Compatibility of Tate cohomology with local base change

A *Thesis Submitted*

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by

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CERTIFICATE

This is to certify that the work contained in this thesis entitled "***Compatibility of Tate cohomology with local base change***" by ***Sabyasachi Dhar***, has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

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October 2024

Declaration

This is to certify that the thesis titled "***Compatibility of Tate cohomology with local base change***" has been authored by me. It presents the research conducted by me under the supervision of ***Prof. Santosh Nadimpalli***.

To the best of my knowledge, it is an original work, both in terms of research content and narrative, and has not been submitted elsewhere, in part or in full, for a degree. Further, due credit has been attributed to the relevant state-of-the-art and collaborations (if any) with appropriate citations and acknowledgements, in line with established norms and practices.

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SYNOPSIS

Name of the Student:	**Sabyasachi Dhar**
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Principle of functoriality, proposed by Langlands, is a central problem in theory of automorphic representations. It involves transferring automorphic representations, Hecke eigenclasses between two reductive algebraic groups defined over a number field. This dissertation draws inspiration from the work of Treumann and Venkatesh ([TV16]), who established a functoriality lifting of mod-p Hecke eigenclasses between two reductive algebraic groups. They also made a conjecture which predicts that the functoriality lifting at ramified places is realized via Tate cohomology. In this thesis, we aim to study the conjecture in the context of cyclic local base change using the techniques coming from local converse theorem.

Let F be a finite extension of $\mathbb{Q}_p$, and let E be a finite Galois extension of F of prime degree l, where $l \neq p$. Fix a generator σ of the cyclic group $\mathrm{Gal}(E/F)$. Let $\overline{\mathbb{Q}}_l$ be an algebraic closure of $\mathbb{Q}_l$ with ring of integers $\overline{\mathbb{Z}}_l$. Let (π, W) be a smooth irreducible l-adic representation of $\mathrm{GL}_n(F)$, where smooth means every vector in $\overline{\mathbb{Q}}_l$-vector space W has an open stabilizer in the topological group $\mathrm{GL}_n(F)$–whose topology is induced from that of F. We further assume that π is integral, i.e., there exists a $\mathrm{GL}_n(F)$ stable $\overline{\mathbb{Z}}_l$-lattice $\mathcal{L}$ in W. We have the mod-l reduction of π, defined as the semi-simplification of the $\overline{\mathbb{F}}_l$-representation $(\pi, \mathcal{L} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l)$. Let (Π, V) be an irreducible, smooth l-adic representation of $\mathrm{GL}_n(E)$, obtained as a base change lift of π. Then Π is also integral and is isomorphic to the twisted representation Π^σ. Let Λ be the ring of integers of the maximal unramified extension of $\mathbb{Q}_l$ in $\overline{\mathbb{Q}}_l$. Let $\mathcal{M}$ be a smooth $\Lambda[\mathrm{GL}_n(E) \rtimes \langle\sigma\rangle]$ module in V. Then, we have the Tate cohomology groups $\widehat{H}^i(\sigma, \mathcal{M})$, $i \in \{0, 1\}$, defined as

$$\widehat{H}^0(\sigma, \mathcal{M}) := \frac{\mathrm{Ker}((1-\sigma)|_{\mathcal{M}})}{N\mathcal{M}},$$

$$\widehat{H}^1(\sigma, \mathcal{M}) := \frac{\mathrm{Ker}(N|_{\mathcal{M}})}{(1-\sigma)\mathcal{M}},$$

where $N = 1 + \sigma + \cdots + \sigma^{l-1}$, viewed as an element of the group ring generated by σ. For each i, the Tate cohomology $\widehat{H}^i(\sigma, \mathcal{M})$ is a mod-l representation of $\mathrm{GL}_n(F)$. Treumann–Venkatesh's conjecture predicts the relation between the mod-l reduction of π and the Tate cohomology groups $\widehat{H}^i(\sigma, \mathcal{M})$. Towards this, N.Ronchetti ([Ron16]) obtained some evidence for depth zero cuspidal representations of $\mathrm{GL}_n(F)$, which had been further extended by T.Feng ([Fen24], [Fen23]) in a more general context, using the recent advances due to the work of V.Lafforgue and Fargues–Scholze. In this thesis, we obtain more precise results. We use the machinery of Rankin–Selberg convolutions to prove the conjecture for certain classes of irreducible smooth representations of $\mathrm{GL}_n(F)$, namely generic and cuspidal representations. The key result is the following:

Theorem. *Let F be a finite extension of $\mathbb{Q}_p$, and let E be a Galois extension of F of degree l, where l and p are distinct odd primes. Let π be an integral, l-adic, generic smooth representation of $\mathrm{GL}_n(F)$ with $J_l(\pi)$ the unique generic sub–quotient of the mod-l reduction of π. Let Π be the base change lifting of π to $\mathrm{GL}_n(E)$.*

1. *Assume that l does not divide the pro-order of $\mathrm{GL}_{n-1}(F)$, for $n \geq 3$. Then the Frobenius twist $J_l(\pi)^{(l)} := J_l(\pi) \otimes_{\mathrm{Frob}} \overline{\mathbb{F}}_l$ occurs as the unique generic sub–quotient of $\widehat{H}^0(\sigma, J_l(\Pi))$.*

2. *Suppose π and Π are l-adic, cuspidal representations of $\mathrm{GL}_3(F)$ and $\mathrm{GL}_3(E)$ respectively, such that the central character ω_π of π is integral. Then the zeroth Tate cohomology group $\widehat{H}^0(\sigma, J_l(\Pi))$ is irreducible, and is isomorphic to $J_l(\pi)^{(l)}$.*

Our method follows the general framework of the proof of the local converse theorem, as in [JPPS79, Proposition 7.5.2] and [Hen93, Theorem 1.1]. The arguments essentially use Kirillov models and completeness of Whittaker models. Let ψ be a non-trivial additive character of F, and let ψ_E be the composition $\psi \circ \mathrm{Tr}_{E/F}$, where $\mathrm{Tr}_{E/F}$ is the trace function of the extension E/F. Let $\mathbb{W}(J_l(\Pi), \psi_E)$ be the Whittaker model $J_l(\Pi)$. By multiplicity one theorem, we get that $\mathbb{W}(J_l(\Pi), \psi_E)$ is stable under the action of $\langle\sigma\rangle$. Then we prove that the Tate cohomology $\widehat{H}^0(\sigma, \mathbb{W}(J_l(\Pi), \psi_E))$ realizes $J_l(\pi)^{(l)}$ as a sub–quotient. The following is the essential content. Let $\mathbb{K}(J_l(\Pi), \psi_E)$ be the Kirillov model of $J_l(\Pi)$, consisting of restrictions of functions in the space $\mathbb{W}(J_l(\Pi), \psi_E)$ to the mirabolic subgroup $P_n(E)$. We consider the following restriction to $P_n(F)$ map

$$\Phi : \mathbb{K}(J_l(\Pi), \psi_E)^{\mathrm{Gal}(E/F)} \xrightarrow{\mathrm{res}_{P_n(F)}} \mathrm{Ind}_{N_n(F)}^{P_n(F)}(\Theta^l),$$

where $N_n(F)$ is the group of unipotent upper triangular matrices in $\mathrm{GL}_n(F)$ and Θ is the non-degenerate character of $N_n(F)$, induced by ψ. Note that the map Φ is $P_n(F)$-equivariant and factors through

$$\widehat{H}^0(\sigma, \mathbb{K}(J_l(\Pi), \psi_E)) \longrightarrow \mathrm{Ind}_{N_n(F)}^{P_n(F)}(\Theta^l).$$

Let X_F be the space $\mathrm{GL}_{n-1}(F)/N_{n-1}(F)$, and let $\varphi \in C_c^\infty(X_F, \overline{\mathbb{F}}_l)$. Typically, for some fixed integer k and a suitably chosen function W in the Tate cohomology space $\widehat{H}^0(\sigma, \mathbb{K}(J_l(\Pi), \psi_E))$, the function φ is equal to

$$\left[\Phi\big(\widehat{H}^0(J_l(\Pi))(w)W\big) - J_l(\pi)^{(l)}(w)\Phi(W)\right]\left(\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right)$$

when $\det(g) = k$ and zero otherwise. Essentially we need to show that φ is zero. In view of completeness of Whittaker models, it is equivalent to prove that

$$\int_{X_F} \varphi(g)W'(g)\, d\mu = 0, \tag{1}$$

for all $W' \in \mathbb{W}(\tau, \psi^{-l})$ and for all generic $\overline{\mathbb{F}}_l$-representation τ of $\mathrm{GL}_{n-1}(F)$. The hypothesis on l in part 1 is required for the completeness of Whittaker models over $\overline{\mathbb{F}}_l$. Using Rankin–Selberg functional equations and the lifting of Rankin–Selberg local zeta integrals from the homogeneous space X_F to the homogeneous space X_E, the assertion (1) then reduces to showing the following identity of local γ-factors

$$\gamma(X, J_l(\Pi), \tau, \psi \circ \mathrm{Tr}_{E/F}) = \gamma(X^l, J_l(\pi)^{(l)}, \tau, \psi^l),$$

which follows from the congruence properties of γ-factors. When the prime l does not divide the pro-order of $\mathrm{GL}_n(F)$, we prove an analogous result for a subclass of non-generic representations of $\mathrm{GL}_n(F)$, namely Zelevinsky subrepresentations. In part 2, we remove the hypothesis on l by exploiting the theory of smooth representations over local Artinian $\overline{\mathbb{F}}_l$-algebras. The reason for considering representations over such rings is to use the local converse theorem in positive characteristic– which, in general, fails for mod-l generic representations. We basically prove the following equality

$$\gamma(X, \widehat{H}^0(\sigma, J_l(\Pi)), \eta, \psi) = \gamma(X, J_l(\pi)^{(l)}, \eta, \psi),$$

for all R-valued characters η of $F^\times$, and for all local Artinian $\overline{\mathbb{F}}_l$-algebras R. Then the local converse theorem over $\overline{\mathbb{F}}_l$ ([Mos21, Theorem 1.1]) gives the required isomorphism.

In this thesis, we also study the compatibility between two homomorphisms of Hecke algebras with coefficients in $\overline{\mathbb{F}}_l$. One of them is introduced by Treumann–Venkatesh ([TV16, Section 4]), and it is an analog of the Brauer homomorphism of modular representation theory. The other one is defined by D.Kazhdan ([Kaz86]). It relates the representation theory of reductive groups defined over a p-adic field F and a local non-Archimedean positive characteristic field F', provided the fields F and F' are sufficiently close. In [TV16, Section 6], the authors defined the notion of linkage, which is the counterpart of Brauer homomorphism in representation theory. Then, using the compatibility between Kazhdan and Brauer homomorphism, we show that the linkage is also compatible under the close local fields F and F'.

This dissertation consists of six chapters, including a general introduction. In Chapter 1, we discuss Treumann–Venkatesh's conjecture and the known results of the conjecture. It has a brief overview of each chapter. In Chapter 2, we recall the notion of smooth representations of p-adic groups and Weil–Deligne groups. It includes various notations, conventions, and some initial results. In Chapter 3, we prove the above theorem for cuspidal representations of $\mathrm{GL}_2(F)$ with no assumption on the odd prime l. Chapter 4 proves part 1 of the above theorem together with the non-generic case. Some crucial results on the Tate cohomology of (twisted) Jacquet modules, including the finite length of the Tate cohomology groups, are also discussed in this chapter. Part 2 of the above theorem is proved in Chapter 5, where we discuss some concepts of smooth representations over Artin local $\overline{\mathbb{F}}_l$-algebras which are relevant to our context. In Chapter 6, we study the compatibility between Kazhdan isomorphism and Brauer homomorphism.

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October 2024 Sabyasachi Dhar

Dedicated

To

My parents

Contents

List of Publications

1. *Santosh Nadimpalli and Sabyasachi Dhar*, ***Tate cohomology of Whittaker lattices and base change of generic representations of*** GL_n, *International Mathematics Research Notices*, DOI. (Chapter 3 and 4)

2. *Sabyasachi Dhar*, ***Compatibility of Tate cohomology with local base change for cuspidal representations of*** GL_3, arXiv. (Chapter 5)

3. *Sabyasachi Dhar*, ***Compatibility of Kazhdan isomorphism and Brauer homomorphism***, *Canadian Mathematical Bulletin*, DOI. (Chapter 6)

Chapter 1

Introduction

Let F be a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers. Let E be a finite Galois extension of degree l, where l and p are distinct primes. This thesis is concerned with the study of the compatibility of Tate cohomology with local base change lifting from $\mathrm{GL}_n(F)$ to $\mathrm{GL}_n(E)$. It is part of a conjecture on Langlands functoriality, made by D. Treumann and A. Venkatesh in their seminal paper [TV16]. We also prove some compatibility results under close local fields in the context of linkage.

1.1 Local Langlands correspondence

Let l and p be two distinct primes. Let F be a finite extension of $\mathbb{Q}_p$ and let $\overline{F}$ be an algebraic closure of F. Let F^{un} be the maximal unramified extension of F contained in $\overline{F}$. We have the quotient map

$$\mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{Gal}(F^{un}/F).$$

Let κ_F (resp. $\kappa_{F^{un}}$) be the residue field of F (resp. F^{un}). Then the group $\mathrm{Gal}(F^{un}/F)$ is isomorphic to the Galois group $\mathrm{Gal}(\kappa_{F^{un}}/\kappa_F)$. Since the field $\kappa_{F^{un}}$ is an algebraic closure of the finite field κ_F, we get a map

$$\mathrm{Gal}(\overline{F}/F) \to \mathrm{Gal}(\kappa_{F^{un}}/\kappa_F) \simeq \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q), \tag{1.1}$$

where q denotes the cardinality of κ_F and $\overline{\mathbb{F}}_q$ is an algebraic closure of $\mathbb{F}_q$. Let Frob_q be the Frobenius automorphism of $\overline{\mathbb{F}}_q$ which sends an element x to x^q. Let $\mathcal{W}_F$ be the inverse image in $\mathrm{Gal}(\overline{F}/F)$ of the group generated by Frob_q. The group $\mathcal{W}_F$ is called the *Weil group* of F.

The kernel of the map (1.1) is $\mathrm{Gal}(\overline{F}/F^{un})$ and it has a natural pro-finite topology from Krull topology. Declaring the open subgroups of $\mathrm{Gal}(\overline{F}/F^{un})$ as open subgroups of $\mathcal{W}_F$ makes $\mathcal{W}_F$ a topological group, whose topology is locally compact and totally disconnected. Moreover, there is a multiplicative norm $\| \ \|$ on $\mathcal{W}_F$, defined as follows: Let x be an element of $\mathcal{W}_F$. Let r be the integer such that the image of x under the map (1.1) is Frob_q^r. Then $\|x\| = q^{-r}$.

Fix an algebraic closure $\overline{\mathbb{Q}}_l$ of $\mathbb{Q}_l$ and a square root of q in $\overline{\mathbb{Q}}_l$. Local class field theory gives us a topological isomorphism $a_F : \mathcal{W}_F^{ab} \xrightarrow{\sim} F^\times$, where $\mathcal{W}_F^{ab}$ is the quotient of $\mathcal{W}_F$ by the closure of the derived subgroup of $\mathcal{W}_F$. The isomorphism a_F induces a bijective correspondence between the set of $\overline{\mathbb{Q}}_l$-valued characters of $\mathcal{W}_F$ (i.e., 1-dimensional representations with open kernel) and that of the multiplicative group $F^\times$. The local Langlands correspondence for the group GL_n is a generalization of the local class field theory. Let $\mathcal{G}_n(F)$ be the set of isomorphism classes of n-dimensional semisimple Weil–Deligne representations of $\mathcal{W}_F$–which are pairs (ρ, N), consisting of a semisimple representation $\rho : \mathcal{W}_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ with open kernel and a nilpotent $n \times n$ matrix N with entries in $\overline{\mathbb{Q}}_l$ such that

$$\rho(x)N\rho(x)^{-1} = \|x\|N,$$

for all $x \in \mathcal{W}_F$. Let $\mathcal{A}_n(F)$ be the set of isomorphism classes of irreducible smooth representations of $\mathrm{GL}_n(F)$ on $\overline{\mathbb{Q}}_l$-vector spaces, where smooth means every vector has an open stabilizer in the topological group $\mathrm{GL}_n(F)$–whose topology is induced from that of F. The local Langlands correspondence (see [HT01] and [Hen00]) gives a bijection

$$\mathcal{A}_n(F) \xrightarrow{\mathrm{LLC}_F} \mathcal{G}_n(F), \tag{1.2}$$

which is uniquely characterized by the certain identities (see Subsection 2.6.1) of local constants known as L-functions and ϵ-factors that one attaches to elements of $\mathcal{A}_n(F)$ and $\mathcal{G}_n(F)$. The constructions in [HT01], [Hen00] are originally over $\mathbb{C}$. By fixing an isomorphism $\iota : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_l$ and twisting the original correspondence by an unramified character of $\mathrm{GL}_n(F)$, we get the l-adic version of the classical local Langlands correspondence (see [Clo90, Conjecture 4.4, Section 4.2], [Hen01, Section 7]), which is independent of ι.

In the article [Vig01, Theorem 1.6], Vignéras constructed a mod-l version of local Langlands correspondence–which gives a map from the set of isomorphism classes of irreducible, smooth, $\overline{\mathbb{F}}_l$-representations of $\mathrm{GL}_n(F)$ to the set of isomorphism classes of n-dimensional, semisimple, $\overline{\mathbb{F}}_l$-representations of $\mathcal{W}_F$. We denote this map by $\mathrm{LLC}_{F,\ell}$, and it is compatible with the l-adic local Langlands correspondence LLC_F under reduction mod-l.

Local base change lifting

Let E be a finite Galois extension of the p-adic field F of degree l. The base change lifting is a transfer map from the set of irreducible smooth $\overline{\mathbb{Q}}_l$-representations of $\mathrm{GL}_n(F)$ to the set of irreducible smooth $\overline{\mathbb{Q}}_l$-representations of $\mathrm{GL}_n(E)$–which is constructed by Arthur and Clozel in their seminal work [AC89]. The base change lifting of smooth irreducible representations of $\mathrm{GL}_n(F)$ to $\mathrm{GL}_n(E)$ is defined by a character identity (see [AC89, Chapter 1, Section 6]).

The local base change lifting is also related to the local Langlands correspondence ([AC89, Chapter 3, Section 6]). Say π_F is an irreducible smooth $\overline{\mathbb{Q}}_l$-representation of $\mathrm{GL}_n(F)$. Let π_E be the irreducible smooth $\overline{\mathbb{Q}}_l$-representation π_F of $\mathrm{GL}_n(F)$ obtained as a base change lift of π_F. If (ρ_F, N_F) and (ρ_E, N_E) are the corresponding semisimple Weil–Deligne $\overline{\mathbb{Q}}_l$-representations of $\mathcal{W}_F$ and $\mathcal{W}_E$ respectively, then we have

$$N_E = N_F \quad \text{and} \quad \rho_E \simeq \mathrm{res}_{\mathcal{W}_E}(\rho_F).$$

1.2 Treumann-Venkatesh's Conjecture

In the article [TV16], Treumann–Venkatesh proved a special instance of Langlands functoriality mod-l. When F is a number field and $\mathbf{G}$ is a connected reductive group defined over F with an automorphism γ of $\mathbf{G}$ of order l, they established a functoriality lifting from the mod-l Hecke eigenclass of $\mathbf{G}^\gamma(F)$ to the mod-l Hecke eigenclass of $\mathbf{G}(F)$. They also made some conjectures for the representation theory of p-adic groups ([TV16, Conjecture 6.3]), which predicts that the functoriality lifting at ramified places is related to Tate cohomology for the action of $\langle\gamma\rangle$. A precise formulation of the conjecture was made by T. Feng ([Fen23, Conjecture 1.2.3]) using the Fargues–Scholze correspondence in [FS21].

We now discuss this conjecture in the case, where $\mathbf{G} = \mathrm{Res}_{E/F}(\mathrm{GL}_n/E)$ for a finite Galois extension E/F of p-adic fields with degree of extension l. Fix a generator γ of the cyclic group $\mathrm{Gal}(E/F)$. There is a natural action of γ on $\mathbf{G}$ and we have $\mathbf{G}^\gamma \simeq \mathrm{GL}_n/F$. In this setup, we state the Treumann–Venkatesh's conjecture using Vignéras's mod-l semisimple local Langlands correspondence–which is equivalent to [Fen23, Conjecture 1.2.3] as the Fargues–Scholze's correspondence (see [Fen23, Subsection 1.3] for the correspondence) for GL_n is compatible with the Vignéras's local Langlands correspondence.

Let (Π, V) be an irreducible smooth $\overline{\mathbb{F}}_l$-representation of $\mathrm{GL}_n(E)$ such that Π is isomorphic to the twisted representation Π^γ. Fix a $\mathrm{GL}_n(E)$ equivariant isomorphism $T_\gamma : \Pi \to \Pi^\gamma$ with $T_\gamma^l = \mathrm{id}$. Thus, the space V is equipped with an action of $\mathrm{Gal}(E/F)$ via the map T_γ. Set $N := 1 + T_\gamma + \cdots + T_\gamma^{l-1}$. Then, we have the following chain complex of $\mathrm{Gal}(E/F)$ modules:

$$\cdots \longrightarrow V \xrightarrow{\mathrm{id}-T_\gamma} V \xrightarrow{N} V \xrightarrow{\mathrm{id}-T_\gamma} V \xrightarrow{N} V \longrightarrow \cdots$$

Associated with the above chain complex, we have the Tate cohomology groups $\widehat{H}^i(\gamma, \Pi)$, for $i \in \{0, 1\}$, defined as

$$\widehat{H}^0(\gamma, \Pi) := \frac{\mathrm{Ker}(1 - T_\gamma)}{\mathrm{Img}(N)}, \tag{1.3}$$

$$\widehat{H}^1(\gamma, \Pi) := \frac{\mathrm{Ker}(N)}{\mathrm{Img}(1 - T_\gamma)}, \tag{1.4}$$

For each $i \in \{0, 1\}$, the action of $\mathrm{GL}_n(E)$ on the space V induces a smooth $\overline{\mathbb{F}}_l$-representation of $\mathrm{GL}_n(F)$ on the Tate cohomology space $\widehat{H}^i(\gamma, \Pi)$. The precise conjecture of Treumann–Venkatesh is as formulated in [Fen23, Conjecture 1.2.3].

Conjecture 1.2.1. *Let F be a finite extension of $\mathbb{Q}_p$, and let E be a finite Galois extension of F of degree l, where l and p are distinct primes. Let Π be a smooth, irreducible $\overline{\mathbb{F}}_l$-representation of* $\mathrm{GL}_n(E)$ *such that $\Pi \simeq \Pi^\gamma$. Then for each $i \in \{0, 1\}$:*

1. *The Tate cohomology group $\widehat{H}^i(\gamma, \Pi)$ has finite length as a representation of* $\mathrm{GL}_n(F)$.
2. *For each irreducible sub-quotient τ of $\widehat{H}^i(\gamma, \Pi)$, the restriction of the semisimple representation $\mathrm{LLC}_{F,\ell}(\tau)$ to the Weil group $\mathcal{W}_E$ is isomorphic to $\mathrm{LLC}_{E,\ell}(\Pi^{(l)})$, where $\Pi^{(l)} := \Pi \otimes_{\overline{\mathbb{F}}_l, \mathrm{Frob}_l} \overline{\mathbb{F}}_l$ is the Frobenius twist of Π, and $\mathrm{LLC}_{,\ell}$ is the mod-l local Langlands correspondence, constructed by Vignéras in [Vig01, Theorem 1.6].*

In the article [Ron16], Ronchetti formulated an integral version of the above conjecture for a subclass of irreducible $\overline{\mathbb{Q}}_l$-representations of $\mathrm{GL}_n(F)$, called cuspidal representations, in terms of the local base change liftings. This is what we will discuss next.

Integral version of Treumann–Venkatesh's Conjecture

We now introduce some notation. Let $\overline{\mathbb{Z}}_l$ be the ring of integers of $\overline{\mathbb{Q}}_l$. Let (π, W) be a smooth integral $\overline{\mathbb{Q}}_l$-representation of $\mathrm{GL}_n(F)$, where integral means that π is of finite length and there exists a $\mathrm{GL}_n(F)$ stable $\overline{\mathbb{Z}}_l$-lattice $\mathcal{L}$ in W. The $\mathrm{GL}_n(F)$ action on the $\overline{\mathbb{F}}_l$-vector space $\mathcal{L}_F \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ is a finite length representation of $\mathrm{GL}_n(F)$. The semi–simplification of this $\overline{\mathbb{F}}_l$-representation is independent of the choice of $\mathcal{L}$ ([Vig96b, Chapter 1]) and it is called the mod-l reduction of π, denoted by $r_l(\pi)$. We further assume that π is cuspidal, i.e., π is irreducible and $r_P(\pi) = 0$ for all proper parabolic subgroups $P = M \ltimes U$ of $\mathrm{GL}_n(F)$, where r_P is the Jacquet restriction functor (see [BZ77, Section 1] for the definition). In this case, the mod-l reduction $r_l(\pi)$ is also cuspidal ([Vig96b, III 4.25 and 5.10]).

Let (Π, V) be a smooth cuspidal $\overline{\mathbb{Q}}_l$-representation of $\mathrm{GL}_n(E)$, obtained as base change lift of (π, W). Then Π is integral (Lemma 2.7.1) and there exists a $\mathrm{GL}_n(E)$ equivariant isomorphism

$T_\gamma : \Pi \to \Pi^\gamma$ such that $T_\gamma^l = \mathrm{id}$. Thus, the space V has an action of the group $\langle\gamma\rangle$ via the isomorphism T_γ. From Proposition 3.3.1, we get that the Tate cohomology $\widehat{H}^0(\gamma, r_l(\Pi))$ is an irreducible $\overline{\mathbb{F}}_l$-representation of $\mathrm{GL}_n(F)$.

Conjecture 1.2.2. [Ron16, Conjecture 2] *Let F be a finite extension of $\mathbb{Q}_p$, and let E be a finite Galois extension of F of degree l, where l and p are distinct odd primes and l does not divide n. Let π be an integral, cuspidal $\overline{\mathbb{Q}}_l$-representation of $\mathrm{GL}_n(F)$, and let Π be the base change lifting of π to $\mathrm{GL}_n(E)$. Then the Tate cohomology group $\widehat{H}^0(\gamma, r_l(\Pi))$ isomorphic to the Frobenius twist $r_l(\pi)^{(l)}$.*

1.3 Known results

In [Ron16, Theorem 1], the author proved the above conjecture for cuspidal $\overline{\mathbb{Q}}_l$-representations of $\mathrm{GL}_n(F)$ of "level zero and minimal-maximal type", i.e., those of the form

$$\mathrm{ind}_{F^\times \mathrm{GL}_n(\mathfrak{o}_F)}^{\mathrm{GL}_n(F)}(\sigma\chi),$$

where $\mathfrak{o}_F$ is the ring of integers of F, σ is an irreducible representation of the compact open subgroup $\mathrm{GL}_n(\mathfrak{o}_F)$, inflated from a cuspidal $\overline{\mathbb{Q}}_l$-representation of the finite group $\mathrm{GL}_n(\mathbb{F}_q)$ via the canonical surjection $\mathrm{GL}_n(\mathfrak{o}_F) \to \mathrm{GL}_n(\mathbb{F}_q)$, and χ is a character of $F^\times$, inflated from the central character of σ via the quotient map $\mathfrak{o}_F^\times \to \mathbb{F}_q^\times$. The condition '$l$ does not divide n' ensures that the base change lift Π is cuspidal, and it is also of level zero and minimal-maximal type.

When F is a local function field, the work of T. Feng ([Fen24]) proved Conjecture 1.2.1 using Genestier–Lafforgue correspondence ([GL17]) in the context of cyclic base change. In a recent preprint [Fen23], a generalized version of part (2) of Conjecture 1.2.1 for arbitrary connected reductive groups over a local non-Archimedean field has been studied by the author, using recent advances of [FS21].

1.4 Organization of the Thesis

This thesis is concerned with the study of Conjecture 1.2.2 for the local base change lifting of integral generic $\overline{\mathbb{Q}}_l$-representations of $\mathrm{GL}_n(F)$ (i.e., those possessing Whittaker models) and a subclass of non-generic representations, namely Zelevinsky subrepresentations of $\mathrm{GL}_n(F)$. The results, in particular, verify Conjecture 1.2.1. Our method is different from those of Ronchetti and Feng. We use the machinery of Rankin–Selberg zeta integrals, local ϵ and γ-factors associated with the representations of both p-adic group and the Weil group. The machinery of local ϵ and γ-factors for both l-adic and mod-l smooth representations of $\mathrm{GL}_n(F)$ as well as over arbitrary

Noetherian rings are available by the seminal works of N. Matringe, R. Kurinczuk, D. Helm and G. Moss (see [KM17], [KM21], [HM18], [Mos16]). We also prove some compatibility results under close local fields. We now give a brief overview of each chapter. The goal is to discuss the results and the method of proof.

1.4.1 Contents of Chapter 2

This chapter is a basic introduction regarding the various notions in the theory of smooth representations of p-adic groups and the Weil–Deligne groups over arbitrary rings. We recall various notations and conventions on Whittaker and Kirillov models. Then, we review Rankin–Selberg theory for smooth representations of $\mathrm{GL}_n(F)$ and the l-adically continuous representations of the Weil group over Noetherian rings. We also collect some results on local constants in l-adic and mod-l settings, which are essential for the later chapters. Then, we review the local Langlands correspondence and the local base change lifting of smooth irreducible representations of $\mathrm{GL}_n(F)$. We also recall and set up some initial results on Tate cohomology.

1.4.2 Results of Chapter 3

In this chapter, we prove Conjecture 1.2.2 for integral cuspidal representations of $\mathrm{GL}_2(F)$. It is a special case of the main results of chapter 4 and 5. Let π_F be a smooth integral cuspidal representation of $\mathrm{GL}_2(F)$ on a $\overline{\mathbb{Q}}_l$-vector space and let π_E be the base change lifting of π_F to $\mathrm{GL}_2(E)$. We further assume that the primes l and p are odd. Then, the base change lift π_E is cuspidal, and so is its mod-l reduction $r_l(\pi_E)$.

