reduction of σ_F and $r_l(\sigma_F)^{(l)}$ is the Frobenius twist of $r_l(\sigma_F)$. Then the mod-l reduction of $Z(\Delta_F)$ ([MS14, Theorem 9.39]) is given by

$$r_l(Z(\Delta_F))=Z(r_l(\Delta_F)).$$

This shows, in particular, that $r_l(Z(\Delta_F))$ is irreducible. Moreover, the Frobenius twist of $r_l(Z(\Delta_F))$ equals $Z(r_l(\Delta_F)^{(l)})$. We conclude this section with the following theorem.

Theorem 7.3. *Let E/F be a finite Galois extension with $[E:F]=l$, where l and p are distinct primes such that l does not divide $|G_{n-1}(\mathbb{F}_q)|$. Let σ_F be an integral cuspidal l-adic representation of $G_m(F)$, and let σ_E be an integral l-adic representation of $G_m(E)$ obtained as a base change of σ_F (Note that σ_E is also cuspidal). Let $\Delta_F=\{\sigma_F,\sigma_F\nu_F,\ldots,\sigma_F\nu_F^{k-1}\}$ and $\Delta_E=\{\sigma_E,\sigma_E\nu_E,\ldots,\sigma_E\nu_E^{k-1}\}$ be two segments (Here $n=km$). Then we have*

$$\widehat{H}^0(r_l(Z(\Delta_E)))\simeq r_l(Z(\Delta_F))^{(l)}.$$

Proof. We use induction on $\ell(\Delta_E)$. For $k=1$, we have $Z(\Delta_E)=\sigma_E$ and $Z(\Delta_F)=\sigma_F$, and the theorem follows from Corollary 6.9. Suppose the result is true for all segments Δ'_F and Δ'_E with $\ell(\Delta'_F)=\ell(\Delta'_E)<k$. Let τ_F and τ_E be the mod-l Zelevinsky representations $Z(r_l(\Delta_F))$ and $Z(r_l(\Delta_E))$, respectively. We denote by τ_F^- and τ_E^- the mod-l representations $Z(r_l(\Delta_F^-))$ and $Z(r_l(\Delta_E^-))$ respectively. Since the restriction $\tau_E|_{P_n(E)}$ is isomorphic to $(\Phi^+)^{m-1}\circ\Psi^+(\tau_E^-)$, it follows from Proposition 5.1 that

$$\widehat{H}^0(\tau_E|_{P_n(E)})\simeq(\Phi^+)^{m-1}\circ\Psi^+(\widehat{H}^0(\tau_E^-)). \tag{7.2}$$

By induction hypothesis, we have

$$\widehat{H}^0(\tau_E^-)\simeq(\tau_F^-)^{(l)}.$$

Thus it follows from (7.2) and [Vig96, Chapter 3, 1.5] that $\widehat{H}^0(\tau_E)$ is an irreducible representation of $P_n(F)$ and hence irreducible as a representation of $G_n(F)$. Let $\lambda=(m,m,\ldots,m)$ be the partition of n and let $P_\lambda=M_\lambda N_\lambda$ be the parabolic subgroup of G_n. The isomorphism implies that $\widehat{H}^0(\tau_E)_{N_\lambda(F)}$ is non-zero. Then, using Lemma 6.2 and Corollary 6.9, we get the following isomorphism of $M_\lambda(F)$-representations

$$(\widehat{H}^0(\tau_E))_{N_\lambda(F)}\simeq((\tau_F^-)^{(l)})_{N_\lambda(F)}. \tag{7.3}$$

The irreducibility of $\widehat{H}^0(\tau_E)$ and the isomorphism (7.3) implies that

$$\widehat{H}^0(\tau_E)\simeq\tau_F^{(l)}$$

as a representation of $G_n(F)$ (see [Vig98, Proposition V.9.1]). □

8. Irreducibility of Tate Cohomology of generic representations

In this section, we discuss the Tate cohomology groups of representations of the form $L(\Delta)$, where $L(\Delta)$ is defined in subsection (2.7.2). We assume that l does not divide the pro-order of $G_n(F)$. We continue with the notation that σ_F is an l-adic cuspidal representation of $G_n(F)$ and σ_E is the base change lift of π_F to $G_n(E)$.

