A NOTE ON BRANCHING OF $V(\rho)$

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Abstract. Let $\mathfrak{g}$ be a simple Lie algebra and let $\mathfrak{g}_0$ be a sub-algebra fixed by a diagram automorphism of $\mathfrak{g}$. Let G be a complex, simply-connected, simple algebraic group with Lie algebra $\mathfrak{g}$, and let G_0 be the connected subgroup of G with Lie algebra $\mathfrak{g}_0$. Let ρ be the half sum of positive roots of $\mathfrak{g}$. In this article, we give a necessary and sufficient condition for a highest weight $\mathfrak{g}_0$-representation $V(d\mu)$ to occur in the representation $\mathrm{res}_{\mathfrak{g}_0} V(d\rho)$, for any saturation factor d of the pair (G_0, G).

1. Introduction

Let $\mathfrak{g}$ be a complex simple Lie algebra, and let ρ be the half sum of positive roots, for some choice of positive roots of the root system of $\mathfrak{g}$ with respect to a Cartan subalgebra $\mathfrak{h}$ and a Borel subalgebra $\mathfrak{b}$, with $\mathfrak{h} \subset \mathfrak{b}$. Let μ be a dominant weight with respect to this chosen set of positive roots, and let $V(\mu)$ be the irreducible representation of $\mathfrak{g}$ with highest weight μ. A conjecture of Kostant predicts that the representation $V(\mu)$ occurs in $V(\rho) \otimes V(\rho)$ for all $\mu \preceq 2\rho$ with respect to the dominance order. In the article [CKM17], the authors show that $V(d\mu)$ occurs in $V(d\rho) \otimes V(d\rho)$ for all $\mu \preceq 2\rho$, and for any *saturation factor* d associated to the Lie algebra $\mathfrak{g}$ (see [Kum14, Definition 11]). The results of the article [CKM17] can be considered as solving the restriction problem for the pair $(\mathfrak{g} = \mathfrak{g}_0 \times \mathfrak{g}_0, \Delta\mathfrak{g}_0)$, for some simple Lie algebra $\mathfrak{g}_0$. In this article, we consider the restriction problem for symmetric pairs associated to diagram automorphisms.

Let Φ be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Let $\{X_\alpha : \alpha \in \Phi\}$ be a Chevalley basis for the Lie algebra $\mathfrak{g}$. Let θ be a non-trivial automorphism fixing the pinning $(\mathfrak{g}, \mathfrak{b}, \mathfrak{h}, \{X_\alpha\})$. Let $\mathfrak{g}_0$ be the fixed point Lie sub-algebra $\{X \in \mathfrak{g} : \theta(X) = X\}$. Let $\mathfrak{h}_0$ be the Cartan sub-algebra $\mathfrak{g}_0 \cap \mathfrak{h}$ of $\mathfrak{g}_0$, and let $\mathfrak{b}_0$ be the Borel sub-algebra $\mathfrak{g}_0 \cap \mathfrak{b}$ of $\mathfrak{g}_0$. Let Φ_0 be the set of roots of $\mathfrak{g}_0$ with respect to $\mathfrak{h}_0$. Let ρ_0 be the half sum of positive roots for the choice of $\mathfrak{b}_0$ as a Borel sub-algebra of $\mathfrak{g}_0$. Let G be a complex, simply-connected, simple algebraic group with Lie algebra $\mathfrak{g}$, and let G_0 be the connected subgroup of G with Lie algebra $\mathfrak{g}_0$.