Let us consider the Kirillov model of $r_l(\pi_E)$, which is a realization of $r_l(\pi_E)$ in the the space of smooth and compactly supported $\overline{\mathbb{F}}_l$-valued functions on $E^\times$, denoted by $C_c^\infty(E^\times, \overline{\mathbb{F}}_l)$. It is equipped with an action of $\mathrm{GL}_2(E)$ (see Section 3.2). If we denote this action by $\mathbb{K}_{r_l(\pi_E)}$, then $r_l(\pi_E)$ is isomorphic to $\mathbb{K}_{r_l(\pi_E)}$. Similarly, we have the Kirillov model $\big(\mathbb{K}_{r_l(\pi_F)^{(l)}}, C_c^\infty(F^\times, \overline{\mathbb{F}}_l)\big)$ associated to the cuspidal representation $r_l(\pi_F)^{(l)}$. Fix a generator γ of $\mathrm{Gal}(E/F)$. The space $C_c^\infty(E^\times, \overline{\mathbb{F}}_l)$ has a natural action of $\langle\gamma\rangle$, by setting

$$(\gamma.f)(x) := f(\gamma^{-1}(x)),$$

for all $x \in E^\times$ and $f \in C_c^\infty(E^\times, \overline{\mathbb{F}}_l)$. The above action is compatible with the action of $\langle\gamma\rangle$ on $\mathrm{GL}_2(E)$. Consider the Tate cohomology group $\widehat{H}^0(\gamma, C_c^\infty(E^\times, \overline{\mathbb{F}}_l))$, defined as (1.3) and (1.4). It is an irreducible $\overline{\mathbb{F}}_l$-representation of $\mathrm{GL}_2(F)$. For $n = 2$, Conjecture 1.2.2 is then equivalent to the following theorem:

Theorem 1.4.1. *The Tate cohomology group $\widehat{H}^0(\gamma, C_c^\infty(E^\times, \overline{\mathbb{F}}_l))$ is isomorphic to the Kirillov space $\big(\mathbb{K}_{r_l(\pi_F)^{(l)}}, C_c^\infty(F^\times, \overline{\mathbb{F}}_l)\big)$, as a representation of $\mathrm{GL}_2(F)$.*

To prove the theorem, we first consider the restriction to $F^\times$ map

$$C_c^\infty(E^\times, \overline{\mathbb{F}}_l)^{\mathrm{Gal}(E/F)} \longrightarrow C_c^\infty(F^\times, \overline{\mathbb{F}}_l),$$

which factorizes through the Tate cohomology space $\widehat{H}^0(\gamma, C_c^\infty(E^\times, \overline{\mathbb{F}}_l))$ and induces the $P_2(F)$ equivariant isomorphism

$$\widehat{H}^0(\gamma, C_c^\infty(E^\times, \overline{\mathbb{F}}_l)) \xrightarrow{\sim} \big(\mathbb{K}_{r_l(\pi_F)^{(l)}}, C_c^\infty(F^\times, \overline{\mathbb{F}}_l)\big),$$

where $P_2(F)$ is the mirabolic subgroup, as defined in Section 2.1. To get the required $\mathrm{GL}_2(F)$ isomorphism, we need to check the following identity

$$\mathbb{K}_{r_l(\pi_E)}(w)(f) = \mathbb{K}_{r_l(\pi_F)^{(l)}}(w)(f), \tag{1.5}$$

for all $f \in C_c^\infty(F^\times, \overline{\mathbb{F}}_l)$, where w is the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Using the explicit action of the linear operators $\mathbb{K}_{r_l(\pi_E)}(w)$ and $\mathbb{K}_{r_l(\pi_F)^{(l)}}(w)$ (see equation (3.2)) in terms of local ϵ-factors, it follows that the assertion (1.5) is equivalent to an identity of local ϵ-factors. The required identity is obtained using various congruence properties of local ϵ-factor.

1.4.3 Results of Chapter 4

In this chapter, we prove Conjecture 1.2.1 for the base change liftings of integral generic $\overline{\mathbb{Q}}_l$-representations of $\mathrm{GL}_n(F)$ and a subclass of non-generic representations of $\mathrm{GL}_n(F)$, namely Zelevinsky subrepresentations $Z(\Delta)$, where Δ is a Zelevinsky segment. The generic case, in particular, proves Conjecture 1.2.2 under the hypothesis that the prime l does not divide the order of the finite group $\mathrm{GL}_{n-1}(\mathbb{F}_q)$ for $n > 2$. When $n = 2$, we do not require the hypothesis on l (see Theorem 1.4.1 for instance). Here, the strategy of the proof is similar to that of Theorem 1.4.1.

Let us introduce some notions to state the precise results. As before, we let E be a finite Galois extension of a p-adic field F of degree l, where l and p are distinct odd primes. Let $\mathcal{K}$ be the maximal unramified extension of $\mathbb{Q}_l$ in $\overline{\mathbb{Q}}_l$, and let Λ be the ring of integers of $\mathcal{K}$. We fix a non-trivial additive character $\psi_F : F \to \Lambda^\times$. Let ψ_E be the composition $\psi_F \circ \mathrm{Tr}_{E/F}$, where $\mathrm{Tr}_{E/F}$ denotes the trace function. Let N_n be the group of unipotent upper triangular matrices in GL_n. Let Θ_F be the non-degenerate character of $N_n(F)$ associated to ψ_F. Let (π, V) be an integral l-adic generic representation of $\mathrm{GL}_n(F)$, where "generic" means π is irreducible and the space of $N_n(F)$ equivariant linear maps

$$\mathrm{Hom}_{N_n(F)}(\pi, \Theta_F)$$

is non-zero. In general, the mod-l reduction $r_l(\pi)$ is not irreducible, and in this case, $r_l(\pi)$ is non-generic but it admits a unique generic sub–quotient, denoted by $J_l(\pi)$ (see [Vig01, Section 1.8.4]).

Let Π be the base change lifting of π to $\mathrm{GL}_n(E)$. Then Π is also integral (Lemma 2.7.1) and generic, and there exists a $\mathrm{GL}_n(E)$ equivariant isomorphism $T_\gamma : \Pi \xrightarrow{\sim} \Pi^\gamma$, $\gamma \in \mathrm{Gal}(E/F)$. Then the generic sub–quotient $J_l(\Pi)$ of the mod-l reduction of Π is also stable under the action of $\mathrm{Gal}(E/F)$ via the isomorphism T_γ. The first main result is as follows:

Theorem 1.4.2. *Let F be a finite extension of $\mathbb{Q}_p$, and let E be a finite Galois extension of F with $[E : F] = l$, where p and l are distinct odd primes such that l does not divide $|\mathrm{GL}_{n-1}(\mathbb{F}_q)|$ whenever $n \geq 3$. Let π be an integral, l-adic generic representation of $\mathrm{GL}_n(F)$, and let Π be the base change lifting of π to $\mathrm{GL}_n(E)$. Then, the Frobenius twist $J_l(\pi)^{(l)}$ occurs as a sub–quotient of $\widehat{H}^0(\gamma, J_l(\Pi))$.*

We note some immediate remarks on the hypothesis in the above theorem. As a consequence of Proposition 4.3.3, the representation $J_l(\pi)^{(l)}$ is the unique generic sub-quotient of the Tate cohomology group $\widehat{H}^0(\gamma, J_l(\Pi))$. The hypothesis that l does not divide $|\mathrm{GL}_{n-1}(\mathbb{F}_q)|$, for $n \geq 3$, is required in the proof of a vanishing result of Rankin-Selberg integrals on $\mathrm{GL}_{n-1}(F)$, which is the analogue of [JPSS81, Lemma 3.5] or [BH03, Proposition 6.2.1]. If π and Π are both cuspidal, then using Kirillov model for cuspidal representation, one observes that the Tate cohomology $\widehat{H}^0(\gamma, r_l(\Pi))$ is an irreducible mod-l representation of $\mathrm{GL}_n(F)$. Thus, it follows from the above theorem that $\widehat{H}^0(\gamma, r_l(\Pi))$ is isomorphic to the Frobenius twist $r_l(\pi)^{(l)}$, which proves Conjecture 1.2.2 under the given hypothesis on l.

When l does not divide the pro-order of $\mathrm{GL}_n(F)$, we obtain a much more precise version of Theorem 1.4.2. Note that the mod-l reductions $r_l(\pi)$ and $r_l(\Pi)$, in this case, are both generic representations, and we have the following result.

Theorem 1.4.3. *Let E/F be a finite Galois extension of p-adic fields with $[E : F] = l$, where l and p are distinct primes, and l does not divide $|\mathrm{GL}_n(\mathbb{F}_q)|$. Let π be an integral, l-adic, generic representation of $\mathrm{GL}_n(F)$, and let Π be the base change lifting of π. Then:*

1. *The Tate cohomology group $\widehat{H}^0(\gamma, r_l(\Pi))$ is isomorphic to the Frobenius twist $r_l(\pi)^{(l)}$.*
2. *For any $\overline{\mathbb{Z}}_l$-lattice $\mathcal{L}$ in Π, stable under the action of $\mathrm{GL}_n(E) \rtimes \mathrm{Gal}(E/F)$, the first Tate cohomology group $\widehat{H}^1(\gamma, \mathcal{L})$ is trivial.*

We now briefly discuss the proof of Theorem 1.4.2. Since $J_l(\Pi)$ is generic, it can be realized as the space $\mathbb{W}(J_l(\Pi), \overline{\psi}_E)$, contained in the smooth induction $\mathrm{Ind}_{N_n(E)}^{\mathrm{GL}_n(E)}(\overline{\Theta}_E)$, where $\overline{\psi}_E$ is the mod-l reduction of ψ_E and $\overline{\Theta}_E$ is the non-degenerate character of $N_n(E)$ corresponding to $\overline{\psi}_E$

(see (2.1) for the definition of $\overline{\Theta}_E$). The space $\mathbb{W}(J_l(\Pi), \overline{\psi}_E)$ is called the Whittaker model of $J_l(\Pi)$. From Lemma 2.3.3, we get that $\mathbb{W}(J_l(\Pi), \overline{\psi}_E)$ is stable under the action of $\mathrm{Gal}(E/F)$. Consider the Kirillov model $\mathbb{K}(J_l(\Pi), \overline{\psi}_E)$, which is the image of $\mathbb{W}(J_l(\Pi), \overline{\psi}_E)$ under restriction to $P_n(E)$ map (see Theorem 2.4.3). The space $\mathbb{K}(J_l(\Pi), \overline{\psi}_E)$ contains the compactly supported functions in $\mathrm{Ind}_{N_n(E)}^{P_n(E)}(\overline{\Theta}_E)$. We have the Tate cohomology group $\widehat{H}^0(\gamma, \mathbb{K}(J_l(\Pi), \overline{\psi}_E))$, defined as $\overline{\mathbb{F}}_l$-representation of the mirabolic subgroup $P_n(F)$. Let Φ be the restriction to $P_n(F)$ map

$$\Phi : \widehat{H}^0(\gamma, \mathbb{K}(J_l(\Pi), \overline{\psi}_E)) \longrightarrow \mathrm{Ind}_{N_n(F)}^{P_n(F)}(\overline{\Theta}_F^l),$$

where $\overline{\Theta}_F$ is the mod-l reduction of Θ_F. Using compactly supported functions, one can show that the inverse image of the Kirillov model $\mathbb{K}(J_l(\pi)^{(l)}, \overline{\psi}_F^l)$ under the map Φ is non-zero, and it is denoted by $\mathcal{M}(\psi_F)$. To prove Theorem 1.4.2, we show that $\mathcal{M}(\psi_F)$ is stable under the action $\mathrm{GL}_n(F)$ and the map

$$\Phi : \mathcal{M}(\psi_F) \longrightarrow \mathbb{K}(J_l(\pi)^{(l)}, \overline{\psi}_F^l)$$

is $\mathrm{GL}_n(F)$ equivariant. These are equivalent to proving the following identity

$$I\big(X, \Phi(\Pi(w_n)W), \sigma(w_{n-1})W'\big) = I\big(X, J_l(\pi)^{(l)}(w_n)\Phi(W), \sigma(w_{n-1})W'\big), \tag{1.6}$$

for all $W \in \mathcal{M}(\psi_F)$ and $W' \in \mathbb{W}(\sigma, \overline{\psi}_F^{-l})$, where w_{n-1}, w_n are defined in the subsection (2.1) of Chapter 2; and σ is arbitrary mod-l generic representation of $\mathrm{GL}_{n-1}(F)$. Here, $I(X, W, W')$ is a mod-l Rankin–Selberg zeta function written as a formal power series in the variable X instead of the traditional q^{-s} ([KM17, Section 3]). We use induction on n to transfer the local Rankin–Selberg zeta functions $I\big(X, \Phi(\Pi(w_n)W), \sigma(w_{n-1})W'\big)$ made from integrals on the homogeneous space $\mathrm{GL}_{n-1}(F)/N_{n-1}(F)$ to Rankin–Selberg zeta functions defined by integrals on $\mathrm{GL}_{n-1}(E)/N_{n-1}(E)$. Then, using local Rankin–Selberg functional equation, we show that the equality (1.6) is equivalent to certain identities of local γ-factors, such as (4.15), which again follows from congruence properties of local γ-factor.

Using the same ideas, we also prove an integral version of Theorem 1.4.2. Recall that $\mathcal{K}$ is the maximal unramified extension of $\mathbb{Q}_l$ in $\overline{\mathbb{Q}}_l$ and Λ denotes the ring of integers of $\mathcal{K}$. Let Π be an integral generic $\mathcal{K}$-representation of $\mathrm{GL}_n(E)$ which is absolutely irreducible (i.e., $\Pi \otimes_{\mathcal{K}} \overline{\mathbb{Q}}_l$ is irreducible). We further assume that $\Pi \simeq \Pi^\gamma$. Let $\mathbb{W}_\Lambda(\Pi, \psi_E)$ be the space of all Λ-valued functions in the Whittaker model of Π. In view of [Vig04, Theorem 2] and Lemma 2.3.3, we get that $\mathbb{W}_\Lambda(\Pi, \psi_E)$ is stable under the action of $\mathrm{GL}_n(E) \rtimes \mathrm{Gal}(E/F)$. Then we have the following theorem.

Theorem 1.4.4. *Let F be a finite extension of $\mathbb{Q}_p$, and let E be a finite Galois extension of F with $[E : F] = l$, where l and p are distinct primes, and l does not divide $|\mathrm{GL}_{n-1}(\mathbb{F}_q)|$ whenever*

$n \geq 3$. Let Π be an integral, generic $\mathcal{K}$-representation of $\mathrm{GL}_n(E)$ which is absolutely irreducible. We further assume that $\Pi \simeq \Pi^\gamma$. Let π be the integral, l-adic generic representation of $\mathrm{GL}_n(F)$ such that $\Pi \otimes_{\mathcal{K}} \overline{\mathbb{Q}}_l$ is the base change lifting of π. Then, the Frobenius twist of $J_l(\pi)$ occurs as a unique generic sub–quotient of the zeroth Tate cohomology group $\widehat{H}^0(\gamma, \mathbb{W}_\Lambda(\Pi, \psi_E))$.

Our method can also be extended to some non-generic representations as well. Especially for those irreducible representations of $\mathrm{GL}_n(E)$, which remain irreducible when restricted to the mirabolic subgroup, denoted by $P_n(E)$. This class of representations is exactly the Zelevinsky sub-representations. Assume that Π is an integral cuspidal l-adic representation of $\mathrm{GL}_n(E)$ obtained as a base change lift of an integral cuspidal representation π of $\mathrm{GL}_n(F)$. Let Δ be a segment (Subsection 2.3.3) on the cuspidal line of Π. We apply Theorem 1.4.2 to compute the Tate cohomology of Zelevinsky subrepresentations $Z(\Delta)$ (see Theorem 4.4.3).

1.4.4 Results of Chapter 5

Recall that the main result of Chapter 4 (i.e., Theorem 1.4.2) assumed that the prime l does not divide the pro-order of $\mathrm{GL}_{n-1}(F)$, for $n \geq 3$. The main goal of this chapter is to remove the hypothesis on l by exploiting the theory of smooth representations over Artin local $\overline{\mathbb{F}}_l$-algebras. To be precise, we prove Theorem 1.4.2 for integral, cuspidal representations of $\mathrm{GL}_3(F)$, where we remove the hypothesis that l does not divide $|\mathrm{GL}_2(\mathbb{F}_q)|$. We use the theory of nilpotent lifts over Artin local $\overline{\mathbb{F}}_l$-algebras–which are infinitesimal deformations of generic $\overline{\mathbb{F}}_l$-representations (see [Mos21, Definition 1.1]). The nilpotent lifts over Artin local $\overline{\mathbb{F}}_l$-algebras deal with the nilpotents in the mod-l Bernstein center. When l does not divide $|\mathrm{GL}_2(\mathbb{F}_q)|$, the mod-$l$ Bernstein center is a reduced ring (Lemma 4.3.5), and in this case, the proof is similar to that of Theorem 1.4.2.

We now state the main result. Let π be an l-adic cuspidal representation of $\mathrm{GL}_3(F)$ such that its central character $\omega_\pi : F^\times \to \overline{\mathbb{Q}}_l^\times$ is integral (i.e., it takes values in $\overline{\mathbb{Z}}_l^\times$); which is equivalent to saying that π is integral. Let E/F be a finite Galois extension with $[E : F] = l$, where l and p are distinct odd primes with $l \neq 3$. Fix a generator γ of the cyclic group $\mathrm{Gal}(E/F)$. Let Π be the base change lifting of π. Then Π is also cuspidal. Moreover, the central character ω_Π of Π is also integral, as $\omega_\Pi = \omega_\pi \circ \mathrm{Nr}_{E/F}$, where $\mathrm{Nr}_{E/F}$ is the norm map of E/F. Since Π is cuspidal and its central character ω_Π is integral, it follows that Π is also integral. Then we have the Tate cohomology group $\widehat{H}^0(\gamma, r_l(\Pi))$, which is an $\overline{\mathbb{F}}_l$-representation of $\mathrm{GL}_3(F)$. We prove the following:

Theorem 1.4.5. *Let F be a finite extension of $\mathbb{Q}_p$ and let E be a finite Galois extension of F with $[E : F] = l$, where l and p are distinct odd primes and $l \neq 3$. Let π be a cuspidal, l-adic representation of $\mathrm{GL}_3(F)$ such that the central character ω_π of π is integral. Let Π be the base change lifting of π to $\mathrm{GL}_3(E)$. Then, the Frobenius twist $r_l(\pi)^{(l)}$ is isomorphic to the Tate cohomology group $\widehat{H}^0(\gamma, r_l(\Pi))$.*

For proving the above theorem, we rely on the local converse theorem over local Artinian $\overline{\mathbb{F}}_l$-algebras, established in [Mos21, Theorem 1.2]. To say more concretely, we first observe that the Tate cohomology group $\widehat{H}^0(\gamma, r_l(\Pi))$ is irreducible (Proposition 3.3.1). Then we use the local converse theorem (see Theorem 5.2.4) for the representations $\widehat{H}^0(\gamma, r_l(\Pi))$ and $r_l(\pi)^{(l)}$, i.e., we prove the following identity of γ-factors:

$$\gamma(X, \widehat{H}^0(\gamma, r_l(\Pi)), \eta, \overline{\psi}_F) = \gamma(X, r_l(\pi)^{(l)}, \eta, \overline{\psi}_F),$$

where η varies over the set of all co-Whittaker $R[F^\times]$ modules (i.e., smooth R-valued characters of $F^\times$) and R is an arbitrary Artin local $\overline{\mathbb{F}}_l$-algebra. To get the above identity of γ-factors, we use the machinery of Rankin–Selberg theory over Noetherian $W(\overline{\mathbb{F}}_l)$-algebras, where $W(\overline{\mathbb{F}}_l)$ is the ring of Witt vectors of $\overline{\mathbb{F}}_l$ (see [Mos16] for a reference). The proof also uses certain properties of Deligne–Langlands γ-factors over Noetherian $W(\overline{\mathbb{F}}_l)$-algebras, which are made available by the seminal work of Helm–Moss ([HM15]).

1.4.5 Results of Chapter 6

The content of this chapter is slightly different from that of the last three chapters. The main objective of this chapter is to prove the compatibility of two maps between Hecke algebras with coefficients in $\overline{\mathbb{F}}_l$.

Let us introduce some notations to state the main results. Let $\mathbf{G}$ be a split connected reductive group over $\mathbb{Z}$. D. Kazhdan conjectured a link between the local Langlands correspondences for $\mathbf{G}(F)$ and $\mathbf{G}(F')$, where F is a p-adic field and F' is a non-Archimedean local field of positive characteristic which are sufficiently close ([Kaz86]). Kazhdan's approach is via isomorphisms between the respective Hecke algebras (with complex coefficients), $\mathcal{H}(\mathbf{G}(F), K)$ and $\mathcal{H}(\mathbf{G}(F'), K')$, for some specific choice of compact open subgroups K and K' of $\mathbf{G}(F)$ and $\mathbf{G}(F')$ respectively. Here the space $\mathcal{H}(\mathbf{G}(F), K)$ consists of all locally constant, compactly supported functions on $\mathbf{G}(F)$ which are bi-K-invariant. To be more precise, let $K_m(F)$ be the congruence subgroup of $\mathbf{G}(F)$ of level m, i.e.,

$$K_m(F) := \mathrm{Ker}\big(\mathbf{G}(\mathfrak{o}_F) \xrightarrow{\mathrm{mod}-\mathfrak{p}_F^m} \mathbf{G}(\mathfrak{o}_F/\mathfrak{p}_F^m)\big),$$

where $\mathfrak{o}_F$ is the ring of integers of F and $\mathfrak{p}_F$ is the unique maximal ideal of $\mathfrak{o}_F$. Similar notations follow over F'. Kazhdan ([Kaz86, Theorem A]) proved that there exists an integer $r \geq m$ such that if F and F' are r-close (i.e., there exists a ring isomorphism $\mathfrak{o}_F/\mathfrak{p}_F^r \xrightarrow{\sim} \mathfrak{o}_{F'}/\mathfrak{p}_{F'}^r$), then there is a Hecke algebra isomorphism

$$\mathrm{Kaz}_m : \mathcal{H}(\mathbf{G}(F), K_m(F)) \longrightarrow \mathcal{H}(\mathbf{G}(F'), K_m(F')).$$

The construction of the algebra isomorphism Kaz_m relies on the finiteness of complex Hecke algebras. For arbitrary Noetherian $\mathbb{Z}_l$-algebras R, where l and p are distinct primes, the work of [DHMK24] gives the finiteness of Hecke algebras over R. Following the ideas of Kazhdan and using the finiteness of Hecke algebras over $\overline{\mathbb{F}}_l$, we formulate a mod-l version of the Kazhdan isomorphism (see Theorem 6.4.5).

Let E/F be a finite Galois extension of prime degree l with $l \neq p$. Let $K_m(E)$ be the m-th congruence subgroup of $\mathbf{G}(E)$. Since $K_m(E)$ is a pro-p subgroup, it is stable under the action of $\mathrm{Gal}(E/F)$. We have $K_m(F) = K_{em}(E) \cap \mathbf{G}(F)$, where e denotes the ramification index of the extension E/F. In the article [TV16, Section 4], Treumann–Venkatesh defined an $\overline{\mathbb{F}}_l$-algebra homomorphism

$$\mathrm{Br} : \mathcal{H}(\mathbf{G}(E), K_{em}(E))^{\mathrm{Gal}(E/F)} \longrightarrow \mathcal{H}(\mathbf{G}(F), K_m(F)),$$

which is the restriction of functions on $\mathbf{G}(E)$ to $\mathbf{G}(F)$. The map Br is called the *Brauer homomorphism* and is conjectured to be compatible with Langlands functoriality.

Given an extension E/F as above. Following Deligne's construction in [Del84], we get a finite Galois extension E'/F' such that the fields E and E' are em-close (see Lemma 6.4.6), where e denotes the ramification index of the extension E/F. The Kazhdan isomorphism Kaz_{em} is then equivariant under the Galois actions (Proposition 6.4.9), and this leads to the following diagram:

$$\begin{tikzcd}
\mathcal{H}(\mathbf{G}(E), K_{em}(E))^{\mathrm{Gal}(E/F)} \arrow[r, "\mathrm{Kaz}_{em}"] \arrow[d, "\mathrm{Br}"'] & \mathcal{H}(\mathbf{G}(E'), K_{em}(E'))^{\mathrm{Gal}(E'/F')} \arrow[d, "\mathrm{Br}'"] \\
\mathcal{H}(\mathbf{G}(F), K_m(F)) \arrow[r, "\mathrm{Kaz}_m"'] & \mathcal{H}(\mathbf{G}(F'), K_m(F'))
\end{tikzcd} \tag{1.7}$$

It brings up the obvious question of commutativity of the above diagram, which we prove in this chapter. The precise result is as follows:

Theorem 1.4.6. *Let F and F' be two non-Archimedean m-close local fields. Let E be a finite Galois extensions of F of prime degree l, and let E' be the Galois extension of F', as indicated above. Then, for any connected split reductive group $\boldsymbol{G}$ defined over $\mathbb{Z}$, we have*

$$\mathrm{Br}' \circ \mathrm{Kaz}^E_{em} = \mathrm{Kaz}^F_m \circ \mathrm{Br}.$$

As a consequence of the above theorem, we get a compatibility of Kazhdan isomorphism with linkage–which is the representation theoretic version of Brauer homomorphism ([TV16, Section 6]). Let us explain it briefly. Say (Π, V) is an irreducible $\overline{\mathbb{F}}_l$-representation of $\mathbf{G}(E)$ such that Π is isomorphic to the twisted representation Π^γ, for all $\gamma \in \mathrm{Gal}(E/F)$. Then, any irreducible sub–quotient of the Tate cohomology groups $\widehat{H}^i(\gamma, V)$, for $i \in \{0,1\}$, is said to be *linked* with Π. Let (ρ, U) be an irreducible smooth $\overline{\mathbb{F}}_l$-representation of $\mathbf{G}(F)$ such that ρ is linked with Π. Let us further assume that the space of $K_m(F)$ and $K_{em}(E)$-fixed vectors $U^{K_m(F)}$ and $V^{K_{em}(E)}$ are non-zero.

Recall that for any compact open subgroup K of $\mathbf{G}(F)$, there is a one-to-one correspondence between the set of all simple $\mathcal{H}(\mathbf{G}(F), K)$ modules and the set of all irreducible smooth $\overline{\mathbb{F}}_l$-representations (π, W) of $\mathbf{G}(F)$ with the space of K-fixed vectors $W^K \neq 0$ ([Vig96b, Chapter 1]). Using this bijective correspondence and the Kazhdan isomorphism, we get an irreducible $\overline{\mathbb{F}}_l$-representation ρ' (resp. Π') of $\mathbf{G}(F')$ (resp. $\mathbf{G}(E')$) associated with ρ (resp. Π). Then, using Theorem 1.4.6, we show that ρ is linked with Π if and only if ρ' is linked with Π' (see Theorem 6.6.2).

Chapter 2

Preliminaries

In this chapter, we recall a few basic notions of representation theory of $\mathrm{GL}_n(K)$ and the Weil–Deligne group over K, where K is a non-Archimedean local field. We review the following:

1. Integral representation and mod-l reduction.
2. Whittaker and Kirillov models.
3. Rankin–Selberg zeta integrals and local constants.
4. Weil–Deligne representations and Deligne–Langlands γ-factors.
5. Local Langlands correspondence and local base change.
6. Tate cohomology.

2.1 Notations

Let K be a non-Archimedean local field, and let $\mathfrak{o}_K$ be the ring of integers of K. Let $\mathfrak{p}_K$ be the maximal ideal of $\mathfrak{o}_K$, and let ϖ_K be a uniformizer of K. We denote by κ_K the finite residue field $\mathfrak{o}_K/\mathfrak{p}_K$. Let q_K be the cardinality of κ_K. Let υ_K be the normalized valuation on K, and let ν_K be the normalized absolute value of K corresponding to υ_K. Let l and p be two distinct odd primes. Let F be a finite extension of $\mathbb{Q}_p$, and let E be a finite Galois extension of F with degree of extension l. We denote by Γ the Galois group $\mathrm{Gal}(E/F)$.

For a commutative ring R with unity, let $M_{r\times s}(R)$ denote the ring of all $r \times s$ matrices with entries in R. Let $\mathrm{GL}_n(K)$ be the subgroup of all invertible elements in $M_{n\times n}(K)$. We use the notation $G_n(K)$ (or simply G_n when the underlying field K is understood) for $\mathrm{GL}_n(K)$. Recall that $G_n(K)$ is equipped with locally compact topology induced from the local field K. We set $P_n(K)$, the mirabolic subgroup, defined as

$$\left\{ \begin{pmatrix} A & M \\ 0 & 1 \end{pmatrix} : A \in G_{n-1}(K),\ M \in K^{n-1} \right\}.$$

Let $B_n(K)$ denote the group of all upper triangular matrices in $G_n(K)$ and let $N_n(K)$ be the unipotent radical of $B_n(K)$. We sometimes simply write P_n, B_n, N_n when the underlying field K is understood from the context. We denote by w_n the following matrix of $G_n(K)$:

$$w_n = \begin{pmatrix} 0 & & & & 1 \\ & & & 1 & \\ & & \cdot & & \\ & \cdot & & & \\ 1 & & & & 0 \end{pmatrix}$$

We fix an algebraic closure $\overline{\mathbb{Q}}_l$ of the field $\mathbb{Q}_l$. Let $\overline{\mathbb{Z}}_l$ be the integral closure of $\mathbb{Z}_l$ in $\overline{\mathbb{Q}}_l$, and let $\mathfrak{P}_l$ be the unique maximal ideal of $\overline{\mathbb{Z}}_l$. We have $\overline{\mathbb{Z}}_l/\mathfrak{P}_l \simeq \overline{\mathbb{F}}_l$. Fix a square root of q_K in $\overline{\mathbb{Q}}_l$, and it is denoted by $q_K^{1/2}$. The choice of $q_K^{1/2}$ is required for transferring the complex local Langlands correspondence to a local Langlands correspondence over $\overline{\mathbb{Q}}_l$ (see [BH06, Chapter 8]). Let $\mathcal{K}$ be the maximal unramified extension of $\mathbb{Q}_l$ in $\overline{\mathbb{Q}}_l$, and let Λ denote the ring of integers of $\mathcal{K}$. Let $W(\overline{\mathbb{F}}_l)$ be the ring of Witt vectors of $\overline{\mathbb{F}}_l$. A prime number l' is called *banal* for $\mathrm{GL}_n(K)$ if l' does not divide $|\mathrm{GL}_n(\kappa_K)|$.