8.1. Keep the notations as in subsection (7.2). Recall that $L(\Delta_E)$ is the unique generic quotient of the parabolically induced representation $\sigma_E \times \sigma_E\nu_E \times \cdots \times \sigma_E\nu_E^{k-1}$. Now fix a $G_m(E)$-invariant lattice $\mathcal{L}_0$ in σ_E. Then we have the $G_n(E)$-invariant lattice $\mathcal{L}_0 \times \cdots \times \mathcal{L}_0$ in $\sigma_E \times \cdots \times \sigma_E\nu_E^{k-1}$, and the image of $\mathcal{L}_0 \times \cdots \times \mathcal{L}_0$ under the surjection

$$\sigma_E \times \sigma_E\nu_E \times \cdots \times \sigma_E\nu_E^{k-1} \longrightarrow L(\Delta_E),$$

say $\mathcal{L}$, is again a $G_n(E)$-invariant lattice in $L(\Delta_E)$. As in subsection (7.3), an isomorphism between σ_E and σ_E^γ induces an isomorphism $T_\gamma : L(\Delta_E) \to L(\Delta_E)^\gamma$ with $T_\gamma^l = \mathrm{id}$ and $T_\gamma(\mathcal{L}) = \mathcal{L}$. Here, the group Γ acts on the lattice $\mathcal{L}$ by T_γ.

Proposition 8.1. *Let $\mathcal{L}$ be a lattice in $L(\Delta_E)$ that is stable under the action of $G_n(E)$ and T_γ. Then $\widehat{H}^1(\mathcal{L}) = 0$.*

Proof. We proceed by induction on $\ell(\Delta_E)$, which equals k. When $\ell(\Delta_E) = 1$, then $L(\Delta_E) = \sigma_E$. In this case, the proposition follows from [Ron16, Theorem 6]. Suppose the result is true for all representations $L(\Delta)$, where $\ell(\Delta)$ is strictly less than k. Let τ be the restriction $\mathrm{res}_{P_n(E)}(L(\Delta_E))$. Consider the filtration of $P_n(E)$-representations:

$$(0) \subseteq \tau_n \subseteq \cdots \subseteq \tau_2 \subseteq \tau_1 = \tau,$$

where $\tau_i/\tau_{i+1} = (\Phi^+)^{i-1} \circ \Psi^+(\tau^{(i)})$ and $\tau^{(i)} = \Psi^- \circ (\Phi^-)^{i-1}(\tau)$. The map T_γ induces an isomorphism between $\tau^{(i)}$ and $(\tau^{(i)})^\gamma$, and also between the representations τ_i and τ_i^γ. Hence, there is an action of Γ on both $\tau^{(i)}$ and τ_i. From [Zel80, Proposition 9.6], we get that

$$\tau^{(j)} = 0, \text{ if } j \text{ is not divisible by } m, \text{ and}$$

$$\tau^{(rm)} = L\big(\{\sigma_E\nu_E^r, \ldots, \sigma_E\nu_E^{k-1}\}\big), \text{ for } r = 0, 1, \ldots, k-1.$$

For each $r \in \{1, 2 \ldots, k-1\}$, let Δ_E' and Δ_E'' be the segments $\{\sigma_E\nu_E^r, \ldots, \sigma_E\nu_E^{k-1}\}$ and $\{\sigma_E, \ldots, \sigma_E\nu_E^{r-1}\}$ respectively. The rm-th derivative of $L(\Delta_E)$ is the composition of the following maps:

$$\tau \xrightarrow{r_{N_{n-rm,rm}(E)}} L(\Delta_E') \otimes L(\Delta_E'') \xrightarrow{\mathrm{id} \otimes r_{N_{rm}(E),\Theta_E}} \tau^{(rm)},$$