Let d be a saturation factor for the pair (G_0, G) (see [Kum14, Definition 12] or see Definition 2.1). Our main theorem gives a necessary and sufficient condition for a $\mathfrak{g}_0$-representation $V(d\mu)$ to occur in the restriction $\mathrm{res}_{\mathfrak{g}_0} V(d\rho)$. Let $p : \mathfrak{h}^* \to \mathfrak{h}_0^*$ be the restriction map. A subset Γ of Φ_0 is said to be a root subsystem if Γ is a root system in the Euclidean space spanned by Γ. We do not necessarily assume that Γ is closed in Φ. For instance, the set of short roots $\Phi_{0,s}$ of Φ_0 is a root subsystem of Φ_0 but it is not necessarily closed. In this note, we show that $V(d\mu)$ occurs in the restriction $\mathrm{res}_{\mathfrak{g}_0} V(d\rho)$ if and only if $\mu \preceq p(\rho)$ in the dominance order of $\Phi_{0,s}$ and $\mu - (2\rho_0 - p(\rho))$ is dominant in $\Phi_{0,s}$. Here, the ordering in $\Phi_{0,s}$ is induced by the Borel sub-algebra $\mathfrak{b}_0$ of $\mathfrak{g}_0$. The condition that $\mu \preceq p(\rho)$ is analogous to $\mu \preceq 2\rho$ in the tensor product case.

Kostant's conjecture is inspired by the $\mathfrak{g}$-representation on the exterior algebra $\wedge^\bullet\mathfrak{g}$. If $\mathfrak{g}$ has a diagram automorphism θ of order two, we set $\mathfrak{g}_1$ to be the -1 eigenspace of θ. We then get $\wedge^\bullet\mathfrak{g} = \wedge^\bullet\mathfrak{g}_0 \otimes \wedge^\bullet\mathfrak{g}_1$. Based on the work Kostant ([Kos97, Proposition 20, Corollary 36]) and Panyushev ([Pan01, Theorem 6.5]), we obtain that

$$\mathrm{res}_{\mathfrak{g}_0} V(\rho) \simeq V(\rho_0) \otimes V(p(\rho) - \rho_0) \tag{1.1}$$

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as $\mathfrak{g}_0$ modules. The results of this article are motivated by the above identity. Indeed if the saturation factor d of the pair (G_0, G) is equal to one, our results describe the decomposition of the tensor product $V(\rho_0) \otimes V(p(\rho) - \rho_0)$ of $\mathfrak{g}_0$-representations.

2. A decomposition of $V(\rho)$

Let B and T be a Borel subgroup and a maximal torus of G such that the Lie algebras of B and T are $\mathfrak{b}$ and $\mathfrak{h}$ respectively. Let Z be the center of G. Let $\Phi^\vee$ be the set of co-roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Let Φ^+ (resp. $(\Phi^\vee)^+$) be the set of positive roots (resp. co-roots) of Φ (resp. $\Phi^\vee$) for the choice of the Borel subalgebra $\mathfrak{b}$. Let Δ and $\Delta^\vee$ be a system of simple roots and simple co-roots in Φ and $\Phi^\vee$ respectively. We denote by B_0 and T_0 the groups $B \cap G_0$ and $T \cap G_0$ respectively. Let Φ_0^+ be the set of positive roots of Φ_0 with respect to the Borel subalgebra $\mathfrak{b}_0$, and let Δ_0 be the set of simple roots in Φ_0^+. Let $\mathrm{X}^*(T)$ (resp. $\mathrm{X}^*(T_0)$ be the character lattice of T (resp. T_0).

Let $P \subset \mathfrak{h}^*$ and $P_0 \subset \mathfrak{h}_0^*$ be the weight lattices of $\mathfrak{g}$ and $\mathfrak{g}_0$ respectively. Let P^+ be the set of dominant weights of $\mathfrak{g}$ with respect to the choice of $\mathfrak{h}$ and $\mathfrak{b}$. For $\lambda \in P^+$, let $V(\lambda)$ be the irreducible representation of $\mathfrak{g}$ with highest weight λ. We denote by $V(\lambda)_\mu$ the μ-weight subspace of $V(\lambda)$. The formal character associated to the representation $V(\lambda)$, defined by the formal sum $\sum_{\mu \in P} \dim(V(\lambda)_\mu) e^\mu$, is denoted by ch_λ. Similarly, let P_0^+ be the set of dominant weights of $\mathfrak{g}_0$ with respect to the choice of $\mathfrak{h}_0$ and $\mathfrak{b}_0$. Recall that we have $p : \mathfrak{h}^* \to \mathfrak{h}_0^*$, the restriction map.