2.2 Smooth representation

Let R be a commutative ring with unity. Let G be a locally compact and totally disconnected group. An $R[G]$ module V is called *smooth* if, for every vector $v \in V$, the stabilizer of v in G is an open subgroup of G. When R is a field, an $R[G]$ module V is usually called an R-representation of G. We interchangeably use these terminologies throughout the thesis. We use the notation (π, V) (or simply π) for an $R[G]$-module V, where π denotes the action of G on the space V. All the representations are assumed to be smooth, and the representation spaces are modules over R. Moreover, a smooth $R[G]$ module V is called *admissible* if for each compact open subgroup H of G, the space of all H-fixed vectors V^H is finitely generated as an R module.

A character is a smooth one-dimensional representation $\chi : G \to R^\times$. For $G = G_n(K)$, any character $\chi : K^\times \to R^\times$ induces a character $\chi \circ \det : G_n(K) \to R^\times$. By abuse of notation, $\chi \circ \det$ is also denoted by χ. In particular, the normalized absolute value ν_K of K gives a character of $G_n(K)$, which is also denoted by ν_K. A representation (π, V) is called l-adic when $R = \overline{\mathbb{Q}}_l$, and

(π, V) is called l-modular when $R = \overline{\mathbb{F}}_l$. We denote by $C_c^\infty(G, R)$ the space of all locally constant and compactly supported R-valued functions on G.

2.2.1 Integral representation and mod-l reduction

Let (π, V) be a smooth l-adic representation of G. A $\overline{\mathbb{Z}}_l$-lattice in V is a $\overline{\mathbb{Z}}_l$ submodule $\mathcal{L}$ such that, for each finite dimensional subspace U of V, the space $\mathcal{L} \cap U$ is a free $\overline{\mathbb{Z}}_l$ module with rank equals the dimension of U. The representation (π, V) is said to be *integral* if it has finite length as a representation of G and there exists a G stable $\overline{\mathbb{Z}}_l$-lattice $\mathcal{L}$ in V. Such a lattice $\mathcal{L}$ is called an *integral structure* of (π, V). We say that a character $\chi : G \to \overline{\mathbb{Q}}_l^\times$ is integral if χ takes values in the ring of integers $\overline{\mathbb{Z}}_l$.

Let (π, V) be an integral l-adic representation of G. Choose a G-stable lattice $\mathcal{L}$ in V. Then the group G acts on the $\overline{\mathbb{F}}_l$-vector space $\mathcal{L} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$, and this gives an l-modular representation, which depends on the choice of the G-stable lattice $\mathcal{L}$. By [Vig96b, II. 5.11.a and 5.11.b], the representation $\left(\pi, \mathcal{L} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l\right)$ is of finite length and its semi-simplification is independent of the choice of $\mathcal{L}$. We denote by $r_l(\pi)$ the semi-simplification of the representation $\left(\pi, \mathcal{L} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l\right)$. The representation $r_l(\pi)$ is called the *mod-l reduction* of the l-adic representation π. We say that an l-modular representation σ lifts to an l-adic integral representation π if there exists a G-invariant lattice $\mathcal{L} \subseteq \pi$ such that $\mathcal{L} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ is isomorphic to σ.

2.2.2 Parabolic induction

In this subsection, we discuss the parabolic induction of smooth R-representations, where $R = \overline{\mathbb{Q}}_l$ or $\overline{\mathbb{F}}_l$. We follow [BZ77] for the definitions. Let H be a closed subgroup of G, and let Ind_H^G and ind_H^G be the smooth induction functor and compact induction functor, respectively.

Let $\lambda = (n_1, n_2, ..., n_t)$ be an ordered partition of the integer n. Let Q_λ be the subgroup of $G_n(K)$, consisting of matrices of the form

$$\begin{pmatrix} A_1 & * & * & * & * \\ & A_2 & * & * & * \\ & & . & * & * \\ & & & . & * \\ & & & & A_t \end{pmatrix},$$

where $A_i \in G_{n_i}(K)$, for all $1 \leq i \leq t$. Then $Q_\lambda = M_\lambda \ltimes U_\lambda$, where M_λ is the group of block diagonal matrices of the form

$$\begin{pmatrix} A_1 & & & & \\ & A_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & A_t \end{pmatrix}, A_i \in G_{n_i}(K),$$

for all $1 \leq i \leq t$, and U_λ is the unipotent radical of Q_λ consisting of matrices of the form

$$U_\lambda = \begin{pmatrix} I_{n_1} & * & * & * & * \\ & I_{n_2} & * & * & * \\ & & \cdot & * & * \\ & & & \cdot & * \\ & & & & I_{n_t} \end{pmatrix},$$

where I_{n_i} is the $n_i \times n_i$ identity matrix.

Let τ be an R-representation of M_λ. Then τ is considered as a representation of Q_λ by inflation via the map $Q_\lambda \to Q_\lambda/U_\lambda \simeq M_\lambda$. The induced representation $\mathrm{ind}_{Q_\lambda}^{G_n(K)}(\tau)$ is called the *parabolic induction* of τ. We denote by $i_{Q_\lambda}^{G_n(K)}(\tau)$ the normalized parabolic induction of τ corresponding to the partition λ. For each $1 \leq i \leq t$, let τ_i be an R-representation of $G_{n_i}(K)$. The parabolic induction $i_{Q_\lambda}^{G_n(K)}(\tau_1 \otimes \cdots \otimes \tau_t)$ is denoted by the product symbol $\tau_1 \times \cdots \times \tau_t$.

Let τ be an integral l-adic smooth representation of M_λ, and let $\mathcal{L}$ be an integral structure of τ. Then the space $i_{Q_\lambda}^{G_n(K)}(\mathcal{L})$, consisting of functions in $i_{Q_\lambda}^{G_n(K)}(\tau)$ taking values in $\mathcal{L}$, is also an integral structure of $i_{Q_\lambda}^{G_n(K)}(\tau)$ (see [Vig96b, I. 9.3]). Moreover, we have

$$i_{Q_\lambda}^{G_n(K)}(\mathcal{L} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l) \simeq i_{Q_\lambda}^{G_n(K)}(\mathcal{L}) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l.$$

Hence, the parabolic induction commutes with mod-l reduction, i.e.,

$$r_l(i_{Q_\lambda}^{G_n(K)}(\tau)) \simeq [i_{Q_\lambda}^{G_n(K)}(r_l(\tau))],$$

where the square bracket denotes the semi-simplification of $i_{Q_\lambda}^{G_n(K)}(r_l(\tau))$.

2.2.3 Cuspidal and Supercuspidal representation

Keep the notations as in the preceding subsection. Here, $R = \overline{\mathbb{Q}}_l$ or $\overline{\mathbb{F}}_l$. An irreducible R-representation π of $G_n(K)$ is called a *cuspidal* representation if for all proper subgroups $Q_\lambda = M_\lambda \ltimes U_\lambda$ of $G_n(K)$ and all irreducible R-representations σ of M_λ, we have

$$\mathrm{Hom}_{G_n(K)}(\pi, i_{Q_\lambda}^{G_n(K)}(\sigma)) = 0.$$

The representation π is called *supercuspidal* if for all proper subgroups $Q_\lambda = M_\lambda \ltimes U_\lambda$ of $G_n(K)$ and all irreducible R-representations σ of M_λ, the representation (π, V) is not a subquotient of $i_{Q_\lambda}^{G_n(K)}(\sigma)$.

Remark 2.2.1. Let k be an algebraically closed field and let π be a k-representation of $G_n(K)$. If the characteristic of k is 0, then π is cuspidal if and only if π is supercuspidal. But, when the characteristic of k is $l > 0$, there are cuspidal k-representations of $G_n(K)$ which are not supercuspidal. For details, see [Vig96b, Section 2.5, Chapter 2].

2.3 Generic representation

In this section, we recall the notion of generic representations of $G_n(K)$. We first discuss the notion of R-representations of Whittaker type, where R denotes either $\overline{\mathbb{Q}}_l$ or $\mathcal{K}$ or a Noetherian $W(\overline{\mathbb{F}}_l)$-algebra. For a reference, see [EH14, Section 3].

We fix a non-trivial additive character $\psi_K : K \to \Lambda^\times$, where Λ is the ring of integers of the maximal unramified extension $\mathcal{K}$. For any Λ-algebra R, we denote by $\psi_{K,R}$ the composition $K \xrightarrow{\psi_K} \Lambda^\times \to R^\times$. Let $\Theta_{K,R}$ be the character of $N_n(K)$, defined by

$$\Theta_{K,R}((u_{ij})_{i,j=1}^n) := \psi_{K,R}(\sum_{i=1}^{n-1} u_{i(i+1)}) \tag{2.1}$$

for all $(u_{ij})_{i,j=1}^n \in N_n$. Let (π, V) be a smooth $R[G_n]$ module. We denote by $V^{(n)}$ the quotient space $V/V(N_n, \Theta_{K,R})$, where $V(N_n, \Theta_{K,R})$ is the R-submodule generated by the set of vectors $\{\pi(u)v - \Theta_{K,R}(u)v : v \in V, u \in N_n\}$.

Definition 2.3.1. A smooth $R[G_n]$ module (π, V) is said to be of Whittaker type if $V^{(n)}$ is free R-module of rank one. When R is a field, an irreducible R-representation of Whittaker type is called *generic*.

2.3.1 Whittaker and Kirillov model

Let (π, V) be a smooth $R[G_n]$ module of Whittaker type. By definition, $\mathrm{Hom}_{N_n}(V, \Theta_{K,R})$ is a free R-module of rank one. We choose a generator of $\mathrm{Hom}_{N_n}(V, \Theta_{K,R})$, denoted by $\mathcal{W}_\pi$. Let $\mathbb{W}(\pi, \psi_{K,R}) \subseteq \mathrm{Ind}_{N_n}^{G_n}(\Theta_{K,R})$ be the space consisting of functions W_v, $v \in V$, where

$$W_v(g) := \mathcal{W}_\pi(\pi(g)v),$$

for all $g \in G_n$. Then the map $v \mapsto W_v$ induces a G_n-equivariant homomorphism from V to $\mathbb{W}(\pi, \psi_K)$. The space $\mathbb{W}(\pi, \psi_{K,R})$ is called the *Whittaker model* of (π, V). When R is a field and (π, V) is irreducible, the map $V \to \mathbb{W}(\pi, \psi_{K,R})$ is an isomorphism. In general, the map $V \to \mathbb{W}(\pi, \psi_{K,R})$ is surjective but not necessarily an isomorphism. Two different $R[G_n]$ modules of Whittaker type can have the same Whittaker model.

The identification $V^{(n)} \simeq R$ also induces a P_n-equivariant map $V \to \mathrm{Ind}_{N_n}^{P_n}(\Theta_{K,R})$. The image of this map is called the *Kirillov model* of (π, V), and it is denoted by $\mathbb{K}(\pi, \psi_{K,R})$. Moreover, $\mathbb{K}(\pi, \psi_{K,R})$ contains $\mathrm{ind}_{N_n}^{P_n}(\Theta_{K,R})$ as $R[P_n]$-submodule. We use the notation $\mathcal{K}(\psi_{K,R})$ for the compact induction $\mathrm{ind}_{N_n}^{P_n}(\Theta_{K,R})$. We have the following result, due to [MM22, Theorem 4.2].

Theorem 2.3.2. *The restriction map $W \mapsto \mathrm{res}_{P_n}(W)$ gives an isomorphism*

$$\mathbb{W}(\pi, \psi_{K,R}) \xrightarrow{\sim} \mathbb{K}(\pi, \psi_{K,R}).$$

The above theorem follows from the Rankin–Selberg functional equations and the completeness of Whittaker models over arbitrary Noetherian $W(\overline{\mathbb{F}}_l)$-algebras (see [HM18, Corollary 3.6]). We will recall the proof in the subsection (2.4.1), where the Rankin–Selberg theory over families is discussed.

2.3.2 Galois invariance of Whittaker models

Here, R will denote either $\overline{\mathbb{Q}}_l$ or $\mathcal{K}$ or $\overline{\mathbb{F}}_l$. We fix a generator γ of Γ, where Γ is the Galois group of the extension E/F. Let (π, V) be an R-representation of $G_n(E)$. The group Γ acts on $G_n(E)$ componentwise, i.e., for any $g = (a_{ij})_{i,j=1}^n \in G_n(E)$, we set

$$\gamma(g) := (\gamma(a_{ij}))_{i,j=1}^n.$$

Let π^γ be the representation of $G_n(E)$ on V, defined by

$$\pi^\gamma(g) := \pi(\gamma(g)), \text{ for } g \in G_n(E).$$

We say that π is Γ-equivariant if the representations π and π^γ are isomorphic. We now prove a lemma concerning the Γ-invariance of the Whittaker model of a Γ-equivariant representation π of $G_n(E)$. We fix a non-trivial additive character $\psi_F : F \to \Lambda^\times$. Let ψ_E be the composition $\psi_F \circ \mathrm{Tr}_{E/F}$, where $\mathrm{Tr}_{E/F}$ is the trace map of the extension E/F. The mod-l reductions of ψ_F and ψ_E are denoted by $\overline{\psi}_F$ and $\overline{\psi}_E$ respectively (Recall $\overline{\psi}_K = \psi_{K,\overline{\mathbb{F}}_l}$). Let Θ_F and Θ_E be the non-degenerate characters of $N_n(F)$ and $N_n(E)$, respectively (see Section 2.3 for the definition). The mod-l reduction of Θ_F (resp. Θ_E) is also denoted by $\overline{\Theta}_F$ (resp. $\overline{\Theta}_E$). We now consider the action of Γ on the space $\mathrm{Ind}_{N_n(E)}^{G_n(E)}(\Theta_{E,R})$, given by

$$(\gamma.\varphi)(g) := \varphi(\gamma^{-1}(g)),$$

for all $g \in G_n(E)$ and $\varphi \in \mathrm{Ind}_{N_n(E)}^{G_n(E)}(\Theta_{E,R})$.

Lemma 2.3.3. *Let π be a generic representation of $G_n(E)$ on an R-vector space V, such that (π, V) is Γ-equivariant. Then the Whittaker model $\mathbb{W}(\pi, \psi_{E,R})$ of π is stable under the action of Γ.*

Proof. Let $\mathcal{W}_\pi$ be a Whittaker functional on the representation (π, V). For all $v \in V$ and $u \in N_n(E)$, we have

$$\mathcal{W}_\pi(\pi^\gamma(u)v) = \Theta_{E,R}(\gamma(u))\mathcal{W}_\pi(v) = (\psi_{F,R} \circ \mathrm{Tr}_{E/F})\big(\textstyle\sum_{i=1}^{n-1} \gamma(u_{i,i+1})\big)\mathcal{W}_\pi(v) = \Theta_{E,R}(u)\mathcal{W}_\pi(v),$$

Thus, $\mathcal{W}_\pi$ is also a Whittaker functional for the representation (π^γ, V). Let $W_v \in \mathbb{W}(\pi, \psi_{E,R})$. Then

$$(\gamma^{-1}.W_v)(g) = \mathcal{W}_\pi(\pi^\gamma(g)v).$$

From uniqueness of the Whittaker model, we have $\gamma^{-1}.W_v \in \mathbb{W}(\pi, \psi_{E,R})$. Hence the lemma. □

For a smooth R-representation (π, V) of $G_n(E)$, we denote by $V_{N_n(E),\Theta_E}$ the quotient space $V/V(N_n(E), \Theta_E)$, where $V(N_n(E), \Theta_E)$ is the space of all $v \in V$ such that

$$\int_{\mathcal{N}_v} \Theta_E(n)^{-1}\pi(n)v\, dn = 0$$

for some compact open subgroup $\mathcal{N}_v$ (depending on v) of $N_n(E)$. Here dn is a Haar measure on $N_n(E)$. Since $N_n(E)$ is Γ-stable, the compact open subgroup $\mathcal{N}_v$ can be chosen to be Γ-stable. Then we have the following lemma.

Lemma 2.3.4. *Let π be a generic R-representation of $G_n(E)$. Then Γ acts freely on the space $\mathbb{W}(\pi, \psi_{E,R})_{N_n(E),\Theta_E}$.*

Proof. Let W_v be an arbitrary element of $\mathbb{W}(\pi, \psi_{E,R})$. We have to show that $\gamma.W_v - W_v \in \mathbb{W}(\pi, \psi_{E,R})(N_n(E), \Theta_E)$ for $\gamma \in \Gamma$. Choose a Γ-stable compact open subgroup $\mathcal{N} \subseteq N_n(E)$. Then

$$\begin{aligned}\int_{\mathcal{N}} \Theta_E(n)^{-1}(\gamma.W_v - W_v)(n)\, dn &= \int_{\mathcal{N}} \Theta_E(\gamma^{-1}(n))W_v(\gamma^{-1}(n))\, dn - \int_{\mathcal{N}} \Theta_E(n)^{-1}W_v(n)\, dn \\ &= c(\gamma)\int_{\mathcal{N}} \Theta_E(n)W_v(n)\, dn - \int_{\mathcal{N}} \Theta_E(n)W_v(n)\, dn,\end{aligned}$$

where $c(\gamma) \in R^\times$. Since $\mathcal{N}$ is compact, we have $c(\gamma) = 1$. Hence the lemma. □

2.3.3 Segments

In this subsection, we recall the notion of segments and its associated representations. Here, R denotes either $\overline{\mathbb{Q}}_l$ or $\overline{\mathbb{F}}_l$. For a precise reference, see [Zel80] for $R = \overline{\mathbb{Q}}_l$, and [MS14], [KM17] for

$R = \overline{\mathbb{F}}_l$. Let r, t be two integers with $r \leq t$. A segment is a sequence $\Delta = (\nu_K^r \sigma, \nu_K^{r+1}\sigma, ..., \nu_K^t \sigma)$ with σ, a cuspidal R-representation of $G_n(K)$. The length of Δ is defined to be $t - r + 1$. The parabolically induced representation

$$\tau = \nu_K^r \sigma \times \nu_K^{r+1}\sigma \times \cdots \times \nu_K^t \sigma.$$

has a quotient $\mathcal{L}(\Delta)$ such that its normalised Jacquet module with respect to the opposite of the parabolic subgroup $P_{(n,\cdots,n)}$ is equal to $\nu_K^r \sigma \otimes \cdots \otimes \nu_K^t \sigma$. Moreover, there is a unique generic sub-quotient of τ, denoted by $\mathrm{St}(\sigma, [r, t])$ and it is called the generalised Steinberg representation associated to Δ. We denote by $\mathrm{St}(\sigma, k)$ the representation $\mathrm{St}(\sigma, [0, k-1])$, for $k \geq 1$.

Let σ be a cuspidal R-representation of $G_n(K)$. The set $\{\nu_K^r \sigma : r \in \mathbb{Z}\}$ is called the *cuspidal line* of σ and the cardinality of this set is denoted by $o(\sigma)$. Recall that [MS14, Section 5.2] defines a positive integer $e(\sigma)$ as follows:

$$e(\sigma) = \begin{cases} +\infty & \text{if } R = \overline{\mathbb{Q}}_l; \\ o(\sigma) & \text{if } R = \overline{\mathbb{F}}_l \text{ and } o(\sigma) > 1; \\ l & \text{if } R = \overline{\mathbb{F}}_l \text{ and } o(\sigma) = 1. \end{cases} \tag{2.2}$$

For a segment $\Delta = (\nu_K^r \sigma, \ldots, \nu_K^t \sigma)$ with $r \leq t$, the representation $\mathcal{L}(\Delta)$ is equal to $\mathrm{St}(\sigma, [r, t])$ if and only if the length of the segment Δ is less than $e(\sigma)$([MS14, Remarque 8.14]). In this case, the segment Δ is called a *generic segment*. For $R = \overline{\mathbb{Q}}_l$, every segment is generic.

Two segments Δ_1 and Δ_2 are said to be *linked* if $\Delta_1 \nsubseteq \Delta_2$, $\Delta_2 \nsubseteq \Delta_1$ and $\Delta_1 \cup \Delta_2$ is a segment. Let π be an R-representation of $G_n(K)$ of the form $\mathcal{L}(\Delta_1) \times \cdots \times \mathcal{L}(\Delta_t)$, where each Δ_j is a generic segment. Then we have a necessary and sufficient condition for the irreducibility of π (see [Zel80, Theorem 9.7] for $R = \overline{\mathbb{Q}}_l$, and [MS14, Theorem 9.10] for $R = \overline{\mathbb{F}}_l$).

Theorem 2.3.5. *The representation π is irreducible if and only if the segments Δ_i and Δ_j are not linked for all i, j with $i \neq j$.*

In the articles [BZ77, Theorem 9.7] and [Vig98, Proposition V.3], it is shown that

Theorem 2.3.6. *An R-representation π of $G_n(K)$ is generic if and only if π is an irreducible R-representation of the form $\mathcal{L}(\Delta_1) \times \cdots \times \mathcal{L}(\Delta_t)$ with each Δ_i a generic segment.*

2.3.4 Generic lift

In this subsection, we fix a standard lift of an l-modular generic representation of $G_n(K)$. First recall that any l-modular cuspidal representation of $G_m(K)$ can be lifted to an l-adic cuspidal representation of $G_m(K)$ (see [Vig96b, Chapter 3, 4.25]). Let ρ be an l-modular cuspidal representation of $G_m(K)$ and let $\Delta = (\rho, \overline{\nu}_K\rho, \ldots, \overline{\nu}_K^{r-1}\rho)$ be a segment associated with ρ, where $\overline{\nu}_K$ is the mod-l reduction of ν_K. Let σ be a cuspidal lift of ρ. Then the segment $D = (\sigma, \nu_K\sigma, \ldots, \nu_K^{r-1}\sigma)$ is called a standard lift of Δ. When $\mathcal{L}(\Delta) = \mathrm{St}(\rho, r)$, then the mod-$l$ representation $\mathcal{L}(\Delta)$ lifts to $\mathcal{L}(D) = \mathrm{St}(\sigma, r)$ ([KM17, Proposition 2.16]).

Let π be a generic l-modular representation of $G_n(K)$. Then π is of the form $\mathcal{L}(\Delta_1)\times\cdots\times\mathcal{L}(\Delta_t)$, where each Δ_i is a generic segment. For each $1 \leq i \leq t$, let D_i be a standard lift of Δ_i. Then the l-adic representation $\tau = \mathcal{L}(D_1)\times\cdots\times\mathcal{L}(D_t)$ is generic, and π lifts to τ ([KM17, Remark 2.31]). The representation τ is a standard lift of π.

2.3.5 Whittaker lattice and Integral Kirillov model

Let (π, V) be an integral l-adic generic representation of $G_n(K)$. Consider the space $\mathbb{W}^0(\pi, \psi_K)$ consisting of $W \in \mathbb{W}(\pi, \psi_K)$, taking values in $\overline{\mathbb{Z}}_l$. The work of [Vig04, Theorem 2] shows that $\mathbb{W}^0(\pi, \psi_K)$ is a $G_n(K)$-invariant lattice in $\mathbb{W}(\pi, \psi_K)$. The lattice $\mathbb{W}^0(\pi, \psi_K)$ is also called the integral Whittaker model or *Whittaker lattice*. Let τ be an l-modular generic representation of $G_n(K)$. An integral l-adic generic representation π of $G_n(K)$ is called *Whittaker lift* of τ if there exists a lattice $\mathcal{L} \subseteq \mathbb{W}^0(\pi, \psi_K)$ such that

$$\mathcal{L} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l \simeq \mathbb{W}(\tau, \overline{\psi}_K),$$

where $\overline{\psi}_K$ is the reduction mod-l of ψ_K. Note that any standard lift of a generic l-modular representation π is a Whittaker lift (see [KM17, Theorem 2.26]). Let (π, V) be an integral l-adic generic representation of $G_n(K)$ defined over the field $\mathcal{K}$. Let V_0 be a $\mathcal{K}$-structure in V. There exists a $G_n(K)$ invariant Λ-lattice in V_0. It follows from [Vig04, Theorem 2] that the set of Λ-valued functions in the Whittaker space $\mathbb{W}(\pi, \psi_K)$, denoted by $\mathbb{W}_\Lambda(\pi, \psi_K)$, is a $G_n(K)$ invariant Λ-lattice.

2.4 Local constants of p-adic representations

In this section, we first review Rankin–Selberg theory over arbitrary Noetherian $W(\overline{\mathbb{F}}_l)$-algebras. For precise reference, see [Mos16, Section 3]. This is analogous to the Rankin–Selberg theory over the fields $\overline{\mathbb{Q}}_l$ and $\overline{\mathbb{F}}_l$ (see [JPSS83] for $\overline{\mathbb{Q}}_l$ and [KM17] for $\overline{\mathbb{F}}_l$). Then we recall and put up some initial results on local constants associated with l-adic and l-modular generic representations of $G_n(K)$.

2.4.1 Rankin-Selberg gamma factors in families

To define the Rankin–Selberg gamma factors in families, we need to recall the notion of co-Whittaker modules.

Definition 2.4.1. Let R be a Noetherian $W(\overline{\mathbb{F}}_l)$-algebra. A smooth admissible $R[G_n(K)]$ module (π, V) is called *co-Whittaker* if $V^{(n)}$ is a free R-module of rank one, and for any non-zero quotient W of V, the space $W^{(n)}$ is non-zero.

A co-Whittaker $R[G_n(K)]$ module (π, V) admits a central character, denoted by $\omega_\pi : F^\times \to R^\times$. By definition, the module (π, V) is of Whittaker type and hence admits a Whittaker model.

We now introduce some notations. Let X_K be the coset space $N_{n-1}(K) \setminus G_{n-1}(K)$. For each positive integer r, we set $G_{n-1}^r(K) = \big\{g \in G_{n-1}(K) : v_K(\det(g)) = r\big\}$. We denote by X_K^r the coset space

$$\Big\{N_{n-1}(K)g : g \in G_{n-1}^r(K)\Big\}.$$

Let A and B be two Noetherian $W(\overline{\mathbb{F}}_l)$-algebras. Let (π, V) and (π', V') be two co-Whittaker $A[G_n(K)]$- and $B[G_{n-1}(K)]$-modules, respectively. For $W \in \mathbb{W}(\pi, \psi_{K,A})$ and $W' \in \mathbb{W}(\pi', \psi_{K,B}^{-1})$, the function $g \mapsto W\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \otimes W'(g)$ is compactly supported on each X_K^r, and takes values in $A \otimes_{W(\overline{\mathbb{F}}_l)} B$. The following integral

$$c_r^K(W, W') := \int_{X_K^r} W\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \otimes W'(g)\, dg$$

is well-defined for all integers r, and it is zero for $r << 0$ (see [Mos16, Section 3]). The formal Laurent series

$$\sum_{r\in\mathbb{Z}} c_r^K(W, W')X^r,$$

is an element of $S^{-1}((A \otimes_{W(\overline{\mathbb{F}}_l)} B)[X, X^{-1}])$, where S is the multiplicative set consisting of polynomials $\sum_{r=t}^{s} a_r X^r$ with a_t and a_s being units in $A \otimes_{W(\overline{\mathbb{F}}_l)} B$. In this thesis, we deal with base change where two different p-adic fields are involved. So, to avoid confusion, we use the notation $c_r^K(W, W')$ instead of $c_r(W, W')$, used in the article [Mos16, Section 5]. Now, let us consider the functions $\widetilde{W}$ and $\widetilde{W'}$, defined as

$$\widetilde{W}(g) = W\big(w_n(g^t)^{-1}\big),$$

$$\widetilde{W'}(x) = W'\big(w_{n-1}(x^t)^{-1}\big),$$

for all $g \in G_n(K)$ and $x \in G_{n-1}(K)$. Making a change of variables, we get the following identity:

$$c_r^K(\widetilde{W}, \widetilde{W'}) = c_{-r}^K\big(\pi(w_n)W, \pi'(w_{n-1})W'\big). \tag{2.3}$$

In [Mos16, Theorem 5.3], the author proved the following functional equation for pair of co-Whittaker modules.

Theorem 2.4.2. *Let π and π' be two co-Whittaker $A[G_n(K)]$- and $B[G_{n-1}(K)]$-modules respectively. Then, there is a unique element $\gamma(X, \pi, \pi', \psi_K)$ in the ring of fraction $S^{-1}((A \otimes_{W(\overline{\mathbb{F}}_l)} B)[X, X^{-1}])$ such that*

$$\sum_{r\in\mathbb{Z}} c_r^K(\widetilde{W}, \widetilde{W'})X^{-r} = \omega_{\pi'}(-1)^{n-1}\gamma(X, \pi, \pi', \psi_K) \sum_{r\in\mathbb{Z}} c_r^K(W, W')X^r, \tag{2.4}$$

for all $W \in \mathbb{W}(\pi, \psi_{K,A})$ and $W' \in \mathbb{W}(\pi', \psi_{K,B}^{-1})$.

The notation in the above theorem is slightly different from that in [JPSS83]. With these, we now recall the proof of Theorem 2.3.2.