where $r_{N_{n-rm,rm}(E)}$ and $\mathrm{id} \otimes r_{N_{rm}(E),\Theta_E}$ are the quotient maps of the corresponding Jacquet module and twisted Jacquet module. Note that the maps $r_{N_{n-rm,rm}(E)}$ and $r_{N_{rm}(E),\Theta_E}$ preserve integral structures (see [Dat05, Proposition 1.4(i)] and [Vig04, Theorem III.2]). Thus, we get that $\mathcal{L}^{(rm)}$, defined as

$$\mathcal{L}^{(rm)} = \Psi^- \circ (\Phi^-)^{rm-1}(\mathcal{L}),$$

is a $G_{n-rm}(E) \rtimes \Gamma$-invariant lattice in $L(\Delta_E)^{(rm)}$. Let $\mathcal{L}_i \subset \tau_i$ be the $P_n(E) \rtimes \Gamma$-invariant $\overline{\mathbb{Z}}_l$-lattice

$$\mathcal{L}_i = (\Phi^+)^{i-1} \circ (\Phi^-)^{i-1}(\mathcal{L}),$$

for all $1 \leq i \leq n$. For $1 \leq r \leq k-1$, we have the short exact sequence of $P_n(E) \rtimes \Gamma$-modules

$$0 \longrightarrow \mathcal{L}_{(r+1)m} \longrightarrow \mathcal{L}_{rm} \longrightarrow (\Phi^+)^{rm-1} \circ \Psi^+(\mathcal{L}^{(rm)}) \longrightarrow 0 \tag{8.1}$$

By induction hypothesis, we have $\widehat{H}^1(\mathcal{L}^{(rm)}) = 0$. Then the long exact sequence of Tate cohomology corresponding to (8.1) gives

$$\cdots \longrightarrow \widehat{H}^1(\mathcal{L}_{(r+1)m}) \longrightarrow \widehat{H}^1(\mathcal{L}_{rm}) \longrightarrow 0 \longrightarrow \widehat{H}^0(\mathcal{L}_{(r+1)m}) \longrightarrow \cdots$$

For $r = k-1$, the representation τ_n, is equal to $\mathrm{ind}_{N_n(E)}^{P_n(E)}\Theta_E$. In this case, $\widehat{H}^1(\mathcal{L}_n) = 0$ by Proposition 5.1. Then, from the above long exact sequence, we get that $\widehat{H}^1(\mathcal{L}_{(k-1)m}) = 0$. Again using the above long exact sequence for $r = k-2$, we get that $\widehat{H}^1(\mathcal{L}_{(k-2)m}) = 0$. Thus, an inductive process gives

$$\widehat{H}^1(\mathcal{L}) = \widehat{H}^1(\mathcal{L}_m) = 0.$$

□

8.2. Let π_E be a generic, integral l-adic representation of $G_n(E)$. Then π_E is of the form

$$\mathcal{L}(\Delta_1) \times \mathcal{L}(\Delta_2) \times \cdots \times \mathcal{L}(\Delta_t),$$

where for each $j \in \{1, 2, \dots, t\}$, the representation $\mathcal{L}(\Delta_j)$ is integral. Let $\mathcal{L}_j$ be a lattice in $L(\Delta_j)$, defined as in subsection (8.1). Let $T_{\gamma,j}$ be the isomorphism between $L(\Delta_j)$ and $L(\Delta_j)^\gamma$ such that $T_{\gamma,j}(\mathcal{L}_j) = \mathcal{L}_j$. Now consider the $\overline{\mathbb{Z}}_l$-module $\mathcal{L} = \mathcal{L}_1 \times \cdots \times \mathcal{L}_t$. Then $\mathcal{L}$ is a lattice in π_E that is stable under the action of $G_n(E)$. Moreover, we have an isomorphism $T_\gamma : \pi_E \to \pi_E^\gamma$, induced by $\left\{T_{\gamma,j}\right\}_{j=1}^t$, such that $T_\gamma(\mathcal{L}) = \mathcal{L}$. Note that l is banal for $G_n(E)$. Since the mod-l reduction of π_E is irreducible, any lattice in π_E is homothetic to $\mathcal{L}$.