Let $\Phi_{0,s}$ (resp. $\Phi_{0,l}$) be the set of short (resp. long) roots in Φ_0. Note that $\Phi_{0,s}$ and $\Phi_{0,l}$ are root subsystems of Φ_0, and the latter subsystem is closed. We set $\Phi_{0,s}^+$ to be the set $\Phi_0^+ \cap \Phi_{0,s}$. Let $P_{0,s}^+$ be the set of dominant weights of the sub-root system $\Phi_{0,s}$ with respect to the choice of positive roots $\Phi_{0,s}^+$. The set $P_{0,s}^+$ is contained in P_0^+. We write $\mu \preceq_s \lambda$, if $\lambda - \mu$ is a non-negative integral linear combination of positive short roots. Let $\rho_s = \frac{1}{2}\sum_{\beta \in \Phi_{0,s}^+} \beta$ and let $\rho_l = \frac{1}{2}\sum_{\beta \in \Phi_{0,l}^+} \beta$. Let $\Delta_{0,s}$ be the set $\Delta_0 \cap \Phi_{0,s}$ and let $\Delta_{0,l}$ be the set of simple roots $\Delta_0 \backslash \Delta_{0,s}$. We then have $\rho_s = \sum_{\alpha \in \Delta_{0,s}} \varpi_\alpha$ and $\rho_l = \rho_0 - \rho_s$. We note that $p(\rho) - \rho_0 = \rho_s$ if θ is of order 2 and $p(\rho) - \rho_0 = 2\rho_s$ if θ has order 3. Let $(\mathfrak{g}_s, \mathfrak{h})$ be the simple Lie algebra with its Cartan sub-algebra corresponding to the root system $\Phi_{0,s}$.

We come to crucial definition of saturation factor associated to the pair (G_0, G).

Definition 2.1. *A positive integer d is called a saturation factor for the pair (G_0, G) if for any $(\mu, \lambda) \in P_0^+ \times P^+$ such that $\mu(z)\lambda(z) = 1$, for all $z \in Z \cap G_0$, and $\mathrm{Hom}_{G_0}(V(N\mu), V(N\lambda)) \neq 0$ for some positive integer N, then $\mathrm{Hom}_{G_0}(V(d\mu), V(d\lambda)) \neq 0$.*

Let $C(G_0, G)$ be the set of all pairs $(\mu, \lambda) \in P_0^+ \times P^+$ such that $\mathrm{Hom}_{G_0}(V(N\mu), V(N\lambda)) \neq 0$ for some positive integer N. The set $C(G_0, G)$ is called an eigencone of the pair (G_0, G). The following theorem generalises the main result of [CKM17] on the components of the form $V(d\lambda)$ in the tensor product $V(d\rho) \otimes V(d\rho)$, where d is a saturation factor of the group G and $\lambda \in P^+$. We use Ressayre's work (see [Res10]) on saturation problem in the context of branching to prove the main theorem. Before stating the main theorem, we recall some notation to be used in the proof.