Theorem 2.4.3. *The restriction map $W \mapsto \mathrm{res}_{P_n}(W)$ gives an isomorphism*

$$\mathbb{W}(\pi, \psi_{K,R}) \xrightarrow{\sim} \mathbb{K}(\pi, \psi_{K,R}).$$

Proof. Let W be an arbitrary element of $\mathbb{W}(\pi, \psi_{K,R})$ such that $\mathrm{res}_{P_n}(W) = 0$. Then, from functional equation (2.4), we get

$$\sum_{r\in\mathbb{Z}} c_r^K(\widetilde{W}, \widetilde{W'})X^{-r} = 0,$$

for all $W' \in \mathbb{W}(\pi', \psi_{K,R}^{-1})$, where π' varies through all co-Whittaker $R'[G_n]$ modules and R' varies through all finitely generated Noetherian Λ-algebras. This shows that $c_r^K(\widetilde{W}, \widetilde{W'}) = 0$, for each $r \in \mathbb{Z}$. Since the restriction W to G_{n-1}^r is compactly supported, it follows from [HM18, Corollary 3.6] that $\widetilde{W} = 0$ on the coset space G_{n-1}^r, for each integer r.

Thus, $\widetilde{W}$ vanishes on G_{n-1} and hence on P_n, as $P_n = N_n \rtimes G_{n-1}$. Using the same argument in other direction, we get that W vanishes on P_n if and only if $\widetilde{W}$ vanishes on P_n. Then the R-submodule $\mathcal{M}$, consisting of Whittaker functions in $\mathbb{W}(\pi, \psi_{K,R})$ with $\mathrm{res}_{P_n}(W) = 0$, is stable under both P_n and the subgroup $\overline{P}_n$, opposite to P_n. It implies that $\mathcal{M}$ is stable under the action of G_n. Hence, we get that $\mathcal{M} = (0)$. $\square$

Next, we define the local constants such as L-factors and ϵ-factors associated with pair of R-representations of Whittaker type, where $R = \overline{\mathbb{Q}}_l$ or $\overline{\mathbb{F}}_l$. We follow the notations of [JPSS83] and [KM17].

2.4.2 L -factors

Let π and π' be two R-representations of $G_n(K)$ and $G_{n-1}(K)$ respectively, both of Whittaker type. For any $W \in \mathbb{W}(\pi, \psi_{K,R})$ and $W' \in \mathbb{W}(\pi', \psi_{K,R}^{-1})$, consider the following power series:

$$I(X, W, W') := \sum_{r \in \mathbb{Z}} c_r^K(W, W') q_K^{r/2} X^r.$$

Note that $I(X, W, W')$ is a rational function of X ([KM17, Theorem 3.5]). Then the R-subspace generated by $I(X, W, W')$ as W varies in $\mathbb{W}(\pi, \psi_{K,R})$ and W' varies in $\mathbb{W}(\pi', \psi_{K,R}^{-1})$, is a fractional ideal of $R[X, X^{-1}]$. It has a unique generator, which is an Euler factor denoted by $L(X, \pi, \pi')$. The generator $L(X, \pi, \pi')$ is called the L-factor associated with π, π' and ψ_K. When $R = \overline{\mathbb{Q}}_l$, then $1/L(X, \pi, \pi') \in \overline{\mathbb{Z}}_l[X]$.

We conclude this subsection with a result [KM17, Theorem 4.3], which describes L-factor of cuspidal representations.

Theorem 2.4.4. *Let π_1 and π_2 be two cuspidal R-representations of $G_n(K)$ and $G_m(K)$ respectively. Then $L(X, \pi_1, \pi_2)$ is equal to 1, except in the following case : π_1 is banal in the sense of* [MS14] *and π_2 is isomorphic to $\chi\pi_1^\vee$, for some unramified character χ of $K^\times$.*

In particular, when $m = n - 1$, it follows from the above theorem that $L(X, \pi_1, \pi_2)$ associated with the cuspidal R-representations π_1 and π_2 is equal to 1.

2.4.3 Functional Equation and Local ϵ-factors

Let π and π' be two R-representations of Whittaker type of $G_n(K)$ and $G_{n-1}(K)$, respectively. Then, there exists an invertible element $\epsilon(X, \pi, \pi', \psi_K)$ in the ring $R[X, X^{-1}]$ such that for all $W \in \mathbb{W}(\pi, \psi_{K,R})$, $W' \in \mathbb{W}(\pi', \psi_{K,R}^{-1})$, we have the following functional equation:

$$\frac{I(q_K^{-1}X^{-1}, \widetilde{W}, \widetilde{W'})}{L(q_K^{-1}X^{-1}, \widetilde{\pi}, \widetilde{\pi'})} = \omega_{\pi'}(-1)^{n-2}\epsilon(X, \pi, \pi', \psi_K)\frac{I(X, W, W')}{L(X, \pi, \pi')},$$

where $\omega_{\pi'}$ denotes the central character of π'. The element $\epsilon(X, \pi, \pi', \psi_K)$ is called the local ϵ-factor associated to π, π' and ψ_K. If π and π' are l-adic, then the factor $\epsilon(X, \pi, \pi', \psi_K)$ is of the form cX^k, for some $c \in \overline{\mathbb{Z}}_l^\times$. In particular, there exists an integer $n(\pi, \pi', \psi_K)$ such that

$$\epsilon(X, \pi, \pi', \psi_K) = (q_K^{\frac{1}{2}}X)^{n(\pi, \pi', \psi_K)}\epsilon(\pi, \pi', \psi_K). \tag{2.5}$$

The local γ-factor associated with π, π' and ψ_K is defined as

$$\gamma(X, \pi, \pi', \psi_K) = \epsilon(X, \pi, \pi', \psi_K)\frac{L(q_K^{-1}X^{-1}, \widetilde{\pi}, \widetilde{\pi'})}{L(X, \pi, \pi')}.$$

2.4.4 Compatibility with reduction modulo l

Here, we recall the compatibility of gamma factors under mod-l reduction. For reference, see [KM17, Section 3.3]. Let τ and τ' be two l-modular representations of $G_n(K)$ and $G_{n-1}(K)$ respectively, both of Whittaker type. Let π and π' be the respective Whittaker lifts of τ and τ'. Then we have

$$r_l((\gamma(X,\pi,\pi',\psi_K)) = \gamma(X,\tau,\tau',\overline{\psi}_K).$$

2.4.5 Generic part of mod-l reduction

Let π be an integral l-adic generic representation of $G_n(K)$. The mod-l reduction $r_l(\pi)$ has a unique generic sub-quotient and it is denoted by $J_l(\pi)$ (see [Vig01, Section 1.8.4]). Let σ be an integral l-adic generic representation of $G_{n-1}(K)$. The functional equation associated with the pair $(J_l(\pi), J_l(\sigma))$ gives

$$I(q_K^{-1}X^{-1},\widetilde{W},\widetilde{W'}) = \varpi_\sigma(-1)^{n-2}\gamma(X,J_l(\pi),J_l(\sigma),\overline{\psi}_K)I(X,W,W'),$$

for all $W \in \mathbb{W}(J_l(\pi),\overline{\psi}_K)$ and $W' \in \mathbb{W}(J_l(\sigma),\overline{\psi}_K^{-1})$. Let us consider the following commutative diagram:

$$\begin{tikzcd} \mathbb{W}^0(\pi,\psi_K) \arrow[r, "\Lambda_\pi"] \arrow[d, "\mathrm{res}_{P_n(K)}"'] & \mathrm{Ind}_{N_n(K)}^{G_n(K)}\overline{\Theta}_K \arrow[d, "\mathrm{res}_{P_n(K)}"] \\ \mathbb{K}^0(\pi,\psi_K) \arrow[r, "\lambda_\pi"'] & \mathrm{Ind}_{N_n(K)}^{P_n(K)}\overline{\Theta}_K \end{tikzcd}$$

Note that that restriction to $P_n(K)$ map on $W^0(\pi,\psi_K)$ is an isomorphism. Here Λ_π and λ_π are pointwise mod-l reduction maps. Since $\mathcal{K}^0(\psi_K)$ is contained in $\mathbb{K}^0(\pi,\psi_K)$ and λ_π maps $\mathcal{K}^0(\psi_K)$ onto $\mathcal{K}(\overline{\psi}_K)$, the $P_n(K)$ equivariant map λ_π is non-zero. Then, it follows from the commutativity of the above diagram that Λ_π is non-zero. Since $J_l(\pi)$ is the unique generic sub-quotient of $r_l(\pi)$, the image of Λ_π contains the Whittaker space $\mathbb{W}(J_l(\pi),\overline{\psi}_K)$. Similarly, the image of Λ_σ contains $\mathbb{W}(J(\sigma),\overline{\psi}_K)$. Let U (resp. U') be an element of $\mathbb{W}^0(\pi,\psi_K)$ (resp. $\mathbb{W}^0(\sigma,\psi_K)$) such that $\Lambda_\pi(U) = W$ (resp. $\Lambda_\tau(U') = W'$). From functional equation for the pair (π,σ), we get the following relation

$$I(q_K^{-1}X^{-1},\widetilde{U},\widetilde{U'}) = \varpi_\sigma(-1)^{n-2}\gamma(X,\pi,\sigma,\psi_K)I(X,U,U').$$

After reducing the above identity mod-l, we have

$$I(q_K^{-1}X^{-1}, \widetilde{W}, \widetilde{W'}) = \varpi_\sigma(-1)^{n-2} r_l(\gamma(X, \pi, \sigma, \psi_K)) I(X, W, W'),$$

Thus, we get that

$$r_l(\gamma(X, \pi, \sigma, \psi_K)) = \gamma(X, J_l(\pi), J_l(\sigma), \overline{\psi}_K). \tag{2.6}$$

2.4.6 Frobenius Twist

Let G be a locally compact and totally disconnected group. Let (π, V) be an l-modular representation of G. Consider the $\overline{\mathbb{F}}_l$-vector space $V^{(l)}$, where the underlying additive group structure of $V^{(l)}$ is same as that of V but the scalar action $*$ on $V^{(l)}$ is given by

$$c * v := c^{\frac{1}{l}} v,$$

for all $c \in \overline{\mathbb{F}}_l$ and $v \in V$. Then the action of G on V induces a representation of G on the space $V^{(l)}$, denoted by $\pi^{(l)}$. The representation $(\pi^{(l)}, V^{(l)})$ is called the *Frobenius twist* of (π, V).

Lemma 2.4.5. *Let ψ be a non-trivial l-modular additive character of K, and let Θ be the non-degenerate character of $N_n(K)$ corresponding to ψ. If (π_1, V_1) and (π_2, V_2) are two l-modular generic representations of $G_n(F)$ and $G_{n-1}(F)$ respectively, then*

$$\gamma(X, \pi_1, \pi_2, \psi)^l = \gamma((\frac{q_K}{l})X^l, \pi_1^{(l)}, \pi_2^{(l)}, \psi^l),$$

where $(\frac{q_K}{l})$ is the Legendre symbol.

Proof. Let $\mathcal{W}_{\pi_1}$ be a Whittaker functional of π_1. Then the composition

$$V_1 \xrightarrow{\mathcal{W}_{\pi_1}} \overline{\mathbb{F}}_l \xrightarrow{x \mapsto x^l} \overline{\mathbb{F}}_l,$$

denoted by $\mathcal{W}_{\pi_1^{(l)}}$, is a Whittaker functional (with respect to $\overline{\psi}^l : K \to \overline{\mathbb{F}}_l^\times$) of the representation $\pi_1^{(l)}$, as we have

$$\mathcal{W}_{\pi_1^{(l)}}(c * v) = \mathcal{W}_{\pi_1}((c^{\frac{1}{l}} v))^l = c\mathcal{W}_{\pi_1^{(l)}}(v)$$

and

$$\mathcal{W}_{\pi_1^{(l)}}(\pi_1^{(l)}(x)v) = \big(\Theta(x)\mathcal{W}_{\pi_1}(v)\big)^l = \Theta(x)^l \mathcal{W}_{\pi_1^{(l)}}(v),$$

for all $v \in V_1$, $c \in \overline{\mathbb{F}}_l$ and $x \in N_n(K)$. Therefore, the Whittaker model $\mathbb{W}(\pi_1^{(l)}, \psi^l)$ consists of the functions W_v^l, where W_v varies in $\mathbb{W}(\pi_1, \psi)$. Similarly, the Whittaker model $\mathbb{W}(\pi_2^{(l)}, \psi^l)$ of $\pi_2^{(l)}$ consists of the functions U_v^l, where U_v varies in the Whittaker model $\mathbb{W}(\pi_2, \psi)$. From the

Rankin–Selberg functional equation (see Subsection 2.4.3), we have

$$\sum_{r\in\mathbb{Z}} c_r^K(\widetilde{W_v},\widetilde{U_v})^l q_K^{-lr/2}X^{-lr} = \omega_{\pi_2}(-1)^{n-2}\gamma(X,\pi_1,\pi_2,\psi)^l \sum_{r\in\mathbb{Z}} c_r^K(W_v,U_v)^l q_K^{lr/2}X^{lr} \tag{2.7}$$

and

$$\sum_{r\in\mathbb{Z}} c_r^K(\widetilde{W_v^l},\widetilde{U_v^l}) q_K^{-r/2}X^{-r} = \omega_{\pi_2^{(l)}}(-1)^{n-2}\gamma(X,\pi_1^{(l)},\pi_2^{(l)},\psi^l) \sum_{r\in\mathbb{Z}} c_r^K(W_v^l,U_v^l) q_K^{r/2}X^{r}. \tag{2.8}$$

Using the identity $q_K^{l/2} = (\frac{q_K}{l})\, q_K^{1/2}$ to the equation (2.7), we get

$$\sum_{r\in\mathbb{Z}} c_r^K(\widetilde{W_v},\widetilde{U_v})^l q_K^{-r/2}\big((\frac{q_K}{l})X^l\big)^{-r} = \omega_{\pi_2}(-1)^{n-2}\gamma(X,\pi_1,\pi_2,\psi)^l \sum_{r\in\mathbb{Z}} c_r^K(W_v,U_v)^l q_K^{r/2}\big((\frac{q_K}{l})X^l\big)^{r}. \tag{2.9}$$

Replacing X by $(\frac{q_K}{l})X^l$ to the equation (2.8), we have

$$\begin{aligned}\sum_{r\in\mathbb{Z}} c_r^K(\widetilde{W_v^l},\widetilde{U_v^l})\, q_K^{-r/2}\big((\frac{q_K}{l})X^l\big)^{-r}& \\ = \omega_{\pi_2^{(l)}}(-1)^{n-2}\gamma((\frac{q_K}{l})X^l,\pi_1^{(l)},\pi_2^{(l)},\psi^l) &\sum_{r\in\mathbb{Z}} c_r^K(W_v^l,U_v^l) q_K^{r/2}\big((\frac{q_K}{l})X^l\big)^{r}.\end{aligned} \tag{2.10}$$

Since $q_K^l = q_K$, we have $c_r^K(W_v,U_v)^l = c_r^K(W_v^l,U_v^l)$ for all $r \in \mathbb{Z}$. Similarly, $c_r^K(\widetilde{W_v},\widetilde{U_v})^l = c_r^K(\widetilde{W_v^l},\widetilde{U_v^l})$. Then, it follows from the identities (2.9) and (2.10) that

$$\gamma(X,\pi_1,\pi_2,\psi)^l = \gamma((\frac{q_K}{l})X^l,\pi_1^{(l)},\pi_2^{(l)},\psi^l).$$

This completes the proof. □

2.5 Review of Weil-Deligne representations

This section discusses the notion of Weil group and the Weil-Deligne representations. We also review the local constants associated with Weil-Deligne representations. For precise references, see [BH06, Chapter 7], [Del73, Chapter 4].

We choose a separable algebraic closure $\overline{K}$ of K. Let Ω_K denote the absolute Galois group $\mathrm{Gal}(\overline{K}/K)$. We have the quotient map

$$\mathrm{Gal}(\overline{K}/K) \longrightarrow \mathrm{Gal}(\overline{\kappa}_K/\kappa_K) \tag{2.11}$$

where $\overline{\kappa}_K$ is an algebraic closure of the finite field κ_K. The kernel of the map (2.11) is called the *inertia* subgroup of Ω_K, denoted by $\mathcal{I}_K$. Let Frob_K denote the Frobenius automorphism of $\overline{\kappa}_K$. Let $\mathcal{W}_K$ be the inverse image in Ω_K of the group generated by Frob_K. The group $\mathcal{W}_K$ is called the *Weil group* of K, and we have

$$\mathcal{W}_K = \mathcal{I}_K \rtimes \mathrm{Frob}_K^{\mathbb{Z}}.$$

There is a natural Krull topology on the absolute Galois group Ω_K and the inertia group $\mathcal{I}_K$, as a subgroup of Ω_K, is equipped with the subspace topology. Declaring the open subgroups of $\mathcal{I}_K$ as open subgroups of $\mathcal{W}_K$ makes $\mathcal{W}_F$ a topological group, whose topology is locally compact and totally disconnected. If K_1/K is a finite seperable extension with $K_1 \subseteq \overline{K}$, then the Weil group $\mathcal{W}_{K_1}$ is considered as a subgroup of $\mathcal{W}_K$.

2.5.1 Semisimple Weil-Deligne representation

In this subsection, R will denote either $\overline{\mathbb{Q}}_l$ or $\overline{\mathbb{F}}_l$. A smooth R-representation ρ of $\mathcal{W}_K$ is called *unramified* if ρ is trivial on the inertia subgroup $\mathcal{I}_K$. Let ν_K be the unramified character of $\mathcal{W}_K$, which satisfies $\nu_K(\mathrm{Frob}_K) = q_K^{-1}$.

An Weil–Deligne R-representation of $\mathcal{W}_K$ is a pair (ρ, U), where ρ is a finite-dimensional R-representation of $\mathcal{W}_K$ and U is a nilpotent endomorphism of the vector space underlying ρ, intertwining the actions of $\nu_K\rho$ and ρ. Moreover, the representation (ρ, U) is called *semisimple* if ρ is semisimple as a representation of $\mathcal{W}_K$. Note that any semisimple representation ρ of $\mathcal{W}_K$ is considered as a semisimple Weil–Deligne representation of the form $(\rho, 0)$. For two Weil–Deligne representations (ρ, U) and (ρ', U') of $\mathcal{W}_K$, set

$$\mathrm{Hom}((\rho, U), (\rho', U')) := \big\{ f \in \mathrm{Hom}_{\mathcal{W}_K}(\rho, \rho') : f \circ U = U' \circ f \big\},$$

We say that (ρ, U) and (ρ', U') are isomorphic if there exists a map $f \in \mathrm{Hom}((\rho, U),\, (\rho', U'))$ such that f is bijective.

2.5.2 Local Constants of Weil-Deligne representation

In this subsection, we consider the local L-factors and local ϵ-factors for l-adic semisimple Weil-Deligne representations of $\mathcal{W}_K$. We follow the notations of the previous subsection.

Let (ρ, U) be an l-adic semisimple Weil-Deligne representation of $\mathcal{W}_K$. Then, the L-factor associated with (ρ, U) is the following rational function in X:

$$L(X, (\rho, U)) = \det((\mathrm{id} - X\rho(\mathrm{Frob}))|_{\mathrm{Ker}(U)^{\mathcal{I}_K}})^{-1}.$$

We choose a self-dual additive Haar measure on K with respect to the character $\psi_K : K \to \overline{\mathbb{Q}}_l^{\times}$. Let ρ be a smooth l-adic representation of $\mathcal{W}_K$. The epsilon factor $\epsilon(X, \rho, \psi_K)$ associated with ρ

and ψ_K is defined in [Del73, Theorem 6.5]. Let K'/K be a finite seperable extension contained in $\overline{K}$. Let $\psi_{K'}$ be the composition $\psi_K \circ \mathrm{Tr}_{K'/K}$. Then, the epsilon factor satisfies the following properties:

1. If ρ_1 and ρ_2 are two l-adic representations of $\mathcal{W}_K$, then
$$\epsilon(X, \rho_1 \oplus \rho_2, \psi_K) = \epsilon(X, \rho_1, \psi_K)\, \epsilon(X, \rho_2, \psi_K).$$

2. If ρ is an l-adic representation of $\mathcal{W}_{K'}$, then
$$\frac{\epsilon\big(X, \mathrm{ind}_{\mathcal{W}_{K'}}^{\mathcal{W}_K}(\rho), \psi_K\big)}{\epsilon(X, \rho, \psi_{K'})} = \left\{ \frac{\epsilon\big(X, \mathrm{ind}_{\mathcal{W}_{K'}}^{\mathcal{W}_K}(1_{K'}), \psi_K\big)}{\epsilon(X, 1_{K'}, \psi_{K'})} \right\}^{\dim(\rho)}, \tag{2.12}$$
where $1_{K'}$ denotes the trivial character of $\mathcal{W}_{K'}$.

3. If ρ is an l-adic representation of $\mathcal{W}_K$, then
$$\epsilon(X, \rho, \psi_K)\, \epsilon(q_K^{-1}X^{-1}, \rho^\vee, \psi_K) = \det(\rho(-1)), \tag{2.13}$$
where $\rho^\vee$ denotes the smooth dual of ρ.

4. For an l-adic representation ρ of $\mathcal{W}_K$, there exists an integer $n(\rho, \psi_K)$ for which
$$\epsilon(X, \rho, \psi_K) = (q_K^{\frac{1}{2}}X)^{n(\rho,\psi_K)}\epsilon(\rho, \psi_K).$$

Now, for an l-adic semisimple Weil-Deligne representation (ρ, U), the ϵ-factor is defined as
$$\epsilon\big(X, (\rho, U), \psi_K\big) = \epsilon(X, \rho, \psi_K)\frac{L(q_K^{-1}X^{-1}, \rho^\vee)}{L(X, \rho)}\frac{L(X, (\rho, U))}{L(q_K^{-1}X^{-1}, (\rho, U)^\vee)},$$
where $(\rho, U)^\vee = (\rho^\vee, -U^\vee)$ and the representation $\rho^\vee$ is contragradient to ρ. Set
$$\gamma(X, (\rho, U), \psi_K) = \epsilon(X, (\rho, U), \psi_K)\frac{L(X, (\rho, U))}{L(q_K^{-1}X^{-1}, (\rho, U)^\vee)}. \tag{2.14}$$
The element $\gamma(X, (\rho, U), \psi_K)$ is called the γ-factor of the semisimple Weil–Deligne representation (ρ, U).

Now, we state a result which concerns the fact that the γ-factors are compatible with mod-l reduction ([KM21, Proposition 5.8]). For any $P \in \overline{\mathbb{Z}}_l[X]$, we denote by $r_l(P) \in \overline{\mathbb{F}}_l[X]$ the polynomial obtained by reduction modulo the maximal ideal $\mathfrak{P}_l$ to the coefficients of P. For $Q \in \overline{\mathbb{Z}}_l[X]$ with $r_l(Q) \neq 0$, we set $r_l(P/Q) = r_l(P)/r_l(Q)$.

Proposition 2.5.1. *Let $\overline{\psi}_K$ be the mod-l reduction of ψ_K. Let ρ be a smooth, integral, l-adic semisimple representation of $\mathcal{W}_K$. Then*

$$r_l\big(\gamma(X,\rho,\psi_K)\big) = \gamma\big(X, r_l(\rho), \overline{\psi}_K\big).$$

We end this subsection with a lemma, which is crucial for the proof of the main theorems.

Lemma 2.5.2. *Let E/F be a finite Galois extension of prime degree l. We further assume that l is odd. Let ρ be an l-adic semisimple representation of $\mathcal{W}_E$ of even dimension. Then*

$$\epsilon(X,\rho,\psi_E) = \epsilon\big(X, \mathrm{ind}_{\mathcal{W}_E}^{\mathcal{W}_F}(\rho), \psi_F\big).$$

Proof. Let $\mathcal{C}_{E/F}(\psi_F) = \dfrac{\epsilon\big(X, \mathrm{ind}_{\mathcal{W}_E}^{\mathcal{W}_F}(1_E), \psi_F\big)}{\epsilon(X, 1_E, \psi_E)}$, where 1_E is the trivial character of $\mathcal{W}_E$. Then $\mathcal{C}_{E/F}(\psi_F)$ is independent of X (see [BH06, Corollary 30.4, Chapter 7]). Using the equality (2.12), we get

$$\frac{\epsilon\big(X, \mathrm{ind}_{\mathcal{W}_E}^{\mathcal{W}_F}(\rho), \psi_F\big)}{\epsilon(X,\rho,\psi_E)} = \big(\mathcal{C}_{E/F}(\psi_F)\big)^{\dim\rho}.$$

In view of the functional equation (2.13), we have

$$\mathcal{C}_{E/F}(\psi_F)^2 = \xi_{E/F}(-1),$$

where $\xi_{E/F} = \det\big(\mathrm{ind}_{\mathcal{W}_E}^{\mathcal{W}_F}(1_E)\big)$, a quadratic character of $\mathcal{W}_F$. Note that $\xi_{E/F}$ factorizes through the quotient group $\mathcal{W}_F/\mathcal{W}_E$–which is a group of prime order l, and this implies that $\xi_{E/F}^l = 1$. Since l is odd, we get that $\xi_{E/F}$ is trivial. Hence the lemma. □

2.5.3 Deligne-Langlands gamma factors in families

In this subsection, we briefly recall Weil-Deligne representation over Noetherian $W(\overline{\mathbb{F}}_l)$-algebras and its associated γ-factors. For details, we refer to [HM15]. The contents of this subsection will be used in Chapter 5.

Let R be a Noetherian $W(\overline{\mathbb{F}}_l)$-algebra. For each non-negetive integer m, let $\mathcal{J}_m$ be the kernel of the map $\mathrm{GL}_n(R) \to \mathrm{GL}_n(R/l^mR)$.

Definition 2.5.3. A representation $\rho : \mathcal{W}_K \to \mathrm{GL}_n(R)$ is said to be *l-adically continuous* if, for all m, the preimage set

$$\rho^{-1}(\mathcal{J}_m) = \{x \in \mathcal{W}_K : \rho(x) \in \mathcal{J}_m\}$$

is an open subgroup of $\mathcal{W}_K$.

In particular, any smooth representation $\rho : \mathcal{W}_K \to \mathrm{GL}_n(R)$ is l-adically continuous. For the above families of representations, Helm in [HM15, Theorem 1.1] defines the Deligne–Langlands gamma factors as follows: Let S be the multiplicative subset of $R[X, X^{-1}]$ consisting of Laurent polynomials whose first and last coefficients are units. Then, for any l-adically continuous representation $\rho : \mathcal{W}_K \to \mathrm{GL}_n(R)$, there exists an unique element $\gamma_R(X, \rho, \psi_K) \in S^{-1}(R[X, X^{-1}])$ with the following properties :

1. For any Noetherian $W(\overline{\mathbb{F}}_l)$-algebra R' and morphism $f : R \to R'$, we have

$$f(\gamma_R(X, \rho, \psi_K)) = \gamma_{R'}(X, \rho \otimes_R R', \psi_K).$$

2. If R is the field of fractions of a complete local domain of characteristic zero with residue field $\overline{\mathbb{F}}_l$, then the gamma factor $\gamma_R(X, \rho, \psi_K)$ coincides with the classical Deligne–Langlands gamma factor, defined as (2.14).

Moreover, the association $(R, \rho, \psi_K) \longmapsto \gamma_R(X, \rho, \psi_K)$ satisfies

1. For any exact sequence $0 \to \rho_1 \to \rho_2 \to \rho_3 \to 0$ of l-adically continuous representations of $\mathcal{W}_K$, we have

$$\gamma_R(X, \rho_2, \psi_K) = \gamma_R(X, \rho_1, \psi_K)\, \gamma_R(X, \rho_3, \psi_K).$$

2. If L/K is a finite seperable extension and ρ is a virtual smooth representation of $\mathcal{W}_L$ of virtual degree zero in the sense of [Del73, Section 1], then

$$\gamma_R\big(X, \rho, \psi_K \circ \mathrm{Tr}_{L/K}\big) = \gamma_R\big(X, \mathrm{ind}_{\mathcal{W}_L}^{\mathcal{W}_K}(\rho), \psi_K\big).$$

2.6 Local Langlands Correspondence

In this section, we discuss the local Langlands correspondence of $\mathrm{GL}_n(K)$, where K is a finite extension of $\mathbb{Q}_p$. Here R denotes either $\overline{\mathbb{Q}}_l$ or $\overline{\mathbb{F}}_l$. For each integer $n \geq 1$, let $\mathrm{Irr}_R(n, K)$ be the set of all irreducible smooth R-representations of $G_n(K)$. Let $\mathcal{A}_R^{gen}(n, K)$ be the subset of $\mathrm{Irr}_R(n, K)$, consisting of all generic R-representations of $G_n(K)$. We denote by $\mathcal{G}_{\overline{\mathbb{Q}}_l}^{ss}(n, K)$ the set of isomorphism classes of n-dimensional, l-adic, semisimple Weil–Deligne representations of $\mathcal{W}_K$. The set of isomorphism classes of n-dimensional, semisimple, l-modular representations of $\mathcal{W}_K$ is denoted by $\mathcal{G}_{\overline{\mathbb{F}}_l}^{ss}(n, K)$.

2.6.1 The l-adic local Langlands correspondence

Here, we recall the local Langlands correspondence over $R = \overline{\mathbb{Q}}_l$. As before, we fix a non-trivial additive character $\psi_K : K \to \overline{\mathbb{Q}}_l^{\times}$. For each integer $n \geq 1$, there exists an unique bijection

$$\Pi_{K,n} : \mathrm{Irr}_{\overline{\mathbb{Q}}_l}(n, K) \longrightarrow \mathcal{G}^{ss}_{\overline{\mathbb{Q}}_l}(n, K)$$

such that, for all $\pi \in \mathrm{Irr}_{\overline{\mathbb{Q}}_l}(n, K)$ and $\pi' \in \mathrm{Irr}_{\overline{\mathbb{Q}}_l}(m, K)$ with $m < n$, we have

$$\epsilon(X, \pi, \pi', \psi_K) = \epsilon(X, \Pi_{K,n}(\pi) \otimes \Pi_{K,m}(\pi'), \psi_K)$$

and

$$L(X, \pi, \pi') = L(X, \Pi_{K.n}(\pi) \otimes \Pi_{K,m}(\pi')).$$

Moreover, the set of all l-adic cuspidal representations of $G_n(K)$ is mapped onto the set of all irreducible, n-dimensional, l-adic representations of the Weil group $\mathcal{W}_K$ via the bijection $\Pi_{K,n}$ (see [HT01], [Hen00] or [Sch13]). Note that the classical local Langlands correspondence is a bijection between $\mathrm{Irr}_{\mathbb{C}}(n, K)$ and the isomorphism classes of n-dimensional, complex, semisimple Weil–Deligne representations of $\mathcal{W}_K$. To get a correspondence over $\overline{\mathbb{Q}}_l$, one twists the original correspondence by the character $\nu_K^{(1-n)/2}$. For details, see [Clo90, Conjecture 4.4, Section 4.2], [Hen01, Section 7] and for $n = 2$, see [BH06, Theorem 35.1].