Corollary 8.2. *Assume that l does not divide $|G_n(\mathbb{F}_q)|$. Let π_E be a generic, integral l-adic representation of $G_n(E)$ as above. Let $\mathcal{L}$ be a lattice in π_E that is stable under the action of $G_n(E)$ and T_γ. Then $\widehat{H}^1(\mathcal{L}) = 0$.*

Proof. Using Proposition 5.1, we have

$$\widehat{H}^1(\mathcal{L}) = \widehat{H}^1(\mathcal{L}_1) \times \cdots \times \widehat{H}^1(\mathcal{L}_t).$$

Now applying Proposition 8.1, we get that $\widehat{H}^1(\mathcal{L}_i) = 0$, for each i. Hence, the theorem. □

Theorem 8.3. *Let E/F be a finite Galois extension with $[E : F] = l$, where l and p are distinct primes such that l does not divide $|G_n(\mathbb{F}_q)|$. Let σ_F be an integral cuspidal l-adic representation of $G_m(F)$, and let σ_E be an integral cuspidal l-adic representation of $G_m(E)$ obtained as a base change of σ_F. Let $\Delta_F = \left\{\sigma_F, \sigma_F\nu_F, \dots, \sigma_F\nu_F^{k-1}\right\}$ and $\Delta_E = \left\{\sigma_E, \sigma_E\nu_E, \dots, \sigma_E\nu_E^{k-1}\right\}$ be two segments (Here $n = km$). Then*

$$\widehat{H}^0(r_l(L(\Delta_E))) \simeq r_l(L(\Delta_F))^{(l)}.$$

Proof. We prove the theorem using induction on $\ell(\Delta_F)$. Since l does not divide $|G_n(\mathbb{F}_q)|$, the mod-l reduction of the irreducible integral representations $L(\Delta_F)$ and $L(\Delta_E)$ are also irreducible, and we have

$$r_l(L(\Delta_F)) = L(r_l(\Delta_F))$$

and

$$r_l(L(\Delta_E)) = L(r_l(\Delta_E))$$

where $r_l(\Delta)$'s are defined as in subsection (7.3). Using the long exact sequence in Tate cohomology for the exact sequence (8.1) we get a filtration

$$\mathrm{res}_{P_n(F)}\, \widehat{H}^0(r_l(L(\Delta_E))) = \eta_1 \supseteq \eta_2 \supseteq \cdots \supseteq \eta_n,$$

such that $\eta_i/\eta_{i+1} \neq 0$ if and only if i is a multiple of m. By induction hypothesis, we get that $\eta_{ms}/\eta_{m(s+1)}$ is an irreducible representation of $P_n(F)$. Theorem 6.7 says that the Frobenius twist $r_l(L(\Delta_F))^{(l)}$ is the unique generic sub-quotient of $\widehat{H}^0(r_l(L(\Delta_E)))$. Since the lengths of $P_n(F)$ representations $\widehat{H}^0(r_l(L(\Delta_E)))$ and $r_l(L(\Delta_F))$ are the same, we get that $r_l(L(\Delta_F))^{(l)}$ is isomorphic to $\widehat{H}^0(r_l(L(\Delta_E)))$. □

Let us continue with the hypothesis as in Theorem 8.3. Let π be an integral l-adic generic smooth representation of $G_n(E)$. Since l does not divide $|G_n(\mathbb{F}_q)|$, the mod-l-reduction $r_l(\pi)$ is irreducible, and hence generic. Then we have

Corollary 8.4. *Let E/F be a finite Galois extension with $[E : F] = l$, where l and p are distinct primes such that l does not divide $|G_n(\mathbb{F}_q)|$. Let π_F be an integral l-adic generic representation of $G_n(E)$, and let π_E be a base change lift of π_F (Note that $\pi_E \simeq \pi_E^\gamma$). Then*

$$\widehat{H}^0(r_l(\pi_E)) \simeq r_l(\pi_F)^{(l)}.$$

Proof. This follows from Proposition 5.1 and Theorem 8.3. □

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