For any G_0-dominant one parameter subgroup δ in T_0, let $P_0(\delta)$ (resp. $P(\delta)$) be the parabolic subgroup associated to δ inside G_0 (resp. G). Let $W_{0,P_0(\delta)}$ (resp. $W_{P(\delta)}$) be the Weyl group of $P_0(\delta)$ (resp. $P(\delta)$) and let $W_0^{P_0(\delta)}$ (resp. $W^{P(\delta)}$) be the set of minimal length representatives in the cosets of $W_0/W_{0,P_0(\delta)}$ (resp. $W/W_{P(\delta)}$). Let $i : G_0/P_0(\delta) \to G/P(\delta)$ be the natural G_0-equivariant embedding and let $i^* : H^*(G/P(\delta), \mathbb{Z}) \to H^*(G_0/P_0(\delta), \mathbb{Z})$ be the corresponding morphism of cohomology rings. For $w' \in W_0^{P_0(\delta)}$ (resp. $w \in W^{P(\delta)}$), let $[X_{w'}] \in H^*(G_0/P_0(\delta), \mathbb{Z})$ (resp. $[X_w] \in H^*(G_0/P_0(\delta), \mathbb{Z})$) be the cohomology class of the sub-variety $X_{w'} := \overline{B_0 w' P_0(\delta)} \subseteq G_0/P_0(\delta)$ (resp. $X_w := \overline{BwP(\delta)} \subseteq G/P(\delta)$). A one parameter subgroup in T_0 is said to be admissible if the hyperplane of $\mathrm{X}^*_{\mathbb{R}}(T_0) := \mathrm{X}^*(T_0) \otimes \mathbb{R}$ which it defines is generated by weights of the representation of T_0 in $\mathfrak{g}/\mathfrak{g}_0$ (see [Bri13, Définition 4.7]).

Theorem 2.2. *Let d be a saturation factor of the pair (G_0, G), and let $\mu \in P_0^+$. The representation $V(d\mu)$ occurs in $\operatorname{res}_{\mathfrak{g}_0} V(d\rho)$ if and only if $\mu = \rho_0 + \beta$ where β occurs as a weight of the $\mathfrak{g}_0$-representation $V(p(\rho) - \rho_0)$.*

Proof. We will use the results proved by Ressayre (see [Res10, Theorem A]), to show that for any dominant weight μ such that $\mu = \rho_0 + \beta$ with $V(p(\rho) - \rho_0)_\beta \neq 0$, the representation $V(d\mu)$ occurs in the representation $\operatorname{res}_{G_0} V(d\rho)$, where d is any saturation factor for the pair (G_0, G). We will use the exposition in Brion's Bourbaki article [Bri13].

The pair $(\lambda, \rho) \in P_0^+ \times P^+$ belongs to $C(G_0, G)$ if and only if, for any admissible and dominant one parameter subgroup δ of T_0 and for any $(w', w) \in W_0^{P_0(\delta)} \times W^{P(\delta)}$ such that

1. $i^*([X_w]).[X_{w'}] = [X_e] \in H^*(G_0/P_0(\delta), \mathbb{Z})$ and
2. $(p(w^{-1}\rho) + p(\rho))(\dot{\delta}) = (\rho_0 - w'^{-1}\rho_0)(\dot{\delta})$,

the following inequalities are satisfied:

$$I_{\delta,w',w} : (p(w^{-1}\rho) + w'^{-1}\lambda)(\dot{\delta}) \leq 0.$$

For the above result we refer to [Bri13, Théorèm 4.9]. We will show that condition (2) already implies the inequalities $I_{\delta,w',w}$, and we will not explicitly use the cohomological condition (1). Note that condition (1) implies that

$$(p(w^{-1}\rho) + p(\rho))(\dot{\delta}) \leq (\rho_0 - w'^{-1}\rho_0)(\dot{\delta}).$$

(see [RR11, Proposition 2.3] and [Kum14, equations (30)-(31)]). For the above results one may as well refer to the survey article: [Kum14, Theorem 33].

Let $(w', w) \in W_0^{P_0(\delta)} \times W^{P(\delta)}$ and δ be as above. Since $(\rho_0 - w'^{-1}\rho_0)(\dot{\delta})$ is equal to $(p(w^{-1}\rho) + p(\rho))(\dot{\delta})$, we have

$$(p(w^{-1}\rho) + w'^{-1}\lambda)(\dot{\delta}) = (\rho_0 - w'^{-1}\rho_0 - p(\rho) + w'^{-1}\lambda)(\dot{\delta}).$$