2.6.2 Mod-l semisimple local Langlands correspondence

This subsection belongs to the discussion of the local Langlands correspondence over $\overline{\mathbb{F}}_l$. Recall that if π is a generic R-representation of $G_n(K)$, then there exists a multiset $\{\pi_1, \pi_2, \ldots, \pi_r\}$, consisting of supercuspidal R-representations of $G_{n_i}(K)$ with $\sum_{i=1}^r n_i = n$, such that π occurs as a sub–quotient of the normalized parabolic induction $\pi_1 \times \cdots \times \pi_r$. If M_λ is the Levi subgroup associated the partition $\lambda = (n_1, n_2, \ldots, n_r)$ (see Subsection 2.2.2 for the definition), then the pair $(M_\lambda, \pi_1 \otimes \cdots \otimes \pi_r)$ is unique upto $G_n(K)$ conjugacy ([Vig98, Section V.4]) and is called the *supercuspidal support* of π, denoted by $\mathrm{scs}(\pi)$. In [Vig01, Theorem 1.6], Vignéras proves the existence of a sequence of bijections

$$\overline{\Pi}_{K,n} : \mathcal{A}^{gen}_{\overline{\mathbb{F}}_l}(n, K) \longrightarrow \mathcal{G}^{ss}_{\overline{\mathbb{F}}_l}(n, K),\ n \geq 1$$

which is compatible with the l-adic local Langlands correspondence $\Pi_{K,n}$ under mod-l reduction, in the sense that if σ and σ' are two integral l-adic cuspidal representations of $G_n(K)$, then

1. $\Pi_{K,n}(\sigma)$ is integral.

2. $r_l(\sigma) \simeq r_l(\sigma') \iff r_l(\Pi_{K,n}(\sigma)) \simeq r_l(\Pi_{K,n}(\sigma'))$.

3. If $\mathrm{scs}(r_l(\sigma)) = \{\sigma_1, \ldots, \sigma_r\}$, then $r_l(\Pi_{K,n}(\sigma)) = \overline{\Pi}_{K,n}(\sigma_1) \oplus \cdots \oplus \overline{\Pi}_{K,n}(\sigma_r)$.

2.7 Local base Change for the extension E/F

Now, we recall local base change lifting for a cyclic extension of a p-adic field. The base change operation on irreducible smooth representations of $G_n(F)$ over complex vector spaces is characterized by a character identities (see [AC89, Chapter 3]). Let us recall the relation between l-adic local Langlands correspondence and local base change lifting of $G_n(F)$.

Let π_F be an irreducible l-adic representation of $G_n(F)$. Let (ρ_F, U_F) be the l-adic semisimple Weil–Deligne representation of $\mathcal{W}_F$ such that $\Pi_{F,n}(\pi_F) = (\rho_F, U)$, where $\Pi_{F,n}$ is the l-adic local Langlands correspondence. Let π_E be the irreducible l-adic representation of $G_n(E)$ with $\Pi_{E,n}(\pi_E) = (\rho_E, U_E)$ such that

$$\mathrm{res}_{\mathcal{W}_E}(\rho_F) \simeq \rho_E \text{ and } U_F = U_E.$$

The representation π_E is the *base change lifting* of π_F, and it is is isomorphic to the twisted representation π_E^γ for all $\gamma \in \Gamma$. If π_F is generic, then so is the base change lifting π_E.

2.7.1 Base change for $L(\Delta)$

Let τ_F be an l-adic cuspidal representation of $G_m(F)$. Let k be a positive integer, and let $\Delta = \{\tau_F, \tau_F\nu_F, ..., \tau_F\nu_F^{k-1}\}$ be a segment. Then we have the generic representation $\mathcal{L}(\Delta)$ of $G_{km}(F)$ and

$$\Pi_F\big(\mathcal{L}(\Delta)\big) = \Pi_F(\tau_F) \otimes \mathrm{Sp}_F(k), \tag{2.15}$$

where $\mathrm{Sp}_F(k)$ is the semisimple Weil–Deligne representation of $\mathcal{W}_F$ (see [BH06, Chapter 7, Subsection 31.1] for the definition). If l does not divide m, then there exists a cuspidal representation τ_E of $G_m(E)$ such that τ_E is the base change lifting of τ_F, i.e.,

$$\mathrm{res}_{\mathcal{W}_E}\big(\Pi_F(\tau_F)\big) \simeq \Pi_E(\tau_E).$$

Let D be the segment $\{\tau_E, \tau_E\nu_E, ..., \tau_E\nu_E^{k-1}\}$. Using the identity (2.15), we get

$$\mathrm{res}_{\mathcal{W}_E}\big(\Pi_F(\mathcal{L}(\Delta))\big) \simeq \Pi_E(\tau_E) \otimes \mathrm{Sp}_E(k) = \Pi_E\big(\mathcal{L}(D)\big).$$

Thus, the representation $\mathcal{L}(D)$ of $G_{km}(E)$ is the base change lifting of $\mathcal{L}(\Delta)$. Next, we prove a lemma about base change lifting and integrality.

Lemma 2.7.1. *Let π_F be an irreducible l-adic representation of $G_n(F)$, and let π_E be the base change lifting of π_F to $G_n(E)$. Then π_F is integral if and only if the base change lifting π_E is integral.*

Proof. Let π be an irreducible l-adic representation of $G_n(K)$. Let $\mathrm{scs}(\pi)$ be the supercuspidal support of π (see [Vig98, III.3.] for the definition). The representation π is integral if and only if $\mathrm{scs}(\pi)$ is integral (see [Vig01, Section 1.4], and for general reductive groups, see [DHMK24, Corollary 1.6] for a reference). The representation $\mathrm{scs}(\pi)$ is integral if and only if the central character of $\mathrm{scs}(\pi)$ is integral.

Assume that π_F is integral. Let (ρ_F, U) be the l-adic, semisimple Weil-Deligne representation of $\mathcal{W}_F$ associated with π_F under the local Langlands correspondence (LLC) Π_F. Under LLC, the representation $\mathrm{scs}(\pi_F)$ corresponds to $\mathcal{W}_F$-representation ρ_F. Note that the central character of $\mathrm{scs}(\pi_F)$, under LLC, corresponds to the product of the determinant characters of irreducible components of ρ_F (see [Hen01, Theorem 1.3] for instance). Since the central character of $\mathrm{scs}(\pi_F)$ is integral, the determinant character of each irreducible component of ρ_F is integral, and hence ρ_F is integral (by [Vig97, Lemma 1.9]). This implies that the restriction $\mathrm{res}_{\mathcal{W}_E}(\rho_F)$ is integral. Under LLC, the supercuspidal support $\mathrm{scs}(\pi_E)$ corresponds to the restriction $\mathrm{res}_{\mathcal{W}_E}(\rho_F)$. Thus, the supercuspidal support $\mathrm{scs}(\pi_E)$ is integral and we get that π_E is integral.

Conversely, assume that π_E is integral. Let (ρ_E, U_E) be the l-adic, semisimple Weil-Deligne representation of $\mathcal{W}_E$ associated with π_E under LLC. Since the supercuspidal support of π_E is integral, the $\mathcal{W}_E$-representation ρ_E (which is $\mathrm{res}_{\mathcal{W}_E}(\rho_F)$) is integral. Let $\mathcal{L}_E$ be a $\mathcal{W}_E$-stable lattice in ρ_E. Then,

$$\sum_{x \in \mathcal{W}_F/\mathcal{W}_E} \rho_F(x)\mathcal{L}_E.$$

is a $\mathcal{W}_F$-stable lattice in ρ_F. Thus, the representation ρ_F is also integral, which implies that the supercuspidal support $\mathrm{scs}(\pi_F)$ is integral. Hence, π_F is integral. □

2.8 Tate Cohomology

In this section, we recall the definition of Tate cohomology groups, and some useful results on Γ-equivariant l-sheaves of Λ-modules on an l-space X equipped with an action of Γ. For precise reference, see [TV16, Section 3]. Fix a generator γ of Γ. Let M be a $\Lambda[\Gamma]$ module, and let T_γ be the automorphism of M, defined by

$$T_\gamma(m) = \gamma.m, \text{ for } \gamma \in \Gamma,\, m \in M.$$

Let $N = \mathrm{id} + T_\gamma + T_{\gamma^2} + \cdots + T_{\gamma^{l-1}}$ be the norm operator. Then the Tate cohomology groups $\widehat{H}^0(M)$ and $\widehat{H}^1(M)$ are defined as

$$\widehat{H}^0(M) := \frac{\mathrm{Ker}(\mathrm{id} - T_\gamma)}{\mathrm{Img}(N)} \text{ and } \widehat{H}^1(M) := \frac{\mathrm{Ker}(N)}{\mathrm{Img}(\mathrm{id} - T_\gamma)}.$$

The above definition is independent of the choice of γ in Γ.

2.8.1 Tate Cohomology of sheaves on l-spaces

Let X be a Hausdorff, locally compact and totally disconnected topological space (in short "l-space") equipped with an action of a finite group $\langle\gamma\rangle$ of order l. Let $C^\infty(X, \Lambda)$ be the space of locally constant Λ-valued functions on X and $C_c^\infty(X, \Lambda)$ the space of functions in $C^\infty(X, \Lambda)$ with compact support.

Definition 2.8.1. An l-sheaf of Λ-modules on X consists of the following data:

1. for each $x \in X$, there is associated a Λ-module $\mathcal{F}_x$,
2. there is defined a family $\mathcal{F}$ of cross-sections, i.e., mappings $\varphi : X \to \bigcup_{x\in X} \mathcal{F}_x$ such that $\varphi(x) \in \mathcal{F}_x$ for each $x \in X$,

satisfying the following conditions:

1. $\mathcal{F}$ is invariant under addition and multiplication by functions in $C^\infty(X, \Lambda)$.
2. If φ is a cross-section that coincides with some cross-section in $\mathcal{F}$ in an open neighbourhood of each point of X, then $\varphi \in \mathcal{F}$.
3. If $\varphi \in \mathcal{F}$, $x \in X$, and $\varphi(x) = 0$, then there exists an open neighbourhood U of x such that $\varphi = 0$ on U.
4. For any $x \in X$ and $v \in \mathcal{F}_x$, there exists a $\varphi \in \mathcal{F}$ such that $\varphi(x) = v$.

Let $\Gamma_c(X; \mathcal{F})$ be the space of compactly supported sections of $\mathcal{F}$, that is, those sections $\varphi \in \mathcal{F}$ for which the set $\{x \in X \mid \varphi(x) \neq 0\}$ is compact. In particular, if $\mathcal{F}$ is the constant sheaf with stalk Λ, then $\Gamma_c(X; \mathcal{F}) = C_c^\infty(X, \Lambda)$. The assignment $\mathcal{F} \mapsto \Gamma_c(X; \mathcal{F})$ is a covariant exact functor. If $\mathcal{F}$ is $\langle\gamma\rangle$-equivariant, then γ can be regarded as a map of restricted sheaves $\mathcal{F}|_{X^\gamma} \to \mathcal{F}|_{X^\gamma}$ and the Tate cohomology is defined as

$$\widehat{H}^0(\mathcal{F}|_{X^\gamma}) := \mathrm{Ker}(1 - \gamma)/\mathrm{Img}(N),$$
$$\widehat{H}^1(\mathcal{F}|_{X^\gamma}) := \mathrm{Ker}(N)/\mathrm{Img}(1 - \gamma).$$

A compactly supported section of $\mathcal{F}$ can be restricted to a compactly supported section of $\mathcal{F}|_{X^\gamma}$.

2.8.2 Tate cohomology of compact induction

Let G be a locally profinite group and γ an automorphism of G of order l. Let H be a closed subgroup of G such that $\gamma(H) = H$ and $(G/H)^\gamma = G^\gamma/H^\gamma$. Let ρ be a smooth $\Lambda[H]$ module equipped with an isomorphism $T : \rho \xrightarrow{\sim} \rho^\gamma$ such that $T^l = \mathrm{id}$. Then we have an isomorphism of the compactly induced representations $\widetilde{T} : \mathrm{ind}_H^G(\rho) \xrightarrow{\sim} \big(\mathrm{ind}_H^G(\rho)\big)^\gamma$, defined by

$$\widetilde{T}(f)(g) := T\big(f(\gamma^{-1}g)\big),$$

for all $g \in G$ and $f \in \mathrm{ind}_H^G(\rho)$. Thus, there is an action of $\langle\gamma\rangle$ on $\mathrm{ind}_H^G(\rho)$–induced by $\widetilde{T}$, and we consider the Tate cohomology groups $\widehat{H}^i(\mathrm{ind}_H^G(\rho))$, $i \in \{0, 1\}$, with respect to this action. Note that the restriction to G^γ map:

$$\big(\mathrm{ind}_H^G(\rho)\big)^{\langle\gamma\rangle} \to \mathrm{ind}_{H^\gamma}^{G^\gamma}(\widehat{H}^i(\rho)),\ \ f \mapsto \mathrm{res}_{G^\gamma}(f) \tag{2.16}$$

factorizes through

$$\widehat{H}^i(\mathrm{ind}_H^G(\rho)) \longrightarrow \mathrm{ind}_{H^\gamma}^{G^\gamma}(\widehat{H}^i(\rho)).$$

Then we have the following result, due to [TV16, Proposition 3.3.1].

Proposition 2.8.2. *Let G be a locally profinite group and H a closed subgroup of G. Let γ be an automorphism of G of order l such that $\gamma(H) = H$. We further assume that $(G/H)^\gamma = G^\gamma/H^\gamma$. Let ρ be a smooth $\Lambda[H]$ module equipped with an isomorphism $T : \rho \to \rho^\gamma$ such that $T^l = \mathrm{id}$. Then, for each $i \in \{0, 1\}$, the restriction map (2.16) induces the following isomorphism of G^γ-representations:*

$$\widehat{H}^i(\mathrm{ind}_H^G(\rho)) \xrightarrow{\sim} \mathrm{ind}_{H^\gamma}^{G^\gamma}(\widehat{H}^i(\rho)).$$

Remark 2.8.3. The above proposition is often useful to calculate the Tate cohomology groups of compactly induced representations. We will now discuss a particular instance of the above proposition, which is crucial for the later chapters. Let E/F be a Galois extension of degree l with Galois group Γ. Let $\psi_F : F \to \overline{\mathbb{Q}}_l^\times$ be a non-trivial additive character, and let ψ_E be the character $\psi_F \circ \mathrm{Tr}_{E/F}$, where $\mathrm{Tr}_{E/F}$ is the trace function. Let Θ_E (resp. Θ_F) be the non-degenerate characters of $N_n(E)$ (resp. $N_n(F)$) associated with ψ_F (resp. ψ_E) (see Subsection 2.3). We will use the notations $\overline{\psi}_E$, $\overline{\psi}_F$, $\overline{\Theta}_E$, $\overline{\Theta}_F$ for the respective mod-l reductions. Recall the notation $\mathcal{K}(\psi_K)$ for the compact induction $\mathrm{ind}_{N_n(K)}^{P_n(K)}(\Theta_K)$. Note that Γ acts on $\mathcal{K}(\overline{\psi}_E)$, by setting

$$(\gamma f)(x) = f(\gamma^{-1}(x)),$$

for all $\gamma \in \Gamma$, $x \in P_n(E)$ and $f \in \mathcal{K}(\overline{\psi}_E)$. Set $Y_K = P_n(K)/N_n(K)$. Since $N_n(E)$ is a Γ-stable subgroup of $P_n(E)$, we have the following exact sequence of non-abelian cohomology (see [Ser, Chapter VII, Appendix]):

$$0 \longrightarrow N_n(F) \longrightarrow P_n(F) \longrightarrow Y_E^\Gamma \longrightarrow H^1(\Gamma, N_n(E)) \longrightarrow H^1(\Gamma, P_n(E))$$

Note that the pointed set $H^1(\Gamma, N_n(E))$ is trivial. Thus, we get the equality $Y_E^\Gamma = Y_F$. Then, applying Proposition 2.8.2, we get that the restriction to $P_n(F)$ map:

$$\mathcal{K}(\overline{\psi}_E)^\Gamma \to \mathcal{K}(\overline{\psi}_F^l),\ f \mapsto \mathrm{res}_{P_n(F)} f$$

induces an isomorphism

$$\widehat{H}^0(\mathcal{K}(\overline{\psi}_E)) \to \mathcal{K}(\overline{\psi}_F^l). \tag{2.17}$$

Proposition 2.8.2 also shows that $\widehat{H}^1(\mathcal{K}^0(\psi_E))$ is trivial. Here, $\mathcal{K}^0(\psi_E)$ is the space of $\overline{\mathbb{Z}}_l$-valued functions in the l-adic representation $\mathcal{K}(\psi_E)$.

2.8.3 Comparison of integrals of smooth functions

In this subsection, we set up an important identity of the integrals on the homogeneous space of F with those on the homogeneous space of E. Let R be an Artin local $\overline{\mathbb{F}}_l$-algebra and $\mathfrak{m}_R$ be its unique maximal ideal. Let X be a set, endowed with an action of the group Γ. Then Γ acts on the space $C_c^\infty(X, R)$, by setting

$$(\gamma\varphi)(x) := \varphi(\gamma^{-1}x),$$

for all $x \in X$ and $\varphi \in C_c^\infty(X, R)$. We denote by $C_c^\infty(X, R)^\Gamma$ the space of functions in $C_c^\infty(X, R)$ fixed under Γ. Let X_K be the coset space $G_{n-1}(K)/N_{n-1}(K)$. We have the following result.

Proposition 2.8.4. *Let R be an Artin local $\overline{\mathbb{F}}_l$-algebra. Let $d\mu_E$ (resp. $d\nu_E$) be the Haar measure on X_E (resp. $M_{r\times s}(E)$), and let $d\mu_F$ (resp. $d\nu_F$) be the corresponding object over F. Then there exists $c \in R^\times$ such that*

$$\int_{M_{r\times s}(E)} \int_{X_E} \varphi\, d\mu_E\, d\nu_E = c \int_{M_{r\times s}(F)} \int_{X_F} \varphi\, d\mu_F\, d\nu_F,$$

for all $\varphi \in C_c^\infty(M_{r\times s}(E) \times X_E, R)^\Gamma$.

Proof. Since $N_{n-1}(E)$ is stable under the action of Γ, we have the following long exact sequence of non-abelian cohomology (see [Ser, Chapter VII, Appendix]):

$$0 \longrightarrow N_{n-1}(E)^\Gamma \longrightarrow G_{n-1}(E)^\Gamma \longrightarrow X_E^\Gamma \longrightarrow H^1\big(\Gamma; N_{n-1}(E)\big) \longrightarrow H^1\big(\Gamma; G_{n-1}(E)\big).$$

Note that $H^1\big(\Gamma, N_{n-1}(E)\big)$ is trivial. Then the above long exact sequence gives the equality $X_E^\Gamma = X_F$. Let Y_K denote the ring of matrices $M_{r\times s}(K)$. Since X_F (resp. Y_F) is closed in X_E (resp. Y_E), we have the following exact sequence of Γ-modules:

$$0 \longrightarrow C_c^\infty\big((Y_E \times X_E) \setminus (Y_F \times X_F), R\big) \longrightarrow C_c^\infty(Y_E \times X_E, R) \longrightarrow C_c^\infty(Y_F \times X_F, R) \longrightarrow 0. \quad (2.18)$$

Since the action of Γ on $(Y_E \times X_E) \setminus (Y_F \times X_F)$ is free, Proposition 2.8.2 gives

$$H^1\big(\Gamma, C_c^\infty\big((Y_E \times X_E) \setminus (Y_F \times X_F), R\big)\big) = 0 \quad (2.19)$$

Using (2.18) and (2.19), we get the following exact sequence:

$$0 \longrightarrow C_c^\infty\big((Y_E \times X_E) \setminus (Y_F \times X_F), R\big)^\Gamma \longrightarrow C_c^\infty(Y_E \times X_E, R)^\Gamma \longrightarrow C_c^\infty(Y_F \times X_F, R) \longrightarrow 0.$$

The free action of Γ on $(Y_E \times X_E) \setminus (Y_F \times X_F)$ gives a fundamental domain $\mathcal{U}$ such that

$$(Y_E \times X_E) \setminus (Y_F \times X_F) = \bigcup_{i=0}^{l-1} \gamma^i \mathcal{U}.$$

Then, for any $\varphi \in C_c^\infty\big((Y_E \times X_E) \setminus (Y_F \times X_F), R\big)^\Gamma$, we have

$$\int_{(Y_E\times X_E)\setminus(Y_F\times X_F)} \varphi \, d\mu_E \, d\nu_E = \sum_{i=0}^{l-1} \int_{\gamma^i\mathcal{U}} \varphi \, d\mu_E \, d\nu_E = 0,$$

Therefore, the linear functionals $d\mu_E$ and $d\nu_E$ induces a $(Y_F \times G_{n-1}(F))$-equivariant linear functional on the space $C_c^\infty(Y_F \times X_F, R)$, and we get following identity:

$$\int_{Y_E}\int_{X_E} \varphi \, d\mu \, d\nu_E = c \int_{Y_F}\int_{X_F} \varphi \, d\mu_F \, d\nu_F,$$

for some non-zero $c \in R$. Now, we will show that $c \in R^\times$. Under mod-$\mathfrak{m}_R$ reduction, the above integral reduces to the following identity

$$\int_{Y_E}\int_{X_E} \overline{\varphi} \, d\mu_E \, d\nu_E = \bar{c} \int_{Y_F}\int_{X_F} \overline{\varphi} \, d\mu_F \, d\nu_F, \quad (2.20)$$

where the function $\overline{\varphi}$ is the pointwise mod-$\mathfrak{m}_R$ reduction of φ. Following [Vig96b, Chapter 1, Section 2.8], we get a surjection $\Psi : C_c^\infty(Y_E \times G_{n-1}(E), \overline{\mathbb{F}}_l) \longrightarrow C_c^\infty(Y_E \times X_E, \overline{\mathbb{F}}_l)$, defined by

$$\Psi(f)(x,g) := \int_{N_{n-1}(E)} f(x, ng) \, dn,$$

for all $f \in C_c^\infty(Y_E \times G_{n-1}(E), \overline{\mathbb{F}}_l)$. Here dn is a Haar measure on $N_{n-1}(E)$. There exists a Γ-invariant compact open subgroup $I \subseteq G_n(E)$ such that $\Psi(1_{M_{r\times s}(\mathfrak{o}_E)\times I}) \neq 0$, where $1_{M_{r\times s}(\mathfrak{o}_E)\times I}$ denotes the characteristic function on $M_{r\times s}(\mathfrak{o}_E) \times I$. So the Haar measure $d\mu_E$ is non-zero on $C_c^\infty(Y_E \times X_E, \overline{\mathbb{F}}_l)^\Gamma$, and we have $\bar{c} \neq 0$. This completes the proof. $\square$

Remark 2.8.5. From now onwards, the Haar measures $d\mu_E$, $d\mu_F$, $d\nu_E$, and $d\nu_F$ on the respective spaces are chosen so as to make $c = 1$. In this case, we have

$$\int_{Y_E}\int_{X_E} \varphi \, d\mu_E \, d\nu_E = \int_{Y_F}\int_{X_F} \varphi \, d\mu_F \, d\nu_F.$$

Let e be the ramification index of the extension E/F. Then, for all $r \notin \{te : t \in \mathbb{Z}\}$, we have

$$\int_{Y_E}\int_{(X_E^r)^\Gamma} \varphi \, d\mu_E \, d\nu_E = 0,$$

and for all $r \in \{te : t \in \mathbb{Z}\}$, we have

$$\int_{Y_E}\int_{(X_E^r)^\Gamma} \varphi \, d\mu_E \, d\nu_E = \int_{Y_F}\int_{X_F^{\frac{r}{e}}} \varphi \, d\mu_F \, d\nu_F.$$

In particular, when $R = \overline{\mathbb{F}}_l$, the following result is a consequence of the above proposition and will be used in the proof of the main result of Chapter 4.

Corollary 2.8.6. *Let $d\mu_E$ and $d\mu_F$ be two Haar measures on X_E and X_F respectively. Then, there exists a non-zero scalar $c \in \overline{\mathbb{F}}_l$ such that for all $\varphi \in C_c^\infty(X_E, \overline{\mathbb{F}}_l)^\Gamma$, we have*

$$\int_{(X_E^r)^\Gamma} \varphi \, d\mu_E = c \int_{X_F^{\frac{r}{e}}} \varphi \, d\mu_F$$

if $r \in \{te : t \in \mathbb{Z}\}$, and

$$\int_{(X_E^r)^\Gamma} \varphi \, d\mu_E = 0$$

if $r \notin \{te : t \in \mathbb{Z}\}$. Moreover, the Haar measures can be chosen so that $c = 1$.

Chapter 3

Tate cohomology of cuspidal representations of GL_2

3.1 Introduction

The main objective of this chapter is to prove Theorem 1.4.2 for the integral l-adic cuspidal representations of $\mathrm{GL}_2(F)$, where F is a p-adic field. Our method relies on Kirillov models of mod-l cuspidal representations of $\mathrm{GL}_2(F)$ and the explicit action of $\mathrm{GL}_2(F)$ on these models. Let us briefly discuss the results of this chapter. Let E be a finite Galois extension of the p-adic field F with $[E : F] = l$, where l and p are distinct odd primes. Let π be an integral l-adic cuspidal representation of $\mathrm{GL}_2(F)$ and let Π be the base change lifting of π to $\mathrm{GL}_2(E)$. Then Π is also cuspidal and integral (by [AC89, Lemma 6.10] and Lemma 2.7.1). Using Proposition 3.3.1, the zeroth Tate cohomology group of the mod-l reduction $r_l(\Pi)$ is an irreducible representation of $\mathrm{GL}_2(F)$ and it is isomorphic to the Frobenius twist $r_l(\pi)^{(l)}$ as a representation of the mirabolic subgroup $P_2(F)$. Comparing the actions of both $r_l(\pi)^{(l)}$ and $r_l(\Pi)$ on the respective Kirillov spaces and using some congruence relations of local ϵ-factors, we get that the Frobenius twist $r_l(\pi)^{(l)}$ is isomorphic to the zeroth Tate cohomology group of $r_l(\Pi)$ (Theorem 3.3.2).

3.2 Kirillov model for GL_2

In this section, we recall the Kirillov model for $n = 2$ and some of its properties. For a precise reference, see [BH06, Chapter 9] and [Vig96a]. We follow the notations of the preceding chapter. Let K be a non-Archimedean local field. Up to isomorphism, any irreducible representation of

$P_2(K)$, which is not a character, is isomorphic to the compactly induced representation

$$J_{\psi_K} := \mathrm{ind}_{N_2(K)}^{P_2(K)}(\psi_K), \tag{3.1}$$

for some non-trivial additive character ψ_K of K, viewed as a character of $N_2(K)$ via the isomorphism $N_2(K) \xrightarrow{\sim} K$, $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mapsto x$. Two different non-trivial characters of $N_2(K)$ induce isomorphic representations of $P_2(K)$. The space (3.1) is identified with the space of all locally constant, compactly supported functions on $K^\times$, to be denoted by $C_c^\infty(K^\times, \overline{\mathbb{Q}}_l)$. The action of $P_2(K)$ on the space $C_c^\infty(K^\times, \overline{\mathbb{Q}}_l)$ is given by

$$\left[J_{\psi_K}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} f\right](y) = f(ay),$$

$$\left[J_{\psi_K}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} f\right](y) = \psi_K(xy)f(y),$$

for all $a, y \in K^\times$ and $x \in K$. For any l-adic cuspidal representation of (π, V) of $G_2(K)$, we get a model for the representation (π, V) on the space $C_c^\infty(K^\times, \overline{\mathbb{Q}}_l)$. The action of $G_2(K)$ on $C_c^\infty(K^\times, \overline{\mathbb{Q}}_l)$ is denoted by $\mathbb{K}_{\psi_K}^\pi$; by definition the restriction of $\mathbb{K}_{\psi_K}^\pi$ to $P_2(K)$ is isomorphic to J_{ψ_K}. The operator $\mathbb{K}_{\psi_K}^\pi(w)$ and the restriction of $\mathbb{K}_{\psi_K}^\pi$ to $P_2(K)$ completely describe the action of $G_2(K)$ on $C_c^\infty(K^\times, \overline{\mathbb{Q}}_l)$, where

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Here, we follow the exposition in [BH06, Section 37.3]. Let χ be a smooth character of $K^\times$ and let k is an integer. Define a function $\xi\{\chi, k\}$ in $C_c^\infty(K^\times, \overline{\mathbb{Q}}_l)$ by setting $\xi\{\chi, k\}(x) = \chi(x)$, for $\nu_K(x) = k$ and zero otherwise. Recall that υ_K is the normalized discrete valuation on K. Then we have

$$\mathbb{K}_{\psi_K}^\pi(w)\xi\{\chi, k\} = \epsilon(\chi^{-1}\pi, \psi_K)\xi\{\chi^{-1}\varpi_\pi, -n(\chi^{-1}\pi, \psi_K) - k\}, \tag{3.2}$$

where ω_π is the central character of π. Here $\epsilon(\pi, \psi_K)$ is the Godement–Jacquet local ϵ-factor associated with a cuspidal representation π and the additive character ψ_K of K.

3.3 Tate cohomology of cuspidal representations of GL_2

In this section, we compute the zeroth Tate cohomology group of mod-l reduction of base change lifting of integral l-adic cuspidal representations of $\mathrm{GL}_2(F)$, where F is a p-adic field. We first demonstrate the irreducibility of the zeroth Tate cohomology group of cuspidal representations of

$\mathrm{GL}_n(F)$, for arbitrary n.

Let E/F be a finite Galois extension with $[E : F] = l$, where l and p are distinct odd primes. We denote by Γ the cyclic group $\mathrm{Gal}(E/F)$. Fix a non-trivial additive character $\psi_F : F \to \Lambda^\times$. Let ψ_E be the character $\psi_F \circ \mathrm{Tr}_{E/F}$, where $\mathrm{Tr}_{E/F}$ denotes the trace map of the extension E/F. As before, we use the notation G_n to denote the group GL_n. From now onwards, for an arbitrary $\Lambda[\Gamma]$ module M, the Tate cohomology group $\widehat{H}^i(\Gamma, M)$ is simply denoted by $\widehat{H}^i(M)$.

Proposition 3.3.1. *Let Π be an integral, cuspidal $\mathcal{K}$-representation (or an l-modular cuspidal representation) of $G_n(E)$. We further assume that Π is stable under the action of Γ. Then, the Tate cohomology group $\widehat{H}^0(\Pi)$ is an irreducible representation of $G_n(F)$.*

Proof. **Case 1**. Here, we consider the case where Π is integral cuspidal $\mathcal{K}$-representation. Let $\mathcal{L}$ be a $G_n(E) \rtimes \Gamma$ stable lattice in Π. Using [Ron16, Theorem 6], we get that $\widehat{H}^1(\mathcal{L}) = 0$. Then, from the following short exact sequence of Γ-modules

$$0 \longrightarrow \mathcal{L} \xrightarrow{\text{mult}.l} \mathcal{L} \longrightarrow \mathcal{L}/l\mathcal{L} \longrightarrow 0,$$

we have $\widehat{H}^0(\mathcal{L}) \simeq \widehat{H}^0(\mathcal{L}/l\mathcal{L})$. Since Π is cuspidal, the mod-l reduction $\mathcal{L}/l\mathcal{L}$ is irreducible (see [Vig96b, Theorem 1.1, Chapter III]), and hence it is independent of the choice of $\mathcal{L}$ by [Vig04, Theorem 1]. Thus, $\widehat{H}^0(\mathcal{L})$ is independent of the choice of $\mathcal{L}$, and it is denoted by $\widehat{H}^0(\Pi)$. Therefore, it is enough to prove the irreducibility of $\widehat{H}^0(\Pi)$ for some specific choice of $\mathcal{L}$. Let $\mathbb{W}_\Lambda(\Pi, \psi_E)$ be the space of all Λ-valued functions in the Whittaker model $\mathbb{W}(\Pi, \psi_E)$. From [Vig04, Theorem 2] and Lemma 2.3.3, it follows that $\mathbb{W}_\Lambda(\Pi, \psi_E)$ is $G_n(E) \rtimes \Gamma$ invariant Λ-lattice in $\mathbb{W}(\Pi, \psi_E)$. As Π is cuspidal, the restriction map $W \mapsto \mathrm{res}_{P_n(E)}(W)$ gives the isomorphism

$$\mathbb{W}(\Pi, \psi_E) \xrightarrow{\sim} \mathrm{ind}_{N_n(E)}^{P_n(E)}(\Theta_E).$$

Moreover, the above isomorphism induces the following isomorphism ([MM22, Theorem 4.2]) of $\Lambda[G_n(E)]$-modules

$$\mathbb{W}^0(\Pi, \psi_E) \xrightarrow{\sim} \left(\mathrm{ind}_{N_n(E)}^{P_n(E)}(\Theta_E)\right)^0,$$

where $\left(\mathrm{ind}_{N_n(E)}^{P_n(E)}(\Theta_E)\right)^0$ denotes the space of all Λ-valued functions in $\mathrm{ind}_{N_n(E)}^{P_n(E)}(\Theta_E)$. Then, using Proposition 2.8.2, we get that

$$\widehat{H}^0\big(\mathbb{W}_\Lambda(\Pi, \psi_E)\big) \simeq \widehat{H}^0\big((\mathrm{ind}_{N_n(E)}^{P_n(E)}(\Theta_E))^0\big) \simeq \mathrm{ind}_{N_n(F)}^{P_n(F)}(\overline{\Theta}_F^l),$$

as representations of $P_n(F)$. Since the compact induction $\mathrm{ind}_{N_n(F)}^{P_n(F)}(\overline{\Theta}_F^l)$ is irreducible ([Vig96b, Theorem 1.1, Chapter III]), the Tate cohomology $\widehat{H}^0(\mathbb{W}_\Lambda(\Pi, \psi_E))$ is an irreducible representation of $P_n(F)$, and hence of $G_n(F)$.

Case 2. Assume that Π is an l-modular cuspidal representation of $G_n(E)$. Since the mod-l reduction $r_l(\Pi)$ is cuspidal, the restriction $\mathrm{res}_{P_n(E)} r_l(\Pi)$ is isomorphic to compact induction $\mathrm{ind}_{N_n(E)}^{P_n(E)}(\overline{\Theta}_E)$, which is an irreducible representation of $P_n(E)$. Then, using Proposition 2.8.2, we get that

$$\widehat{H}^0(r_l(\Pi)) \simeq \mathrm{ind}_{N_n(F)}^{P_n(F)}(\overline{\Theta}_F^l),$$

as a representation of $P_n(F)$. This implies that $\widehat{H}^0(r_l(\Pi))$ is irreducible as a representaton of $G_n(F)$. This completes the proof. $\square$

Theorem 3.3.2. *Let F be a finite extension of $\mathbb{Q}_p$, and let E be a finite Galois extension of F with $[E:F]=l$, where l and p are distinct odd primes. Let π_F be an integral l-adic cuspidal representation of $G_2(F)$ and let π_E be the base change lift of π_F to $G_2(E)$. Then*

$$\widehat{H}^0(r_l(\pi_E)) \simeq r_l(\pi_F)^{(l)}.$$

Proof. First, note that the base change lifting π_E is integral and cuspidal, and hence its mod-l reduction $r_l(\pi_E)$ is also cuspidal. Let $\big(\mathbb{K}_{\overline{\psi}_E}^{r_l(\pi_E)}, C_c^\infty(E^\times, \overline{\mathbb{F}}_l)\big)$ be the Kirillov model of the representation $r_l(\pi_E)$. We denote by $\widehat{H}^0(r_l(\pi_E))$ the Tate cohomology group $\widehat{H}^0\big(C_c^\infty(E^\times,\overline{\mathbb{F}}_l)\big)$. Using Proposition 2.8.2, we have

$$\widehat{H}^0(r_l(\pi_E)) \simeq C_c^\infty(F^\times, \overline{\mathbb{F}}_l).$$

Thus, the space $\widehat{H}^0(r_l(\pi_E))$ is isomorphic to $\mathrm{ind}_{N_2(F)}^{P_2(F)}(\overline{\psi}_F^l)$ as a representation of $P_2(F)$, where $\overline{\psi}_F$ is the mod-l reduction of ψ_F; and the induced action of the operator $\mathbb{K}_{\overline{\psi}_E}^{r_l(\pi_E)}(w)$ on the space $\widehat{H}^0(r_l(\pi_E))$ is denoted by $\overline{\mathbb{K}_{\overline{\psi}_E}^{r_l(\pi_E)}(w)}$. The theorem now follows from the following claim.

Claim. $\overline{\mathbb{K}_{\overline{\psi}_E}^{r_l(\pi_E)}}(w)(f) = \mathbb{K}_{\overline{\psi}_F^l}^{r_l(\pi_F)^{(l)}}(w)(f)$, for all $f \in C_c^\infty(F^\times, \overline{\mathbb{F}}_l)$.

Now, for a function $f \in C_c^\infty(F^\times, \overline{\mathbb{F}}_l)$, any covering of $\mathrm{supp}(f)$ by open subsets of $F^\times$ has a finite refinement of pairwise disjoint open compact subgroups of $F^\times$. So we may assume that $\mathrm{supp}(f) \subseteq \varpi_F^r x U_F^1$, where $r \in \mathbb{Z}$ and x is a unit in $k_F^\times$ embedded in $F^\times$. Therefore, it is sufficient to prove the claim for functions $f \in C_c^\infty(F^\times, \overline{\mathbb{F}}_l)$ with $\mathrm{supp}(f) \subseteq U_F^1$, and we have

$$f = \sum_{\chi_F \in \widehat{U_F^1}} c_{\chi_F}\, \xi\{\chi_F, 0\},$$

where $c_{\chi_F} \in \overline{\mathbb{F}}_l$ and $\widehat{U_F^1}$ is the group of smooth characters of $F^\times = \langle \varpi_F \rangle \times k_F^\times \times U_F^1$, which are trivial on $k_F^\times$ and ϖ_F. We now prove the claim for the functions $\xi\{\chi_F, 0\}$ for $\chi_F \in \widehat{U_F^1}$. There exists a character $\chi_0 \in \widehat{U_F^1}$ such that $\chi_0^l = \chi_F$. Let $\widetilde{\chi}_0$ be the l-adic lift of χ_0. Define the characters $\widetilde{\chi}_E$ and χ_E of $E^\times$ as follows :

$$\widetilde{\chi}_E(x) = \widetilde{\chi}_0(\mathrm{Nr}_{E/F}(x)),$$

$$\chi_E(x) = \chi_0(\mathrm{Nr}_{E/F}(x)),$$

for $x \in E^{\times}$. Here $\mathrm{Nr}_{E/F} : E^{\times} \to F^{\times}$ denotes the norm map. Note that both $\widetilde{\chi}_E$ and χ_E extend the character χ_F. We have the following relations:

$$\overline{\mathbb{K}^{r_l(\pi_E)}_{\overline{\psi}_E}(w)}\big(\xi\{\chi_F, 0\}\big) = \epsilon(\chi_E^{-1} r_l(\pi_E), \overline{\psi}_E)\xi\Big\{\chi_F, \frac{-n(\chi_E^{-1} r_l(\pi_E), \overline{\psi}_E)}{e}\Big\} \tag{3.3}$$

and

$$\mathbb{K}^{r_l(\pi_F)^{(l)}}_{\overline{\psi}^l_F}(w)\big(\xi\{\chi_F, 0\}\big) = \epsilon(\chi_F^{-1} r_l(\pi_F)^{(l)}, \overline{\psi}^l_F)\xi\{\chi_F, -n(\chi_F^{-1} r_l(\pi_F)^{(l)}, \overline{\psi}^l_F)\}, \tag{3.4}$$

where e is the ramification index of the extension E/F. Next, we aim to prove the following identity of ϵ-factors

$$\epsilon(X^f, \chi_E^{-1} r_l(\pi_E), \overline{\psi}_E) = \epsilon(X^l, \chi_F^{-1} r_l(\pi_F)^{(l)}, \overline{\psi}^l_F),$$

where f is the residue degree of the extension E/F. From Theorem 2.4.4, it follows that the ϵ-factor is the same as the γ-factor in both l-adic and l-modular cases. Now, using the identity in [AC89, Proposition 6.9], we get

$$\epsilon(X^f, \widetilde{\chi}_E^{-1}\pi_E, \psi_E) = \prod_{\eta} \epsilon(X, \widetilde{\chi}_0^{-1}\pi_F \otimes \eta, \psi_F), \tag{3.5}$$

where η runs over the characters of the group $F^{\times}/\mathrm{Nr}_{E/F}(E^{\times})$, which is isomorphic to Γ via the local class field theory. Using the identity (2.6), we have

$$r_l\big(\epsilon(X^f, \widetilde{\chi}_E^{-1}\pi_E, \psi_E)\big) = \epsilon(X^f, \chi_E^{-1} r_l(\pi_E), \overline{\psi}_E)$$

and

$$r_l\big(\epsilon(X, \widetilde{\chi}_0^{-1}\pi_F \otimes \eta, \psi_F)\big) = \epsilon(X, \chi_0^{-1} r_l(\pi_F), \overline{\psi}_F),$$

for each character η. Using the above relations, it follows from (3.5) that

$$\epsilon(X^f, \chi_E^{-1} r_l(\pi_E), \overline{\psi}_E) = \epsilon(X, \chi_0^{-1} r_l(\pi_F), \overline{\psi}_F)^l.$$

In view of Lemma 2.4.5, we obtain the following identity

$$\epsilon(X^f, \chi_E^{-1} r_l(\pi_E), \overline{\psi}_E) = \epsilon\big((\frac{q_F}{l})X^l, \chi_F^{-1} r_l(\pi_F)^{(l)}, \overline{\psi}^l_F\big),$$

where $(\frac{q_F}{l})$ denotes the Legendre symbol, valued in $\{-1, 1\}$. Now, using the identity (2.5) and comparing the degree of X from the above relation, we get

$$\frac{n(\chi_E^{-1}r_l(\pi_E), \overline{\psi}_E)}{e} = n(\chi_F^{-1}r_l(\pi_F)^{(l)}, \overline{\psi}_F^l)$$

and

$$\epsilon(\chi_E^{-1}r_l(\pi_E), \overline{\psi}_E) = \epsilon\big(\chi_F^{-1}r_l(\pi_F)^{(l)}, \overline{\psi}_F^l\big).$$

Thus, it follows from (3.3) and (3.4) that

$$\overline{\mathbb{K}_{\overline{\psi}_E}^{r_l(\pi_E)}(w)}\big(\xi\{\chi_F, 0\}\big) = \mathbb{K}_{\overline{\psi}_F^l}^{r_l(\pi_F)^{(l)}}(w)\big(\xi\{\chi_F, 0\}\big).$$

Hence the theorem. □

Chapter 4

Tate cohomology of Whittaker lattices and Base change of generic representations of GL_n

4.1 Introduction

In this chapter, we study the Tate cohomology groups of base change lifts of more general irreducible l-adic representations of $\mathrm{GL}_n(F)$, namely generic representations and Zelevinsky subrepresentations. The results are precisely a generalization of Theorem 3.3.2 in the previous chapter. A similar strategy as that of Theorem 3.3.2 has been adapted to prove the main theorem.

Let E be a finite Galois extension of the p-adic field F with $[E : F] = l$, where l and p are distinct odd primes. Let π_F be an integral l-adic generic representation of $G_n(F)$, and let π_E be the base change lifting of π_F. Then, π_E is generic and is equipped with an action of $\mathrm{Gal}(E/F)$ via the isomorphism $\pi_E \simeq \pi_E^{\gamma}$, for all $\gamma \in \mathrm{Gal}(E/F)$. Moreover, the base change lift π_E is also integral (Lemma 2.7.1). Consider the unique generic sub–quotient $J_l(\pi_E)$ of the mod-l reduction of π_E. Then, the representation $J_l(\pi_E)$ is stable under the action of $\mathrm{Gal}(E/F)$, and we have the Tate cohomology groups $\widehat{H}^i(J_l(\pi_E))$, $i \in \{0, 1\}$, which are mod-l representations of $G_n(F)$. We show that each $\widehat{H}^i(J_l(\pi_E))$ has finite length as a representation of $G_n(F)$ (see Proposition 4.2.2). In Subsection 4.3.1, we set up some initial results on compatibility of (twisted) Jacquet modules with Tate cohomology–which implies that the zeroth Tate cohomology group $\widehat{H}^0(J_l(\pi_E))$ admits a unique generic sub–quotient (Proposition 4.3.3). The main result of this chapter is as follows.

Theorem 4.1.1. *Let F be a finite extension of $\mathbb{Q}_p$, and let E be a finite Galois extension of F with $[E : F] = l$, where p and l are distinct primes such that l does not divide $|G_{n-1}(\kappa_F)|$. Let π_F be an integral, l-adic generic representation of $G_n(F)$, and let π_E be the base change lifting of π_F to $G_n(E)$. Then, the Frobenius twist $J_l(\pi_F)^{(l)}$ is the unique generic sub–quotient of the Tate cohomology group $\widehat{H}^0(J_l(\pi_E))$.*

To prove the theorem, we rely on Rankin–Selberg integrals and use various properties of local ϵ and γ-factors both in l-adic and mod-l situations associated with representations of the p-adic group and the Weil group. Using similar ideas, we also prove an integral version of the above theorem (Corollary 4.3.7) by considering the Whittaker lattice of π_E over Λ. When l does not divide the pro-order of $G_n(F)$, we obtain a much more precise version of the above theorem. We can show that the Tate cohomology group $\widehat{H}^0(r_l(\pi_E))$ is an irreducible mod-l representation of $G_n(F)$ (Theorem 4.5.3 and Corollary 4.5.4) and hence isomorphic to $r_l(\pi_F)^{(l)}$. We also prove that the first Tate cohomology group of any $\mathrm{Gal}(E/F)$ stable integral structure $\mathcal{L}$ of the generic representation π_E is trivial. Our method can also be extended to Zelevinsky subrepresentations as well. This is the class of irreducible representations of $\mathrm{GL}_n(E)$ which remain irreducible when restricted to the mirabolic subgroup denoted by $P_n(E)$ (see [Zel80, Remark 3.6] for instance). We apply Theorem 4.1.1 to compute the Tate cohomology groups of Zelevinsky subrepresentations (Theorem 4.4.3).

4.2 Finiteness of Tate cohomology

In this part, we prove some finiteness results of the Tate cohomology groups of irreducible representations of GL_n. First, we recall the Bernstein–Zelevinsky derivative functors. For precise reference, see [BZ77, Section 3] for $R = \overline{\mathbb{Q}}_l$, and [Vig96b, Chapter III, Section 1] for $R = \overline{\mathbb{F}}_l$. Let $U_n(K)$ be the subgroup of $G_n(K)$, given by

$$U_n(K) = \left\{ \begin{pmatrix} 1 & C \\ 0 & 1 \end{pmatrix} : C \in K^{n-1} \right\}$$

respectively. Note that $U_n(K)$ is contained in the mirabolic subgroup $P_n(K)$. We use the short notation $Z_{K,n}$ for the coset space $P_n(K)/P_{n-1}(K)U_n(K)$. We denote by θ_K the restriction $\mathrm{res}_{U_n(K)}(\Theta_K)$, where Θ_K is the character of $N_n(K)$, defined by (2.1). Let $\mathrm{Rep}_R(G)$ be the category of smooth R-representations of a locally profinite group G, where $R = \overline{\mathbb{Q}}_l$ or $\overline{\mathbb{F}}_l$.

1. For $\tau \in \mathrm{Rep}_R(P_{n-1})$, let $i_{U_n(K),\theta_K}(\tau)$ denote the representation of $P_n(K)$, defined as the normalized compact induction $i_{P_{n-1}(K)U_n(K)}^{P_n(K)}(\tau \otimes \theta_K)$.

2. For any $\rho \in \mathrm{Rep}_R(G_{n-1})$, we denote by $i_{U_n(K),\mathrm{id}_K}(\rho)$ the normalized compact induction $i_{G_{n-1}(K)U_n(K)}^{P_n(K)}(\tau \otimes \mathrm{id}_K)$, where id_K is the trivial character of $U_n(K)$.

3. For $(\pi, V) \in \mathrm{Rep}_R(P_n)$, let $V(U_n(K), \theta_K)$ be the space generated by the set of vectors $\{\pi(u)v - \theta_K(u)v : v \in V, u \in U_n(K)\}$. We denote by $r_{U_n(K),\theta_K}(\pi)$ the representation of $P_{n-1}(K)$, defined on the quotient space $V/V(U_n(K), \theta_K)$, by setting
$$r_{U_n(K),\theta_K}(\pi)(p)(\overline{v}) = \delta_{U_n(K)}^{1/2}(p')\overline{\pi(p)v}, \tag{4.1}$$
for all $v \in V$ and $p \in P_{n-1}(K)$. Here $\delta_{U_n(K)}$ is the modulus character of $U_n(K)$, and $\overline{v}$ denotes the class of v in the quotient space $V/V(U_n(K), \theta_K)$.

4. For $(\pi, V) \in \mathrm{Rep}_R(P_n)$, we denote by $r_{U_n(K),\mathrm{id}_K}(\pi)$ the representation of $G_{n-1}(K)$ on the quotient space $V/V(U_n(K), \mathrm{id}_K)$, defined as (4.1).

Then we have four fundamental functors
$$\Psi^- : \mathrm{Rep}_R(P_n) \to \mathrm{Rep}_R(G_{n-1}), \quad \Psi^+ : \mathrm{Rep}_R(G_{n-1}) \to \mathrm{Rep}_R(P_n)$$
$$\Phi^- : \mathrm{Rep}_R(P_n) \to \mathrm{Rep}_R(P_{n-1}), \quad \Phi^+ : \mathrm{Rep}_R(P_{n-1}) \to \mathrm{Rep}_R(P_n).$$
where $\Psi^- = r_{U_n(K),\mathrm{id}_K}$, $\Phi^- = r_{U_n(K),\theta_K}$, $\Psi^+ = i_{U_n(K),\mathrm{id}_K}$, and $\Phi^+ = i_{U_n(K),\theta_K}$.

Let τ be a smooth R-representation of $P_n(K)$. The m-th derivative of τ, denoted by $\tau^{(m)}$, is defined as the representation $\Psi^-(\Phi^-)^{m-1}(\tau)$ of $G_{n-m}(K)$. There is a functorial filtration on τ, given by
$$0 \subseteq \tau_n \subseteq \tau_{n-1} \subseteq \cdots \subseteq \tau_2 \subseteq \tau_1 = \tau,$$
where $\tau_m = (\Phi^+)^{m-1}(\Phi^-)^{m-1}(\tau)$ and $\tau_m/\tau_{m+1} = (\Phi^+)^{m-1}(\Psi^+)(\tau^{(m)})$. We have the following lemma.

Lemma 4.2.1. Let ρ be a finite length R-representation of $G_t(K)$, where $1 \leq t < n$. Then $(\Phi^+)^{n-t-1}(\Psi^+)(\rho)$ has finite length as a representation of $P_n(K)$.

Proof. It is an immediate consequence of [Vig96b, Chapter III, Subsection 1.5] and the exactness of the functor $(\Phi^+)^{n-t-1}(\Psi^+)$. $\square$

Now, we prove

Proposition 4.2.2. *Let Π be a finite length l-modular representation of $G_n(E)$ with an isomorphism $T : \Pi \to \Pi^\gamma$ and $T^l = \mathrm{id}$. Then the Tate cohomology $\widehat{H}^i(\Pi)$, with respect to the operator T, is a finite length representation of $G_n(F)$.*

Proof. We prove the proposition using induction on the integer n. The case $n = 1$ is clear. So assume that the proposition is true for all finite length l-modular representations of $G_t(E)$ and for all $t < n$. Now, we consider Π as a representation of $P_n(E)$. Since $\overline{\psi}_E(\gamma(x)) = \overline{\psi}_E(x)$, for $x \in E$, we get the isomorphism

$$\Phi^-(\Pi^\gamma) \simeq \Phi^-(\Pi)^\gamma, \tag{4.2}$$

as representation of $P_{n-1}(E)$. Similarly, for any smooth l-modular representation τ of $P_{n-1}(E)$, we have $P_n(E)$-equivariant isomorphism

$$\Phi^+(\tau^\gamma) \simeq \Phi^+(\tau)^\gamma. \tag{4.3}$$

Using (4.2) and (4.3), and the isomorphism T, we get an isomorphism between Π_m and Π_m^γ, and also between the representations $\Pi^{(m)}$ and $(\Pi^{(m)})^\gamma$, for all $m \leq n$.

Recall that $Z_{E,m}$ denote the coset space $P_m(E)/P_{m-1}(E)U_m(E)$. Since $P_{m-1}(E)U_m(E)$ is a Γ-stable subgroup of $P_m(E)$, we have the following long exact sequence of non-abelian cohomology ([Ser, Appendix, Proposition 1]):

$$0 \to P_{m-1}(F)U_m(F) \to P_m(F) \to Z_{E,m}^\Gamma \to H^1(\Gamma, P_{m-1}(E)U_m(E)) \to H^1(\Gamma, P_m(E)). \tag{4.4}$$

Since $P_m(E) = G_{m-1}(E) \ltimes U_m(E)$, we have the following exact sequence by [Ser, Appendix, Proposition 1]:

$$0 \to U_m(F) \to P_m(F) \to G_{m-1}(E) \to H^1(\Gamma, U_m(E)) \to H^1(\Gamma, P_m(E)) \to H^1(\Gamma, G_{m-1}(E)).$$

Using Hilbert's theorem 90 ([Ser, Proposition 1 and Proposition 3]), we get that the pointed sets $H^1(\Gamma, U_m(E))$ and $\widehat{H}^1(\Gamma, G_{m-1}(E))$ are trivial. Therefore, it follows from the above exact sequence that $H^1(\Gamma, P_m(E)) = 0$ for all positive integers m. Now, consider the following short exact sequence of non-abelian Γ-modules

$$0 \longrightarrow U_m(E) \longrightarrow P_{m-1}(E)U_m(E) \longrightarrow P_{m-1}(E) \longrightarrow 0. \tag{4.5}$$

Since the pointed sets $H^1(\Gamma, U_m(E))$ and $H^1(\Gamma, P_m(E))$ are trivial, it follows from the long exact sequence of non-abelian cohomology corresponding to (4.5) that $H^1(\Gamma, P_{m-1}(E)U_m(E)) = 0$. Hence, the long exact sequence (4.4) gives the equality $Z_{E,m}^\Gamma = Z_{F,m}$. Now, applying Proposition 2.8.2 repeatedly $(m-1)$-times, we get the $P_n(F)$ equivariant isomorphism

$$\widehat{H}^i(\Pi_m/\Pi_{m+1}) \simeq (\Phi^+)^{m-1}(\Psi^+)\big(\widehat{H}^i(\Pi^{(m)})\big).$$

Using Leibniz forumula for derivatives ([Vig96b, Lemma 1.10, Chapter 3]), we get that the m-th derivative $\Pi^{(m)}$ is a finite length representation of $G_{n-m}(E)$. By induction hypothesis, the $G_{n-m}(F)$-representation $\widehat{H}^i(\Pi^{(m)})$ is of finite length, for all $m < n$. In view of Lemma 4.2.1, it follows from the above isomorphism that the $P_n(F)$-representation $\widehat{H}^i(\Pi_m/\Pi_{m+1})$ is of finite length, for all $m < n$. Now, for each $m \in \{1, 2, \ldots, n-1\}$, consider the short exact sequence of $P_n(E)$-representations

$$0 \longrightarrow \Pi_{m+1} \longrightarrow \Pi_m \longrightarrow \Pi_m/\Pi_{m+1} \longrightarrow 0.$$

Since Γ is cyclic, the corresponding long exact sequence of Tate cohomology gives the following diagram:

$$\begin{tikzcd}
& \widehat{H}^0(\Pi_{m+1}) \arrow[r] & \widehat{H}^0(\Pi_m) \arrow[dr] & \\
\widehat{H}^1(\Pi_m/\Pi_{m+1}) \arrow[ur] & & & \widehat{H}^0(\Pi_m/\Pi_{m+1}) \arrow[dl] \\
& \widehat{H}^1(\Pi_m) \arrow[ul] & \widehat{H}^1(\Pi_{m+1}) \arrow[l] &
\end{tikzcd}$$

We denote the above exact sequence by $S(m)$. Consider the largest integer r for which Π_r is non-zero. By induction hypothesis, each Tate cohomology group $\widehat{H}^i(\Pi_r)$ has finite length as a representation of $P_n(F)$. Now, from the hexagonal diagram $S(r-1)$, we get the following two exact sequences:

$$H^0(\Pi_r) \longrightarrow H^0(\Pi_{r-1}) \longrightarrow H^0(\Pi_{r-1}/\Pi_r)$$

and

$$H^0(\Pi_r) \longrightarrow H^0(\Pi_{r-1}) \longrightarrow H^0(\Pi_{r-1}/\Pi_r).$$

Since both the representations $\widehat{H}^i(\Pi_{r-1}/\Pi_r)$ and $\widehat{H}^i(\Pi_r)$, for $i \in \{0, 1\}$, have finite length, the Tate cohomology group $\widehat{H}^i(\Pi_{r-1})$ is a finite length representation of $P_n(F)$. Again, using the finiteness of both $\widehat{H}^i(\Pi_{r-1})$ and $\widehat{H}^i(\Pi_{r-2}/\Pi_{r-1})$, it follows from the hexagonal diagram $S(r-2)$ that $\widehat{H}^i(\Pi_{r-2})$ is a finite length representation of $P_n(F)$. Thus, inductively, we finally get that $\widehat{H}^i(\Pi)$ has finite length as a representation of $P_n(F)$, and hence of $G_n(F)$. $\square$

As a consequence, we have

Corollary 4.2.3. *Let (π, V) be an integral $\mathcal{K}$-representation of $G_n(E)$ with an isomorphism $T : (\pi, V) \to (\pi^\gamma, V)$ and $T^l = \mathrm{id}$. Then, for any $G_n(E) \rtimes \Gamma$ stable lattice $\mathcal{L}$ in V (Here, Γ acts on V via T), the Tate cohomology group $\widehat{H}^i(\mathcal{L})$ has finite length as a representation of $G_n(F)$.*

Proof. Consider the short exact sequence of $G_n(E) \rtimes \Gamma$-modules

$$0 \longrightarrow \mathcal{L} \xrightarrow{\text{mult}.l} \mathcal{L} \longrightarrow \mathcal{L}/l\mathcal{L} \longrightarrow 0.$$

Since π is of finite length, the l-modular representation $\mathcal{L}/l\mathcal{L}$ is of finite length (see Subsection 2.2.1). Then, using Proposition 4.2.2, we get that $\widehat{H}^i(\mathcal{L}/l\mathcal{L})$ has finite length as a representation of $G_n(F)$. Now, the long exact sequence of Tate cohomology corresponding to the above short exact sequence gives

$$0 \to \widehat{H}^0(\mathcal{L}) \to \widehat{H}^0(\mathcal{L}/l\mathcal{L}) \to \widehat{H}^1(\mathcal{L}) \to 0.$$

and

$$0 \to \widehat{H}^1(\mathcal{L}) \to \widehat{H}^1(\mathcal{L}/l\mathcal{L}) \to \widehat{H}^0(\mathcal{L}) \to 0.$$

Thus, we get that $\widehat{H}^i(\mathcal{L})$ is of finite length. $\square$

4.3 Tate Cohomology of Whittaker Lattice

Let (π, V) be a generic R-representation of $G_n(E)$, where R denotes either $\overline{\mathbb{Q}}_l$ or $\mathcal{K}$ or $\overline{\mathbb{F}}_l$. We further assume that $\pi \simeq \pi^\gamma$, for all $\gamma \in \Gamma$. Let $W(\pi, \psi_{E,R})$ be the Whittaker model of π. For $W \in \mathbb{W}(\pi, \psi_{E,R})$, we recall that $\gamma.W$ is a function given by

$$(\gamma.W)(g) = W(\gamma^{-1}(g)),$$

for all $g \in G_n(E)$. Note that $\gamma.W \in \mathbb{W}(\pi, \psi_{E,R})$ (by Lemma 2.3.3). We define

$$T_\gamma : \mathbb{W}(\pi, \psi_{E,R}) \to \mathbb{W}(\pi, \psi_{E,R})$$

by setting $T_\gamma(W) = \gamma.W$, for all $W \in \mathbb{W}(\pi, \psi_{E,R})$. The map T_γ gives an isomorphism between (π, V) and (π^γ, V), as we have

$$[T_\gamma(\pi(g)W)](h) = [\pi^\gamma(g)T_\gamma(W)](h),$$

for all $g, h \in G_n(E)$. In particular, when $R = \mathcal{K}$, the lattice $\mathbb{W}_\Lambda(\pi, \psi_E)$ is also stable under the action of Γ, where $\gamma \in \Gamma$ acts via the isomorphism T_γ.