We apply these results to the present case, i.e., $\lambda = \mu$, where $\mu = \rho_0 + \beta$ and $V(p(\rho) - \rho_0)_\beta \neq 0$. Note that $\mu \in X^*(T_0) \subseteq P_0$. We have, $w'^{-1}\mu = w'^{-1}\rho_0 + w'^{-1}\beta$. Since $V(p(\rho) - \rho_0)_{w'^{-1}\beta} \neq 0$ we have $w'^{-1}\beta \preceq p(\rho) - \rho_0$. Thus,

$$\begin{aligned}(p(w^{-1}\rho) + w'^{-1}\mu)(\dot{\delta}) =& (\rho_0 - w'^{-1}\rho_0 - p(\rho) + w'^{-1}\rho_0 + w'^{-1}\beta)(\dot{\delta}) \\ &\leq (\rho_0 - w'^{-1}\rho_0 - p(\rho) + w'^{-1}\rho_0 + p(\rho) - \rho_0)(\dot{\delta}) = 0.\end{aligned}$$

Hence, (μ, ρ) belongs to the eigencone $C(G_0, G)$. From the assumption that $\mu = \rho_0 + \beta$, where $V(p(\rho) - \rho_0)_\beta \neq 0$, we get that $p(\rho) - \mu$ is contained in the root lattice of G_0. Thus we get that $\mu(z)\rho(z) = 1$, for all $z \in Z \cap G_0$. We conclude that $V(d\mu)$ occurs as a sub-representation of $\operatorname{res}_{\mathfrak{g}_0} V(d\rho)$, for any saturation factor d of the pair (G_0, G).

Conversely, assume that $V(d\mu)$ is contained in the representation $\operatorname{res}_{\mathfrak{g}_0} V(d\rho)$. Note that $\Phi_0^+ = \Phi_{0,s}^+ \cup \Phi_{0,l}^+$. From the Weyl Character formula, we have

$$p(ch_{d\rho}) = \prod_{\beta \in \Phi_0^+} (e^{d\beta/2} + \cdots + e^{-d\beta/2})^{m(\beta)},$$

where $m(\beta)$ is the order of θ if β is short, and $m(\beta) = 1$ if β is long. Hence, we get that

$$p(ch_{d\rho}) = \prod_{\beta \in \Phi_{0,s}^+} (e^{d\beta/2} + \cdots + e^{-d\beta/2})^{m(\beta)-1} ch_{d\rho_0}.$$

Since $p(ch_{d\rho})/ch_{d\rho_0}$ is invariant under the Weyl group, W_0, of G_0, we get that

$$\prod_{\beta \in \Phi_{0,s}^+} (e^{d\beta/2} + \cdots + e^{-d\beta/2})^{m(\beta)-1} = \sum_i n_i ch_{\mu_i},$$

for some $n_i \in \mathbb{Z}$ and $\mu_i \preceq dp(\rho) - d\rho_0$. Therefore, we get that $V(d\mu)$ is contained in $V(\mu_i) \otimes V(d\rho_0)$, where $\mu_i \preceq dp(\rho) - d\rho_0$. Hence, the weight $d\mu$ is equal to $d\rho_0 + \beta$ where $V(\mu_i)_\beta \neq 0$. Now $d\mu - d\rho_0$ belongs to the convex hull of $W_0(\mu_i)$. Then, $d\mu - d\rho_0$ belongs to the convex hull of $W_0(dp(\rho) - d\rho_0)$.

Thus we get that $\mu - \rho_0$ belongs to the convex hull of $W_0(p(\rho) - \rho_0)$. Since $\mu - \rho_0$ is a weight, and it belongs to $W_0(p(\rho) - \rho_0)$, we get that $\mu - \rho_0$ is comparable to $p(\rho) - \rho_0$ (see [Hal03, Proposition 8.44]). Thus, $V(p(\rho) - \rho_0)_{\mu-\rho_0} \neq 0$. Here, $V(p(\rho) - \rho_0)$ is considered as a $\mathfrak{g}_0$-representation. This completes the proof of the theorem. □

Remark 2.3. The above theorem holds true for any automorphism of order 2 with some modification in the set up. We need to choose a θ-stable Cartan subalgebra $\mathfrak{h}$ and a θ-stable Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ and the rest of the notations are as defined above. The proof is also the same. However, in what follows, we focus on the case of diagram automorphisms–in which case we describe the weights β occurring in the $\mathfrak{g}_0$-representation $V(p(\rho) - \rho_0)$ such that $\rho_0 + \beta$ is dominant.

Remark 2.4. Let us consider the case where $\mathfrak{g}$ is a simple Lie algebra of type D_4, and let $\mathfrak{g}_0$ be a sub-algebra of type G_2, fixed by a triality automorphism. Using the methods as in the above theorem, we can see that the irreducible sub-representations of $\mathrm{res}_{\mathfrak{g}_0} V(\rho)$ are exactly the sub-representations of

$$V(\rho_0) \otimes V(p(\rho) - \rho_0).$$

In this small rank case, using Klimyk's formula, we check that the irreducible sub-representations which occur in the above tensor product are precisely the following set:

$$\{V(\rho_0 + \beta) : \rho_0 + \beta \in P_0^+ \text{ and } V(p(\rho) - \rho_0)_\beta \neq 0\}.$$

In the following Lemma and Proposition, we assume that θ is of order 2. The following lemma gives a characterisation of the weights occurring in the $\mathfrak{g}_0$-representation $V(p(\rho) - \rho_0)$. For any subset $\Psi \subseteq \Phi_0$, we denote by $\mathrm{sum}(\Psi)$ the vector $\sum_{v\in\Psi} v$.

Lemma 2.5. *The space $V(p(\rho) - \rho_0)_\beta \neq 0$ if and only if $\beta = p(\rho) - \rho_0 - \mathrm{sum}(\Psi)$, where Ψ is a subset of $\Phi_{s,0}^+$.*

Proof. Note that the character of the $\mathfrak{g}_0$-representation $V(p(\rho) - \rho_0)$ is given by

$$\prod_{\beta\in\Phi_{s,0}^+} (e^{\beta/2} + e^{-\beta/2}).$$

The above is the formal character of the $\mathfrak{g}_s$-representation $V(\rho_s)$. The lemma now follows from the result [Kos61, Lemma 5.9] of Kostant. □

Proposition 2.6. *Let $\mu \in P_0^+$. Then, $V(p(\rho) - \rho_0)_{\mu-\rho_0} \neq 0$ if and only if $\mu \preceq_s p(\rho)$ and $\mu - (2\rho_0 - p(\rho)) \in P_{0,s}^+$.*

Proof. Assume that $\mu \preceq_s p(\rho)$. Then $\mu - (2\rho_0 - p(\rho)) \preceq_s p(\rho) - (2\rho_0 - p(\rho)) = 2\rho_s$. Since $\mu - (2\rho_0 - p(\rho)) \in P_{0,s}^+$, using the result [CKM17, Proposition 9] we get that

$$\mu - (2\rho_0 - p(\rho)) = \rho_s + \beta$$

where $V(\rho_s)_\beta \neq 0$. Here, $V(\rho_s)$ is the $\mathfrak{g}_s$-representation with highest weight ρ_s. Since the weights occurring in the $\mathfrak{g}_s$-representation $V(\rho_s)$ are the same as the weights occurring in the $\mathfrak{g}_0$-representation $V(p(\rho) - \rho_0)$, we have $\mu = \rho_0 + \beta$ where $V(p(\rho) - \rho_0)_\beta \neq 0$.

Conversely, assume that $\mu = \rho_0 + \beta$ where $V(p(\rho) - \rho_0)_\beta \neq 0$. By Lemma 2.5, we have $\beta = p(\rho) - \rho_0 - \mathrm{sum}(\Psi)$, where Ψ is a subset of $\Phi_{s,0}^+$. So $p(\rho) - \mu = p(\rho) - \rho_0 - \beta = \rho_s - \beta = \rho_s - p(\rho) + \rho_0 + \mathrm{sum}(\Psi) = \mathrm{sum}(\Psi)$. So, $p(\rho) - \mu$ is a sum of positive short roots. Hence $\mu \preceq_s p(\rho)$.