4.3.1 Jacquet-functors and Tate cohomology

In this subsection, we establish some elementary results on the compatibility of Jacquet (twisted Jacquet) functors with Tate cohomology. Let $(\pi, \mathcal{L})$ be a smooth $\Lambda[G_n(E) \rtimes \Gamma]$-module. Let

$\lambda = (n_1, n_2, \ldots, n_r)$ be a partition of n and let $P_\lambda = M_\lambda N_\lambda$ be a parabolic subgroup of G_n with N_λ its unipotent radical and M_λ a standard Levi subgroup. Let $\psi_F : F \to \Lambda^\times$ be an additive character, and let ψ_E be the composition $\psi_F \circ \mathrm{Tr}_{E/F}$. Let Θ_F (resp. Θ_E) be the character of $N_n(F)$ (resp. $N_n(E)$), defined as (2.1) . Let $\mathcal{L}(N_\lambda(E), \Theta_E)$ be the space spanned by the set of vectors

$$\big\{\Theta_E(n)v - \pi(n)v : v \in \mathcal{L}, n \in N_\lambda(E)\big\}.$$

Note that the space $\mathcal{L}(N_\lambda(E), \Theta_E)$ is stable under the action of Γ. Then the natural inclusion $\mathcal{L}(N_\lambda(E), \Theta_E) \hookrightarrow \mathcal{L}$ induces a map $\widehat{H}^0(\mathcal{L}(N_\lambda(E), \Theta_E)) \to \widehat{H}^0(\mathcal{L})$, denoted by ϕ.

Lemma 4.3.1. *The image of the natural map ϕ is equal to $\widehat{H}^0(\mathcal{L})(N_\lambda(F), \Theta_F^l)$.*

Proof. Let v be an element in the image of ϕ. Let $\tilde{v}$ be a lift of v in $\mathcal{L}(N_\lambda(E), \Theta_E)^\Gamma$. There exists a compact open subgroup $\mathcal{N}$ of $N_\lambda(E)$ such that

$$\int_{\mathcal{N}} \Theta_E(n)^{-1}\pi(n)\tilde{v}\, dn = 0. \tag{4.6}$$

Since π and Θ_E are both smooth, there exists a compact open subgroup $\mathcal{N}' \subseteq \mathcal{N}$ of finite index such that

$$\int_{\mathcal{N}} \Theta_E(n)^{-1}\pi(n)\tilde{v}\, dn = \sum_{n \in \mathcal{N}/\mathcal{N}'} \Theta_E(n)^{-1}\pi(n)\tilde{v}.$$

Since $N_\lambda(E)$ has a filtration of Γ-stable compact open subgroups, we may assume that $\mathcal{N}$ is Γ-stable. If X denotes the coset space $\mathcal{N}/\mathcal{N}'$, then we have

$$\sum_{x \in X} \Theta_E(x)^{-1}\pi(x)\tilde{v} = \sum_{y \in X^\Gamma} \Theta_F(y)^{-l}\pi(y)\tilde{v} \;+ \sum_{z \in X \setminus X^\Gamma} \Theta_E(z)^{-1}\pi(z)\tilde{v}. \tag{4.7}$$

Since the Γ-action on $X \setminus X^\Gamma$ is free, there exists a subset $\mathcal{U}$ such that $X \setminus X^\Gamma$ is the disjoint union of $\gamma^i\mathcal{U}$, $1 \le i \le l$. As $\tilde{v}$ is Γ-invariant, we have

$$\sum_{z \in X \setminus X^\Gamma} \Theta_E(z)^{-1}\pi(z)\tilde{v} = \sum_{i=1}^{l}\sum_{u \in \mathcal{U}} \Theta_E(u)^{-1}\pi(\gamma^i(u))\tilde{v} = N\big(\sum_{u \in \mathcal{U}} \Theta_E(u)^{-1}\pi(u)\tilde{v}\big),$$

where $N = 1 + \gamma + \cdots + \gamma^{l-1}$. This shows that

$$\sum_{z \in X \setminus X^\Gamma} \Theta_E(z)^{-1}\pi(z)v = 0$$

in $\widehat{H}^0(\mathcal{L})$. Therefore, it follows from (4.6) and (4.7) that

$$\sum_{y\in X^\Gamma}(\Theta_F^l(y))^{-1}\pi(y)v=0.$$

This implies that v belongs to $\widehat{H}^0(\mathcal{L})(N_\lambda(F),\Theta_F^l)$. Conversely, let w be an element of the space $\widehat{H}^0(\mathcal{L})(N_\lambda(F),\Theta_F^l)$. Let $\mathcal{N}_F$ be the compact open subgroup of $N_\lambda(F)$ with

$$\int_{\mathcal{N}_F}\Theta_F(n)^{-l}\pi(n)w\,dn=0.$$

Choose a compact open subgroup $\mathcal{N}'_F\subseteq\mathcal{N}_F$ of finite index such that

$$\int_{\mathcal{N}_F}\Theta_F(n)^{-l}\pi(n)w\,dn=\sum_{n\in\mathcal{N}_F/\mathcal{N}'_F}\Theta_F(n)^{-l}\pi(n)w.$$

Let $\tilde{w}$ be a lift of w in $\mathcal{L}^\Gamma$. Consider the element

$$\tilde{w}_1=\frac{1}{|\mathcal{N}_F/\mathcal{N}'_F|}\sum_{n\in\mathcal{N}_F/\mathcal{N}'_F}\Theta_E(n)\tilde{w}-\frac{1}{|\mathcal{N}_F/\mathcal{N}'_F|}\sum_{n\in\mathcal{N}_F/\mathcal{N}'_F}\pi(n)\tilde{w}.$$

Then $\tilde{w}_1$ belongs to $\mathcal{L}(N_\lambda(E),\Theta_E)^\Gamma$ and $\phi(\tilde{w}_1)=w$. Moreover, we have

$$\int_{\mathcal{N}_F}\Theta_F(n)^{-l}\pi(n)\tilde{w}_1\,dn=\sum_{n\in\mathcal{N}_F/\mathcal{N}'_F}\Theta_F(n)^{-l}\pi(n)\tilde{w}_1=0.$$

This completes the proof. □

Lemma 4.3.2. *Let (π,V) be an l-modular representation of $G_n(E)\rtimes\Gamma$ such that $\widehat{H}^1(V_{N_\lambda(E)})=0$, $\widehat{H}^0(V)_{N_\lambda(F)}$ is non-zero and $\widehat{H}^0(V_{N_\lambda(E)})$ is irreducible under $M_\lambda(F)$. Then, the $M_\lambda(F)$-representation $\widehat{H}^0(V)_{N_\lambda(F)}$ is isomorphic to $\widehat{H}^0(V_{N_\lambda(E)})$.*

Proof. The long exact sequence of Tate cohomology groups corresponding to the short exact sequence

$$0\longrightarrow V(N_\lambda(E))\longrightarrow V\longrightarrow V_{N_\lambda(E)}\longrightarrow 0,$$

gives the following exact sequence

$$0\longrightarrow\widehat{H}^0(V(N_\lambda(E)))\xrightarrow{\phi}\widehat{H}^0(V)\longrightarrow\widehat{H}^0(V_{N_\lambda(E)})\longrightarrow\widehat{H}^1(V(N_\lambda(E)))\longrightarrow\widehat{H}^1(V)\longrightarrow 0.$$

Using Lemma 4.3.1 for $\psi_F=1_F$, the trivial character of F, we get that $\phi\big(\widehat{H}^0(V(N_\lambda(E)))\big)$ is equal to $\widehat{H}^0(V)(N_\lambda(F))$, and therefore the $N_\lambda(F)$-coinvariants $\widehat{H}^0(V)_{N_\lambda(F)}$ is isomorphic to a subrepresentation of $\widehat{H}^0(V_{N_\lambda(E)})$. Since $\widehat{H}^0(V)_{N_\lambda(F)}$ is non-zero, the lemma follows from the irreducibility of $\widehat{H}^0(V_{N_\lambda(E)})$. $\square$

Using similar ideas, we can prove that the zeroth Tate cohomology group of a generic representation has a unique generic subquotient. For any integral l-adic generic representation (π, V) of $G_n(K)$, defined over $\mathcal{K}$, we recall that $\mathbb{W}_\Lambda(\pi, \psi_K)$ denotes the set of all Λ-valued functions in $\mathbb{W}(\pi, \psi_K)$. The Λ-module $\mathbb{W}_\Lambda(\pi, \psi_K)$ gives a Λ-structure of π (see Subsection 2.3.5).

Proposition 4.3.3. *Let π_E be an l-modular generic representation (or an integral l-adic generic representation defined over $\mathcal{K}$) of $G_n(E)$. Assume that π_E is stable under the action of Γ. Then, there exists a unique generic $G_n(F)$ sub-quotient of the Tate cohomology group $\widehat{H}^0(\mathbb{W}(\pi_E, \psi_E))$ (resp. $\widehat{H}^0(\mathbb{W}_\Lambda(\pi_E, \psi_E))$).*

Proof. We consider only the l-adic case. The mod-l case is similar, where the character ψ_E is replaced by its mod-l reduction $\overline{\psi}_E$. Let $\mathcal{W}$ be the Λ-lattice $\mathbb{W}_\Lambda(\pi_E, \psi_E)$ (resp. $\mathbb{W}(\pi_E, \overline{\psi}_E)$). Let $\mathcal{W}(N_n(E), \Theta_E)$ be the Λ-span of the set of vectors

$$\big\{\Theta_E(n)v - \pi_E(n)v : v \in \mathcal{W}, n \in N_n(E)\big\}.$$

We have the following exact sequence of Γ-modules :

$$0 \longrightarrow \mathcal{W}(N_n(E), \Theta_E) \longrightarrow \mathcal{W} \to \mathcal{W}_{N_n(E),\Theta_E} \longrightarrow 0.$$

Note that $\mathcal{W}_{N_n(E),\Theta_E}$ is a free Λ-module of rank one. Then the long exact sequence of Tate cohomology gives

$$\widehat{H}^0(\mathcal{W}(N_n(E), \Theta_E)) \xrightarrow{f} \widehat{H}^0(\mathcal{W}) \xrightarrow{g} \widehat{H}^0(\mathcal{W}_{N_n(E),\Theta_E}) \to \widehat{H}^1(\mathcal{W}(N_n(E), \Theta_E)) \to \widehat{H}^1(\mathcal{W}).$$

Using Lemma 4.3.1, the image of the homomorphism f is equal to $\widehat{H}^0(\mathcal{W})(N_n(F), \Theta_F^l)$. The Tate cohomology of the Λ-lattice $\mathbb{K}_\Lambda(\pi_E, \psi_E)$ contains $\mathcal{K}(\overline{\psi}_F^l)$ as $P_n(F)$ subrepresentation (see 4.3.3, Part 1 of Theorem 4.3.6). Since Γ acts trivially on $\mathcal{W}_{N_n(E),\Theta_E}$ (Lemma 2.3.4), the Tate cohomology space $\widehat{H}^0(\mathcal{W}_{N_n(E),\Theta_E})$ is a one dimensional vector space over $\overline{\mathbb{F}}_l$. Hence, the map g induces the isomorphism:

$$\widehat{H}^0(\mathcal{W})_{N_n(F),\overline{\Theta}_F^l} \simeq \widehat{H}^0(\mathcal{W}_{N_n(E),\Theta_E}).$$

Now, it follows from the exactness of twisted Jacquet functors and Corollary 4.2.3, that $\widehat{H}^0(\mathcal{W})$ admits a unique generic sub-quotient. $\square$

Remark 4.3.4. The above lemmas will be used to compute the Tate cohomology group of base change lift of mod-l the Zelevinsky sub-representation $Z(\Delta)$. The Jacquet functor of $Z(\Delta)$ with respect to the parabolic subgroup of type $(n/k, n/k, \dots, n/k)$, where k is the length of Δ, is an l-modular cuspidal representation and the hypothesis in Lemma 4.3.2 are applicable. The precise definition will be recalled in the next section.

4.3.2 Completeness of Whittaker models

Our main result uses the following lemma, which is the analog of the completeness of Whittaker models in the complex case.

Lemma 4.3.5. *Assume that l does not divide $|G_n(\kappa_K)|$ and let $\overline{\psi}_K$ be the mod-l reduction of ψ_K. Let $\overline{\Theta}_K$ be the non-degenerate character of $N_n(K)$ associated with $\overline{\psi}_K$ (see Section 2.3). Let $\varphi \in \mathrm{ind}_{N_n(K)}^{G_n(K)}(\overline{\Theta}_K)$. If*

$$\int_{N_n(K)\backslash G_n(K)} \varphi(t)W(t)\,dt = 0,$$

for all $W \in \mathbb{W}(\sigma, \overline{\psi}_K^{-1})$ and for all l-modular generic representations σ of $G_n(K)$, then $\varphi = 0$.

Proof. We will prove the contrapositive. Suppose φ is non-zero. Let $\mathcal{Z}$ be the integral Bernstein center, i.e., the center of the category $\mathrm{Rep}_{W(\overline{\mathbb{F}}_l)}(G_n(K))$, consisting of smooth $W(\overline{\mathbb{F}}_l)[G_n(K)]$-modules (see [Hel16b] for details). Let W_n be the smooth $W(\overline{\mathbb{F}}_l)[G_n(K)]$-module $\mathrm{ind}_{N_n(K)}^{G_n(K)}(\Theta_K)$. Recall that for any primitive idempotent e in $\mathcal{Z}$, the space eW_n is a co-Whittaker $e\mathcal{Z}_n[G_n(K)]$ module. According to [Mos21, Corollary 4.3], there exists a primitive idempotent e' in $\mathcal{Z}$ and an element $U \in \mathbb{W}(e'W_n \otimes_{W(\overline{\mathbb{F}}_l)} \overline{\mathbb{F}}_l, \overline{\psi}_K^{-1})$ such that the integral

$$\langle \phi, U\rangle := \int_{N_n(K)\backslash G_n(K)} \phi(t) \otimes U(t)\,dt$$

is non-zero in $\overline{\mathbb{F}}_l \otimes_{W(\overline{\mathbb{F}}_l)} e'\mathcal{Z}_n$. We use the notation $\mathcal{A}$ for the ring $\overline{\mathbb{F}}_l \otimes_{W(\overline{\mathbb{F}}_l)} e'\mathcal{Z}_n$. The work of [Hel16a] shows the primitive idempotent e' corresponds to an inertial equivalence class $[M, \pi]$, where M is a Levi subgroup of $G_n(K)$ and π is an l-modular supercuspidal representation of M.

For the inertial equivalence class $\mathfrak{s} = [M, \pi]$, consider the subcategory $\mathrm{Rep}_{\mathfrak{s}}(G_n(K))$, consisting of objects Π in $\mathrm{Rep}_{W(\overline{\mathbb{F}}_l)}(G_n(K))$ such that the irreducible sub-quotients of Π have mod-l inertial supercuspidal support $\mathfrak{s}$ (see [Hel16a, Definition 4.12] for the definition). Let $A_{[M,\pi]}$ denote the center of $\mathrm{Rep}_{\mathfrak{s}}(G_n(K))$. We have $A_{[M,\pi]} = e'\mathcal{Z}_n$. Since l does not divide $|G_n(\kappa_K)|$, it follows from [Hel16a, Example 13.9] that

$$A_{[M,\pi]} = C_{[M,\pi]},$$

where $C_{[M,\pi]}$ is a $W(\overline{\mathbb{F}}_l)$-subalgebra of $A_{[M,\pi]}$, as defined in [Hel16a, Theorem 12.5]. There is an isomorphism of $C_{[M,\pi]} \otimes_{W(\overline{\mathbb{F}}_l)} \overline{\mathbb{F}}_l$ with the reduced quotient of $\mathcal{A}$ (see [Hel16a, Corollary 12.13]). Hence, we get that the $W(\overline{\mathbb{F}}_l)$-algebra $\mathcal{A}$ is reduced. In particular, the element $\langle \phi, U\rangle$ is not nilpotent. Therefore, the basic open set $D(\langle\phi, U\rangle)$, being non-empty, intersects the dense set of closed points of the affine $W(\overline{\mathbb{F}}_l)$-scheme associated with $\mathcal{A}$, and this implies that there exists a map $f : \mathcal{A} \to \overline{\mathbb{F}}_l$ such that the image of $\langle\phi, U\rangle$ under f, which equals

$$\int_{N_n(K)\backslash G_n(K)} \phi(t)W_0(t)\,dt$$

for some $W_0 \in \mathbb{W}\big(e'W_n \otimes_{\mathcal{A},f} \overline{\mathbb{F}}_l, \overline{\psi}_K^{-1}\big)$, is non-zero in $\overline{\mathbb{F}}_l$. Note that the co-Whittaker module $e'W_n \otimes_{\mathcal{A},f} \overline{\mathbb{F}}_l$, as mod-$l$ representation, admits a generic quotient with same Whittaker space. Hence, the lemma. $\square$

4.3.3 The general case

We will now prove the first main result of this chapter.

Theorem 4.3.6. *Let F be a finite extension of $\mathbb{Q}_p$, and let E be a finite Galois extension of F with $[E : F] = l$, where p and l are distinct primes such that l does not divide $|G_{n-1}(\kappa_F)|$. Let π_F be an integral, l-adic generic representation of $G_n(F)$ and let π_E be the base change lifting of π_F to $G_n(E)$. Then, the representation $J_l(\pi_F)^{(l)}$ is the unique generic sub–quotient of the Tate cohomology group $\widehat{H}^0(J_l(\pi_E))$.*

Proof. We begin with a summary of the proof. We prove the above theorem using induction on the integer n. The proof is divided into four parts. In the first part, we isolate a subspace $\mathcal{M}(\pi_F, \psi_F)$ of the Tate cohomology of the Kirillov model of $J_l(\pi_E)$ which will eventually give $J_l(\pi_F)^{(l)}$ as a quotient. In the second part, we set up a comparison of Rankin–Selberg integrals on homogeneous spaces of F with those on homogeneous spaces of E. In the third part, we reduce the theorem to an identity of local γ-factors. In the fourth part, we deal with these local γ-factor identities, and we show that $\mathcal{M}(\pi_F, \psi_F)$ is stable under the action of $G_n(F)$. At the end of the fourth part, we get a natural surjection from $\mathcal{M}(\pi_F, \psi_F)$ to the Kirillov space $\mathbb{K}(J_l(\pi_F)^{(l)}, \overline{\psi}_F^l)$ as $G_n(F)$-representations.

Part 1.

Notations on Whittaker and Kirillov models are defined in subsection (2.3.1). Consider the Whittaker model $\mathbb{W}(J_l(\pi_E), \overline{\psi}_E)$ of the unique generic sub-quotient $J_l(\pi_E)$. The restriction map $W \longmapsto \mathrm{Res}_{P_n(E)}(W)$ is an isomorphism from $\mathbb{W}(J_l(\pi_E), \overline{\psi}_E)$ onto $\mathbb{K}(J_l(\pi_E), \overline{\psi}_E)$ ([MM22, Theorem 4.2]). Recall that $\mathcal{K}(\overline{\psi}_E)$ denotes the compactly induced representation $\mathrm{ind}_{N_n(E)}^{P_n(E)}(\overline{\Theta}_E)$. Let I_n be the following natural map:

$$I_n : \widehat{H}^0(\mathcal{K}(\overline{\psi}_E)) \to \widehat{H}^0(\mathbb{K}(J_l(\pi_E), \overline{\psi}_E)).$$

Let $\Phi_n : \mathbb{K}(J_l(\pi_E), \overline{\psi}_E)^\Gamma \to \mathrm{Ind}_{N_n(F)}^{P_n(F)}(\overline{\Theta}_F^l)$ be the restriction to $P_n(F)$ map. Note that the map Φ_n factorizes through

$$\Phi_n : \widehat{H}^0(\mathbb{K}(J_l(\pi_E), \overline{\psi}_E)) \to \mathrm{Ind}_{N_n(F)}^{P_n(F)}(\overline{\Theta}_F^l).$$

The composition $\Phi_n \circ I_n$ is induced by the restriction to $P_n(F)$ map from $\mathcal{K}(\overline{\psi}_E)^\Gamma$ to $\mathcal{K}(\overline{\psi}_F^l)$, and hence, $\Phi_n \circ I_n$ is an isomorphism onto the space $\mathcal{K}(\overline{\psi}_F^l)$ by Proposition 2.8.2 (see Remark 2.8.3). This implies that the image of Φ_n contains the space $\mathcal{K}(\overline{\psi}_F^l)$. Let $\mathcal{M}(\pi_F, \psi_F)$ be the space $\Phi_n^{-1}(\mathbb{K}(J_l(\pi_F)^{(l)}, \overline{\psi}_F^l))$. The space $\mathcal{M}(\pi_F, \psi_F)$ is a non-zero $P_n(F)$ sub-representation of $\widehat{H}^0(\mathbb{K}(J_l(\pi_E), \overline{\psi}_E))$, and the map

$$\Phi_n : \mathcal{M}(\pi_F, \psi_F) \to \mathbb{K}(J_l(\pi_F)^{(l)}, \overline{\psi}_F^l)$$

is non-zero. Then, using induction on the integer n, we will show that the space $\mathcal{M}(\pi_F, \psi_F)$ is stable under the action of $G_n(F)$ and the map Φ_n is $G_n(F)$ equivariant.

Part 2.

Let $\overline{J_l(\pi_E)}(w_n)$ denotes the induced action of $J_l(\pi_E)(w_n)$ on $\widehat{H}^0(\mathbb{K}(J_l(\pi_E), \overline{\psi}_E))$. Let V be an element in $\mathcal{M}(\pi_F, \psi_F)$. Then there exists $W \in \mathbb{W}(J_l(\pi_E), \overline{\psi}_E)^\Gamma$ such that W is mapped to V under the map

$$\mathbb{W}(J_l(\pi_E), \overline{\psi}_E)^\Gamma \longrightarrow \mathbb{K}(J_l(\pi_E), \overline{\psi}_E)^\Gamma \longrightarrow \widehat{H}^0\big(\mathbb{K}(J_l(\pi_E), \overline{\psi}_E)\big). \tag{4.8}$$

Let $\overline{\sigma}_F$ be an arbitrary l-modular generic representation of $G_{n-1}(F)$, and let σ_F be its l-adic lift. In this case, the generic mod-l representation $J_l(\sigma_F)$ is equal to $\overline{\sigma}_F$. Let σ_E be an l-adic generic representation of $G_{n-1}(E)$ obtained as a base change of σ_F. Note that the map

$$\tilde{\Phi}_{n-1} : \widehat{H}^0(\mathbb{W}(J_l(\sigma_E), \overline{\psi}_E^{-1})) \longrightarrow \mathrm{Ind}_{N_{n-1}(F)}^{G_{n-1}(F)}(\overline{\Theta}_F^{-l})$$

is non-zero. Here, $\tilde{\Phi}_{n-1}$ is the restriction to $G_{n-1}(F)$ map on $\mathrm{Ind}_{N_{n-1}(E)}^{G_{n-1}(E)}(\overline{\Theta}_E)$. Assuming the induction hypothesis for $n-1$ and using the fact that $\widehat{H}^0(\mathbb{W}(J_l(\sigma_E), \overline{\psi}_E^{-1}))$ admits an unique generic sub-quotient (Proposition 4.3.3), the image of $\tilde{\Phi}_{n-1}$ contains $\mathbb{W}(\overline{\sigma}_F^{(l)}, \overline{\psi}_F^{-l})$. Thus, for any $W' \in \mathbb{W}(\overline{\sigma}_F^{(l)}, \overline{\psi}_F^{-l})$, there exists an element $\mathcal{S} \in \mathbb{W}(J_l(\sigma_E), \overline{\psi}_E^{-1})^\Gamma$ such that $\tilde{\Phi}_{n-1}(\mathcal{S}) = W'$ and

$$\tilde{\Phi}_{n-1}\big(\overline{J_l(\sigma_E)}(w_{n-1})\mathcal{S}\big) = \overline{\sigma}_F^{(l)}(w_{n-1})W'.$$

Now, the functional equation in (2.4.3) gives the following relation

$$\sum_{r\in\mathbb{Z}} c_r^E(\widetilde{W}, \widetilde{\mathcal{S}}) q_F^{-\frac{r}{2}f} X^{-fr} = \omega_{J_l(\sigma_E)}(-1)^{n-2}\gamma\big(X^f, J_l(\pi_E), J_l(\sigma_E), \overline{\psi}_E\big) \sum_{r\in\mathbb{Z}} c_r^E(W, \mathcal{S}) q_F^{\frac{r}{2}f} X^{fr}, \tag{4.9}$$

where f denotes the residue degree of the extension E/F. Note that $\omega_{J_l(\sigma_E)}(-1) = \omega_{\overline{\sigma}_F}(-1)$ as l is an odd prime. Applying Corollary 2.8.6, we have

$$\begin{aligned}\int_{(X_E^r)^\Gamma} W\begin{pmatrix} g & 0 \\ 0 & 1\end{pmatrix}\mathcal{S}(g)\,dg &= \int_{X_F^{\frac{r}{e}}} W\begin{pmatrix} g & 0 \\ 0 & 1\end{pmatrix}\mathcal{S}(g)\,dg,\\ \int_{(X_E^r)^\Gamma} \widetilde{W}\begin{pmatrix} g & 0 \\ 0 & 1\end{pmatrix}\widetilde{\mathcal{S}}(g)\,dg &= \int_{X_F^{\frac{r}{e}}} \widetilde{W}\begin{pmatrix} g & 0 \\ 0 & 1\end{pmatrix}\widetilde{\mathcal{S}}(g)\,dg,\end{aligned} \tag{4.10}$$

for each $r \in \mathbb{Z}$. Using the above equalities, the functional equation (4.9) after reduction mod-l, becomes

$$\sum_{r\in\mathbb{Z}} c_r^F(\widetilde{W},\widetilde{\mathcal{S}})q_F^{-\frac{lr}{2}}X^{-lr} = \omega_{\overline{\sigma}_F}(-1)^{n-2}\gamma\big(X^f, J_l(\pi_E), J_l(\sigma_E), \overline{\psi}_E\big)\sum_{r\in\mathbb{Z}} c_r^F(W,\mathcal{S})q_F^{\frac{lr}{2}}X^{lr}.$$

Using the formula $q_F^{l/2} = (\frac{q_F}{l})\, q_F^{1/2}$ (mod–l) and the modification as in (2.3), the above identity becomes

$$\begin{aligned}&\sum_{r\in\mathbb{Z}} c_{-r}^F\big(\overline{J_l(\pi_E)}(w_n)W, \overline{\sigma}_F^{(l)}(w_{n-1})W'\big)q_F^{-\frac{r}{2}}\big((\frac{q_F}{l})X^l\big)^{-r} =\\ &\qquad \omega_{\overline{\sigma}_F}(-1)^{n-2}\gamma\big(X^f, J_l(\pi_E), J_l(\sigma_E), \overline{\psi}_E\big)\sum_{r\in\mathbb{Z}} c_r^F(W,W')\,q_F^{\frac{r}{2}}\big((\frac{q_F}{l})X^l\big)^r.\end{aligned} \tag{4.11}$$

<u>**Part 3.**</u>

For any $V \in \mathcal{M}(\pi_F,\psi_F)$, we show that

$$\Phi_n(\overline{J_l(\pi_E)}(w_n)V) = J_l(\pi_F)^{(l)}(w_n)\Phi_n(V). \tag{4.12}$$

Let U be an element of $\mathbb{W}(J_l(\pi_F)^{(l)}, \overline{\psi}_F^l)$ such that the restriction of U to $P_n(F)$ maps to the element $\Phi_n(V)$. By Lemma 4.3.5, the assertion (4.12) is equivalent to the following equality:

$$\begin{aligned}&\sum_{r\in\mathbb{Z}} c_{-r}^F\big(\overline{J_l(\pi_E)}(w_n)W,\, \overline{\sigma}_F^{(l)}(w_{n-1})W'\big)q_F^{-r/2}X^{-r} =\\ &\qquad \sum_{r\in\mathbb{Z}} c_{-r}^F\big(J_l(\pi_F)^{(l)}(w_n)U,\, \overline{\sigma}_F^{(l)}(w_{n-1})W'\big)q_F^{-r/2}X^{-r},\end{aligned} \tag{4.13}$$

for all $W' \in \mathbb{W}(\overline{\sigma}_F, \overline{\psi}_F^{-l})$ and for all l-modular generic representations $\overline{\sigma}_F$ of $G_{n-1}(F)$. Now, consider an l-modular generic representation $\overline{\sigma}_F$ of $G_{n-1}(F)$ and take the l-adic lift of $\overline{\sigma}_F$, say σ_F (see Subsection 2.3.4). Note that $J_l(\sigma_F) = \overline{\sigma}_F$. Let σ_E be the l-adic generic representation of $G_{n-1}(E)$ obtained as a base change of σ_F. From the functional equation with its modifications as in (2.3), we have

$$\sum_{r\in\mathbb{Z}} c_{-r}^F\big(J_l(\pi_F)^{(l)}(w_n)U, \overline{\sigma}_F^{(l)}(w_{n-1})W'\big)(\frac{q_F}{l})^{-r} q_F^{-\frac{r}{2}}\big((\frac{q_F}{l})X^l\big)^{-r}$$
$$= \omega_{\overline{\sigma}_F}(-1)^{n-2}\gamma\big((\frac{q_F}{l})X^l, J_l(\pi_F)^{(l)}, \overline{\sigma}_F^{(l)}, \overline{\psi}_F^l\big)\sum_{r\in\mathbb{Z}} c_r^F(U,W')\, q_F^{\frac{r}{2}}\big((\frac{q_F}{l})X^l\big)^r,$$

where we replace the variable X by $(\frac{q_F}{l})X^l$. Note that the mod-l reduction of the function $\mathrm{res}_{P_n(F)}(W)$ is equal to $\mathrm{res}_{P_n(F)}(U)$. Thus, comparing the above functional equation with (4.11), the relation (4.12) is now equivalent to the following identity of gamma factors:

$$\gamma\big(X^f, J_l(\pi_E), J_l(\sigma_E), \overline{\psi}_E\big) = \gamma\big((\frac{q_F}{l})X^l, J_l(\pi_F)^{(l)}, \overline{\sigma}_F^{(l)}, \overline{\psi}_F^l\big).$$

Part 4.