Write $\rho_0 = \rho_s + \rho_l$. Then $\mu = \rho_s + \rho_l + \beta$ and $\mu - (2\rho_0 - p(\rho)) = \rho_s + \beta$. For $\alpha \in \Delta_{0,s}$, we have $\langle \rho_l, \alpha\rangle = 0$, and hence,

$$\langle \rho_s + \beta, \alpha\rangle = \langle \mu - \rho_l, \alpha\rangle = \langle \mu, \alpha\rangle \geq 0.$$

This completes the proof of the proposition. □

Remark 2.7. If $p(\rho) = 2\rho_0$, for instance in the case where $(\mathfrak{g}, \mathfrak{g}_0)$ is of the form $(\mathfrak{g}_0 \times \mathfrak{g}_0, \Delta\mathfrak{g}_0)$, the hypotheses in Proposition 2.6 are equivalent to saying that $\mu \preceq 2\rho_0$ (see [CKM17, Proposition 9]). However, in the outer involution case, i.e., for the pair $(\mathfrak{g}, \mathfrak{g}_0)$, where $\mathfrak{g}_0$ is the fixed point sub-algebra of an outer automorphism of a simple Lie algebra $\mathfrak{g}$, there exist dominant weights μ

such that $p(\rho) - \mu$ is a non-negative integral linear combination of short positive roots in Φ_0 but $V(p(\rho) - \rho_0)_{\mu-\rho_0} = 0$.

Note that $p(\rho)$ is in the root lattice except in the case where $\mathfrak{g}$ is of the type A_{2n-1} and $\mathfrak{g}_0$ is of the type C_n, for n is odd. When $p(\rho)$ is in the root lattice, we take $\mu = 0$. In the case where the pair $(\mathfrak{g}, \mathfrak{g}_0)$ is of type (A_{2n-1}, C_n), for n is odd, we take $\mu = (2n-1)\varpi_1$; here, ϖ_1 is the first fundamental weight of $\mathfrak{g}_0$. Here the numbering of fundamental weights is according to the conventions in [Bou81, PLANCHE III]. Then it is easy to see that $p(\rho) - \mu$ is a non-negative integral linear combination of short positive roots. But, $V(p(\rho) - \rho_0)_{\mu-\rho_0} = 0$ since $p(\rho) - \mu$ is not of the form $p(\rho) - \rho_0 - \mathrm{sum}(\Psi)$, for any $\Psi \subseteq \Phi^+_{s,0}$.

Remark 2.8. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be the eigendecomposition of θ. From Kostant's description of $\wedge^\bullet \mathfrak{g}$, and Panyushev's work on the description of $\wedge^\bullet \mathfrak{g}_1$, we get that

$$\mathrm{res}_{\mathfrak{g}_0} V(\rho) \simeq V(\rho_0) \otimes V(p(\rho) - \rho_0)$$

as $\mathfrak{g}_0$ modules. If $V(\mu)$ occurs in $\mathrm{res}_{\mathfrak{g}_0} V(\rho)$, then $\mu = \rho_0 + \beta$, where $V(p(\rho) - \rho_0)_\beta \neq 0$. If $d = 1$, for instance, in the cases where $(\mathfrak{g}, \mathfrak{g}_0)$ are of the type: $(\mathrm{D}_n, \mathrm{B}_{n-1})$, $(\mathrm{E}_6, \mathrm{F}_4)$ and $(\mathrm{A}_{2n-1}, \mathrm{C}_n)$, for $2 \leq n \leq 5$, (see [PR13]) then the main theorem gives the converse of this statement. In general, using computations on SAGE, we observed that the representation $V(\mu)$ occurs in $\mathrm{res}_{\mathfrak{g}_0} V(\rho)$ if and only if $V(p(\rho) - \rho_0)_{\mu-\rho_0} \neq 0$.

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