Recall that

$$\gamma(X^f, \pi_E, \sigma_E, \psi_E) = \epsilon(X^f, \pi_E, \sigma_E, \psi_E)\frac{L(q_E^{-1}X^{-f}, \pi_E^\vee, \sigma_E^\vee)}{L(X^f, \pi_E, \sigma_E)}.$$

Now, using the identity in [AC89, Proposition 6.9], we have

$$L(X^f, \pi_E, \sigma_E) = \prod_\eta L(X, \pi_F, \sigma_F\otimes\eta)$$

and

$$\epsilon(X^f, \pi_E, \sigma_E, \psi_E) = \mathcal{C}_{E/F}(\psi_F)^{n(n-1)}\prod_\eta \epsilon(X, \pi_F, \sigma_F\otimes\eta, \psi_F),$$

where η runs over all the characters of the group $F^\times/\mathrm{Nr}_{E/F}(E^\times)$–which is isomorphic to Γ via the local class field theory. Here $\mathcal{C}_{E/F}(\psi_F)$ is the Langlands constant, defined as in the proof of Lemma 2.5.2, and the proof of the lemma shows that $\mathcal{C}_{E/F}(\psi_F)^2 = 1$. Therefore, we have $\mathcal{C}_{E/F}(\psi_F)^{n(n-1)} = 1$. From the above relations, we get

$$\gamma(X^f, \pi_E, \sigma_E, \psi_E) = \prod_\eta \gamma(X, \pi_F, \sigma_F\otimes\eta, \psi_F). \tag{4.14}$$

Now, using the identity (2.6), we have

$$r_l(\gamma(X^f, \pi_E, \sigma_E, \psi_E)) = \gamma(X^f, J_l(\pi_E), J_l(\sigma_E), \overline{\psi}_E),$$

$$r_l(\gamma(X, \pi_F, \sigma_F\otimes\eta, \psi_E)) = \gamma(X, J_l(\pi_F), \overline{\sigma}_F, \overline{\psi}_F),$$

for each character η of Γ. Now, taking mod-l reduction to the identity (4.14) and using these relations, we get

$$\gamma(X^f, J_l(\pi_E), J_l(\sigma_E), \overline{\psi}_E) = \gamma(X, J_l(\pi_F), \overline{\sigma}_F, \overline{\psi}_F)^l. \tag{4.15}$$

Finally, it follows from Lemma 2.4.5 that

$$\gamma(X^f, J_l(\pi_E), J_l(\sigma_E), \overline{\psi}_E) = \gamma\big((\frac{q_F}{l})X^l, J_l(\pi_F)^{(l)}, \overline{\sigma}_F^{(l)}, \overline{\psi}_F^l\big).$$

The identity (4.12) shows that the space $\mathcal{M}(\pi_F, \psi_F)$ is stable under the action of $G_n(F)$ and the map

$$\Phi_n : \mathcal{M}(\pi_F, \psi_F) \longrightarrow \mathbb{K}(J_l(\pi_F)^{(l)}, \overline{\psi}_F^l)$$

is surjective. Using Proposition 4.3.3, we get that the $G_n(F)$ representation $\widehat{H}^0(\mathbb{W}(J_l(\pi_E), \overline{\psi}_E))$ has a unique generic sub-quotient, which is necessarily equal to $J(\pi_F)^{(l)}$. This completes the proof. $\square$

Now, we deduce some corollaries of Theorem 4.3.6. We keep the same assumptions that E/F is a finite Galois extension of p-adic fields with $[E : F] = l$, where $l \neq p$ and l does not divide $|G_{n-1}(\kappa_F)|$.

Corollary 4.3.7. *Let π_E be an integral generic $\mathcal{K}$-representation of $G_n(E)$ which is absolutely irreducible. Assume that $\pi_E^\gamma \simeq \pi_E$, for all $\gamma \in \Gamma$. Let $\mathbb{W}_\Lambda(\pi_E, \psi_E)$ be the space of all Λ-valued functions in the Whittaker model of π_E. Let π_F be the integral, l-adic generic representation of $G_n(F)$ such that $\pi_E \otimes_{\mathcal{K}} \overline{\mathbb{Q}}_l$ is the base change lifting of π_F. Then, the Frobenius twist $J_l(\pi_F)^{(l)}$ occurs as the unique generic sub–quotient of the zeroth Tate cohomology group $\widehat{H}^0(\mathbb{W}_\Lambda(\pi_E, \psi_E))$.*

Proof. The outline of the proof is the same as Theorem 4.3.6. For completeness, we discuss some crucial steps. As one can observe, the previous and the present theorems are similar in spirit to the local converse theorem for $(n, n-1)$, we precisely use Theorem 4.3.6 at the $(n-1)$ step.

It follows from [Vig04, Theorem 2] and Lemma 2.3.3 that the Λ-lattice $\mathbb{W}_\Lambda(\pi_E, \psi_E)$ is invariant under the action of $G_n(E) \rtimes \Gamma$. Consider the integral Kirillov model $\mathbb{K}_\Lambda(\pi_E, \psi_E)$. Then the restriction map $W \mapsto \mathrm{res}_{P_n(E)}(W)$ is a bijection between $\mathbb{W}_\Lambda(\pi_E, \psi_E)$ and $\mathbb{K}_\Lambda(\pi_E, \psi_E)$. Let Φ_n be the following $P_n(F)$ equivariant map, defined as the composition of restriction to $P_n(F)$ map and (pointwise) mod-l reduction map

$$\Phi_n : \mathbb{K}_\Lambda(\pi_E, \psi_E)^\Gamma \longrightarrow \mathbb{K}(J_l(\pi_F)^{(l)}, \overline{\psi}_F^l).$$

Then Φ_n is non-zero, and it factorizes through the Tate cohomology space $\widehat{H}^0(\mathbb{K}_\Lambda(\pi_E, \psi_E))$. As before, we consider the non-zero space $\Phi_n^{-1}(\mathbb{K}(J_l(\pi_F)^{(l)}, \overline{\psi}_F^l))$ and denote it by $\mathcal{M}(\pi_F, \psi_F)$. To prove the above corollary, it is sufficient to prove that

$$\Phi_n(\overline{\pi_E(w_n)}V) = J_l(\pi_F)^{(l)}(w_n)\Phi_n(V),$$

for all $V \in \mathcal{M}(\pi_F, \psi_F)$. Here, $\overline{\pi_E(w_n)}$ denotes the induced action of $\pi_E(w_n)$ on $\widehat{H}^0(\mathbb{K}_\Lambda(\pi_E, \psi_E))$. Because of Lemma 4.3.5, it is enough to prove the following identity of Laurent series

$$\sum_{r\in\mathbb{Z}} c_r^F\left(\Phi_n(\overline{\pi_E(w_n)}V), W'\right)q_F^{r/2}X^r = \sum_{r\in\mathbb{Z}} c_r^F\left(J_l(\pi_F)^{(l)}(w_n)\Phi_n(V), W'\right)q_F^{r/2}X^r, \tag{4.16}$$

for all $W' \in \mathbb{W}(\overline{\sigma}_F^{(l)}, \overline{\psi}_F^{-l})$ and for all l-modular generic representations $\overline{\sigma}_F$ of $G_{n-1}(F)$. Take such mod-l generic representation $\overline{\sigma}_F$ of $G_{n-1}(F)$. Let σ_F be an l-adic lift of $\overline{\sigma}_F$ and let σ_E be the base change of σ_F to $G_{n-1}(E)$. Theorem 4.3.6 gives a $G_{n-1}(F)$ stable subspace $\mathcal{N}(\sigma_F, \psi_F)$ of the Tate cohomology group $\widehat{H}^0(\mathbb{W}(J_l(\sigma_E), \overline{\psi}_E))$ with the following $G_{n-1}(F)$ equivariant surjection

$$\Phi_{n-1} : \mathcal{N}(\sigma_F, \psi_F) \longrightarrow \mathbb{W}(\overline{\sigma}_F^{(l)}, \overline{\psi}_F^{-l}).$$

Now, lifting W' via Φ_{n-1} and the function V to the respective Γ-invariant Kirillov models and using the identities (4.10), the relation (4.16) is then equivalent to the following identity local γ-factors

$$r_l(\gamma(X^f, \pi_E, \sigma_E, \psi_E)) = \gamma\big((\frac{q_K}{l})X^l, J_l(\pi_F)^{(l)}, \overline{\sigma}_F^{(l)}, \overline{\psi}_F^{l}\big),$$

where f is the residue degree of the extension E/F. This follows from the arguments of the subsection (4.3.3) in the proof of Theorem 4.3.6. □

Corollary 4.3.8. *Let E/F be a finite Galois extension of p-adic fields with $[E : F] = l$, where l and p are distinct odd primes. Let π_F be an integral, l-adic cuspidal representation of $G_n(F)$, where $n \geq 3$. We further assume that l does not divide $|G_{n-1}(\kappa_F)|$. Let π_E be the base change lifting of π_F to $G_n(E)$. Then we have*

$$\widehat{H}^0(r_l(\pi_E)) \simeq r_l(\pi_F)^{(l)}.$$

Proof. Since l does not divide n, the base change lift π_E is cuspidal. As the Kirillov model $\mathbb{K}(r_l(\pi_E), \overline{\psi}_E)$ is equal to $\mathcal{K}(\overline{\psi}_E)$, we get that $\widehat{H}^0(\mathbb{K}(r_l(\pi_E), \overline{\psi}_E))$ is equal to $\mathcal{K}(\overline{\psi}_F^{l})$ (Remark 2.8.3). Thus, the action of $G_n(F)$ on $\widehat{H}^0(\mathbb{K}(r_l(\pi_E), \overline{\psi}_E))$ is irreducible, and the corollary follows from Theorem 4.3.6. □

4.4 Tate cohomology of $Z(\Delta)$

In this section, we study the Tate cohomology of the base change of the Zelevinsky subrepresentations of the form $Z(\Delta)$. In [Zel80], Zelevinsky uses the notation $\langle\Delta\rangle$ for $Z(\Delta)$. In this section, we continue with the assumptions in Corollary 4.3.8, i.e., $l \neq p$ and l does not divide $|G_{n-1}(\mathbb{F}_q)|$ and the integer n. Recall that q is the cardinality of the residue field of F. We will crucially use the fact that $Z(\Delta)$ remains irreducible under the restriction to P_n, and is characterized by this property.

4.4.1 Zelevinsky subrepresentation

Keeping the notations as in Subsection 2.3.3, let $\Delta = \left\{\sigma, \sigma\nu_K, \dots, \sigma\nu_K^{r-1}\right\}$ be a segment, where K is a finite extension of $\mathbb{Q}_p$ and σ is an l-adic cuspidal representation of $G_m(K)$. We denote by $\ell(\Delta)$ the length of Δ, i.e., the integer r. The parabolic induction

$$\sigma \times \sigma\nu_K \times \cdots \sigma\nu_K^{r-1}$$

admits a unique irreducible subrepresentation, denoted by $Z(\Delta)$. Moreover, $Z(\Delta)$ can be characterised as those irreducible representation of $G_{rm}(K)$ that remain irreducible after restricting to $P_{rm}(K)$, and the restriction is isomorphic to $(\Phi^+)^{m-1} \circ \Psi^+(Z(\Delta^-))$, where $\Delta^- = \Delta \setminus \left\{\sigma\nu_K^{r-1}\right\}$. For definitions of the functors $\Phi^\pm$ and $\Psi^\pm$, we refer to [BZ77, Section 3] and [Vig96b, Section 1.2, Chapter III]. For the definition of Zelevinsky sub-representation $Z(\Delta)$ and its restriction to $P_n(K)$, we refer to [Zel80, Section 3].

4.4.2 Base change for $Z(\Delta)$

Let F be a finite extension of $\mathbb{Q}_p$, and let E/F be a finite Galois extension of prime degree l with $l \neq p$. Let Γ be the cyclic group $\mathrm{Gal}(E/F)$ with generator, say γ. Let σ_F and σ_E be the integral l-adic cuspidal representations of $G_m(F)$ and $G_m(E)$ respectively such that σ_E is a base change lifting of σ_F. Consider the segments

$$\Delta_F = \left\{\sigma_F, \sigma_F\nu_F, \dots, \sigma_F\nu_F^{k-1}\right\}$$

$$\Delta_E = \left\{\sigma_E, \sigma_E\nu_E, \dots, \sigma_E\nu_E^{k-1}\right\}.$$

Then we have the irreducible l-adic representations $Z(\Delta_F)$ and $Z(\Delta_E)$ of $G_n(F)$ and $G_n(E)$ respectively, where $n = km$. If we let σ'_F (resp. σ'_E) to be the representation $\sigma_F\nu_F^{k-1}$ (resp.

$\sigma_E\nu_E^{k-1})$, then we have

$$\Pi_F(Z(\Delta_F)) = \Pi_F(\sigma'_F) \oplus \Pi_F(\sigma'_F\nu_F^{-1}) \oplus \cdots \oplus \Pi_F(\sigma_F)$$

and

$$\Pi_E(Z(\Delta_E)) = \Pi_E(\sigma'_E) \oplus \Pi_E(\sigma'_E\nu_E^{-1}) \oplus \cdots \oplus \Pi_F(\sigma_E),$$

where Π_F and Π_E are the l-adic local Langlands correspondences defined as in Subsection 2.6.1. This shows that

$$\mathrm{Res}_{\mathcal{W}_E}\big(\Pi_F(Z(\Delta_F))\big) \simeq \Pi_E(Z(\Delta_E)).$$

Thus, the representation $Z(\Delta_E)$ is the base change lift of $Z(\Delta_F)$.

4.4.3 Integral structure of $Z(\Delta)$ and mod-l reduction

Let $\mathcal{L}_0$ be a $G_m(E)$ invariant lattice in σ_E, and let $S_\gamma : \sigma_E \to \sigma_E^\gamma$ be an isomorphism with $S_\gamma^l = \mathrm{id}$ and $S_\gamma(\mathcal{L}_0) = \mathcal{L}_0$. Recall that the representation $\pi_E := \sigma_E \times \sigma_E\nu_E \times \cdots \times \sigma_E\nu_E^{k-1}$ admits a $G_n(E)$ invariant lattice, say $\mathcal{L}'$, which is induced via $\mathcal{L}_0$. Then $\mathcal{L} = \mathcal{L}' \cap Z(\Delta_E)$ is a $G_n(E)$-invariant lattice in $Z(\Delta_E)$. Now, the map S_γ induces an isomorphism $T_\gamma : Z(\Delta_E) \to Z(\Delta_E)^\gamma$ such that $T_\gamma^l = \mathrm{id}$ and T_γ stabilizes $\mathcal{L}$. Moreover, choosing a $G_m(E)$ invariant, S_γ-stable lattice in σ_E is equivalent to choosing a $G_n(E)$ invariant, T_γ-stable lattice in $Z(\Delta_E)$.

Remark 4.4.1. Since σ_E is cuspidal, the restriction $\mathrm{res}_{P_n(E)}(\sigma_E)$ is isomorphic to the compact induction $\mathcal{K}(\psi_E)$ as $\overline{\mathbb{Q}}_l$-representations. So, the restriction of $\mathcal{L}_0$ to the subgroup $P_m(E)$ is isomorphic to the space of $\overline{\mathbb{Z}}_l$-valued functions in $\mathcal{K}(\psi_E)$. This implies that $\widehat{H}^1(\mathcal{L}_0) = 0$. For details, see [Ron16, Theorem 6]. From this, we now deduce the following result.

Proposition 4.4.2. *Let $\mathcal{L}$ be a $\overline{\mathbb{Z}}_l$-lattice in $Z(\Delta_E)$ that is stable under the action of $G_n(E)$ and the isomorphism T_γ. Then we have $\widehat{H}^1(\mathcal{L}) = 0$.*

Proof. We prove the above claim using induction on $\ell(\Delta_E)$. If the length of Δ_E is 1, then the proposition clearly follows from Remark 4.4.1. Recall that

$$\mathrm{res}_{P_n(E)}(Z(\Delta_E)) \simeq (\Phi^+)^{m-1} \circ \Psi^+(Z(\Delta_E^-)).$$

Here, $Z(\Delta_E)^-$ is identified with the m-th derivative of $Z(\Delta_E)$, which is the composition of the following maps:

$$Z(\Delta_E) \xrightarrow{r_{N_{n-m,m}(E)}} Z(\Delta_E^-) \otimes \sigma_E \xrightarrow{\mathrm{id}\otimes r_{N_m(E),\Theta_E}} Z(\Delta_E^-),$$

where $r_{N_{n-m,m}(E)}$ and $\mathrm{id} \otimes r_{N_m(E),\Theta_E}$ are the natural quotient maps of corresponding Jacquet module and twisted Jacquet module. Note that the maps $r_{N_{n-m,m}(E)}$ and $r_{N_m(E),\Theta_E}$ preserve integral structures (see [Dat05, Proposition 1.4(i)] for $r_{N_{n-m,m}(E)}$, and [Vig04, Theorem III.2] for $r_{N_m(E),\Theta_E}$). Thus, we get that $\mathcal{L}^-$ defined as

$$\mathcal{L}^- = \Psi^- \circ (\Phi^-)^{m-1}(\mathrm{res}_{P_n(E)}(\mathcal{L}))$$

is a lattice in $Z(\Delta_E^-)$. We have the following isomorphism of $P_n(E) \rtimes \Gamma$-modules

$$\mathrm{res}_{P_n(E)}(\mathcal{L}) \simeq (\Phi^+)^{m-1} \circ \Psi^+(\mathcal{L}^-).$$

Applying Proposition 2.8.2 repeatedly $(m-1)$-times, we get that

$$\widehat{H}^1(\mathrm{res}_{P_n(E)}(\mathcal{L})) \simeq (\Phi^+)^{m-1} \circ \Psi^+\big(\widehat{H}^1(\mathcal{L}^-)\big). \tag{4.17}$$

When k is 2, we have $\Delta_E = \{\sigma_E, \sigma_E\nu_E\}$ and in this case, the representation $Z(\Delta_E^-)$ is equal to σ_E and $\mathcal{L}^-$ is a $G_m(E) \rtimes \Gamma$-stable lattice in σ_E. Then, it follows from Remark 4.4.1 and the isomorphism (4.17) that $\widehat{H}^1(\mathrm{res}_{P_n(E)}(\mathcal{L})) = 0$. Suppose the result is true for all $Z(\Delta)$'s where the length of Δ is strictly less than k. Recall that $\ell(\Delta_E^-) = k-1$. By induction hypothesis, we get that $\widehat{H}^1(\mathcal{L}^-) = 0$. Then, using the isomorphism (4.17), we have $\widehat{H}^1(\mathcal{L}) = 0$. □

We now recall the mod-l reduction of the integral representation $Z(\Delta_F)$. Let us introduce the following notations :

$$r_l(\Delta_F) = \{r_l(\sigma_F), r_l(\sigma_F)\overline{\nu}_F, \dots, r_l(\sigma_F)\overline{\nu}_F^{k-1}\}$$

and

$$r_l(\Delta_F)^{(l)} = \{r_l(\sigma_F)^{(l)}, (r_l(\sigma_F)\overline{\nu}_F)^{(l)}, \dots, (r_l(\sigma_F)\overline{\nu}_F^{k-1})^{(l)}\},$$

where $r_l(\sigma_F)$ is the mod-l reduction of σ_F and $r_l(\sigma_F)^{(l)}$ is the Frobenius twist of $r_l(\sigma_F)$. Then the mod-l reduction of $Z(\Delta_F)$ ([MS14, Theorem 9.39]) is given by

$$r_l(Z(\Delta_F)) = Z(r_l(\Delta_F)).$$

This shows, in particular, that $r_l(Z(\Delta_F))$ is irreducible. Moreover, the Frobenius twist of the representation $r_l(Z(\Delta_F))$ equals $Z(r_l(\Delta_F)^{(l)})$. We end this section with the following theorem.

Theorem 4.4.3. *Let E/F be a finite Galois extension with $[E:F] = l$, where l and p are distinct primes such that l does not divide $|G_{n-1}(\kappa_F)|$. Let σ_F be an integral, l-adic cuspidal representation of $G_m(F)$, and let σ_E be an integral, l-adic representation of $G_m(E)$ obtained as a base change lift of σ_F (Note that σ_E is also cuspidal). Let $\Delta_F = \{\sigma_F, \sigma_F\nu_F, \dots, \sigma_F\nu_F^{k-1}\}$ and $\Delta_E = \{\sigma_E, \sigma_E\nu_E, \dots, \sigma_E\nu_E^{k-1}\}$ be two segments (Here $n = km$). Then we have*

$$\widehat{H}^0\big(r_l(Z(\Delta_E))\big) \simeq r_l(Z(\Delta_F))^{(l)}.$$

Proof. We use induction on $\ell(\Delta_E)$. For $k = 1$, we have $Z(\Delta_E) = \sigma_E$ and $Z(\Delta_F) = \sigma_F$, and the theorem follows from Corollary 4.3.8. Suppose the result is true for all segments Δ'_F and Δ'_E with $\ell(\Delta'_F) = \ell(\Delta'_E) < k$. Let τ_F and τ_E be the mod-l Zelevinsky representations $Z(r_l(\Delta_F))$ and $Z(r_l(\Delta_E))$, respectively. We denote by τ_F^- and τ_E^- the representations $Z(r_l(\Delta_F^-))$ and $Z(r_l(\Delta_E^-))$ respectively. Since $\mathrm{res}_{P_n(E)}(\tau_E)$ is isomorphic to $(\Phi^+)^{m-1} \circ \Psi^+(\tau_E^-)$, it follows from Proposition 2.8.2 that

$$\widehat{H}^0(\mathrm{res}_{P_n(E)}(\tau_E)) \simeq (\Phi^+)^{m-1} \circ \Psi^+(\widehat{H}^0(\tau_E^-)). \tag{4.18}$$

By induction hypothesis, we have

$$\widehat{H}^0(\tau_E^-) \simeq (\tau_F^-)^{(l)}.$$

Thus, it follows from (4.18) and [Vig96b, Chapter 3, 1.5] that the Tate cohomology group $\widehat{H}^0(\tau_E)$ is an irreducible as a representation of $P_n(F)$ and hence of $G_n(F)$. Let $\lambda = (m, m, \ldots, m)$ be the partition of n and let $P_\lambda = M_\lambda N_\lambda$ be the parabolic subgroup of G_n. The isomorphism implies that $\widehat{H}^0(\tau_E)_{N_\lambda(F)}$ is non-zero. Then, using Lemma 4.3.2 and Corollary 4.3.8, we get the following isomorphism of $M_\lambda(F)$ representations

$$\widehat{H}^0(\tau_E)_{N_\lambda(F)} \simeq ((\tau_F^-)^{(l)})_{N_\lambda(F)}.$$

This isomorphism and the irreducibility of $\widehat{H}^0(\tau_E)$ together implies that $\widehat{H}^0(\tau_E)$ is isomorphic to $\tau_F^{(l)}$, as a representation of $G_n(F)$ (see [Vig98, Proposition V.9.1]). This completes the proof. □

Remark 4.4.4. Additionally, we now give a proof of the fact that $\widehat{H}^0(\tau_E)$ is a sub–representation of $r_l(\sigma_F)^{(l)} \times \cdots \times r_l(\sigma_F)^{(l)} \overline{\nu}_F^{l(k-1)}$. We again use induction on the length of Δ_E and Δ_F. For $k = 1$, the result follows from Theorem 4.3.8. Assume that the result is true for all $Z(\Delta)$'s with $\ell(\Delta) < k$. By induction hypothesis, we have

$$\widehat{H}^0(\tau_E^-) \hookrightarrow r_l(\sigma_F)^{(l)} \times \cdots \times (r_l(\sigma_F)\overline{\nu}_F^{k-2})^{(l)}. \tag{4.19}$$

Applying the composition functor $(\Phi^+)^{m-1} \circ \Psi^+$ to (4.19), and then using Proposition 2.8.2, we get

$$\mathrm{res}_{P_n(F)}(\widehat{H}^0(\tau_E)) \hookrightarrow \mathrm{res}_{P_n(F)}\big(r_l(\sigma_F)^{(l)} \times \cdots \times (r_l(\sigma_F)\overline{\nu}_F^{k-1})^{(l)}\big)$$

Now, the irreducibility of the Tate cohomology group $\widehat{H}^0(\tau_E)$ shows that

$$\widehat{H}^0(\tau_E) \hookrightarrow r_l(\sigma_F)^{(l)} \times \cdots \times (r_l(\sigma_F)\overline{\nu}_F^{k-1})^{(l)}. \tag{4.20}$$

Moreover, the relation (4.20) directly concludes the above theorem under the assumption that l does not divide $|G_n(\kappa_F)|$.

4.5 Irreducibility of Tate Cohomology of generic representations

In this section, we discuss the Tate cohomology of representations of the form $L(\Delta)$, where $L(\Delta)$ is defined in Subsection 2.3.3. We assume that l does not divide the pro-order of $G_n(F)$. It is important for the induction step in the proof of the main theorem. We continue with the notations that σ_F is an integral l-adic cuspidal representation of $G_m(F)$ and σ_E is the base change lifting of σ_F to $G_m(E)$, where $n = km$.

4.5.1 Integral structure of $L(\Delta)$

Keep the notations as in Subsection 4.4.2. Recall that the representation $L(\Delta_E)$ is the unique generic quotient of the parabolically induced representation $\sigma_E \times \sigma_E\nu_E \times \cdots \times \sigma_E\nu_E^{k-1}$. Now fix a $G_m(E)$ invariant lattice $\mathcal{L}_0$ in σ_E. Then, we have the $G_n(E)$ invariant lattice $\mathcal{L}_0 \times \cdots \times \mathcal{L}_0$ in $\sigma_E \times \cdots \times \sigma_E\nu_E^{k-1}$. The image of $\mathcal{L}_0 \times \cdots \times \mathcal{L}_0$ under the surjection

$$\sigma_E \times \sigma_E\nu_E \times \cdots \times \sigma_E\nu_E^{k-1} \longrightarrow L(\Delta_E),$$

say $\mathcal{L}$, is again a $G_n(E)$ invariant lattice in $L(\Delta_E)$ that is stable under action of $G_n(E)$. As in Subsection 4.4.3, an isomorphism between σ_E and σ_E^γ induces an isomorphism $T_\gamma : L(\Delta_E) \to L(\Delta_E)^\gamma$ with $T_\gamma^l = \mathrm{id}$ and $T_\gamma(\mathcal{L}) = \mathcal{L}$. Here, the group Γ acts on $\mathcal{L}$ via the isomorphism T_γ.

Proposition 4.5.1. *Let $\mathcal{L}$ be a $\overline{\mathbb{Z}}_l$-lattice in $L(\Delta_E)$ that is stable under the action of $G_n(E)$ and the operator T_γ. Then we have $\widehat{H}^1(\mathcal{L}) = 0$.*

Proof. We proceed by induction on $\ell(\Delta_E)$, which equals k. When $\ell(\Delta_E) = 1$, then $L(\Delta_E) = \sigma_E$, and in this case the proposition follows from [Ron16, Theorem 6]. Suppose the result is true for all representations $L(\Delta)$, where $\ell(\Delta)$ is strictly less than k. Let $\tau = \mathrm{res}_{P_n(E)}(L(\Delta_E))$. Consider the filtration of $P_n(E)$-representations:

$$(0) \subseteq \tau_n \subseteq \cdots \subseteq \tau_2 \subseteq \tau_1 = \tau,$$

where $\tau_i/\tau_{i+1} = (\Phi^+)^{i-1} \circ \Psi^+(\tau^{(i)})$ and $\tau^{(i)}$ is the i-th derivative of τ. According to [Zel80, Proposition 9.6], we have

$$\tau^{(j)} = 0, \text{ if } j \text{ is not divisible by } m, \text{ and}$$

$$\tau^{(rm)} = L\big(\{\sigma_E\nu_E^r, \dots, \sigma_E\nu_E^{k-1}\}\big), \text{ for } r = 0, 1, \dots, k-1.$$

For $1 \leq r \leq k-1$, let Δ'_E and Δ''_E be the segments $\{\sigma_E \nu_E^r, \dots, \sigma_E \nu_E^{k-1}\}$ and $\{\sigma_E, \dots, \sigma_E \nu_E^{r-1}\}$ respectively. The rm-th derivative of τ is the composition of the following maps:

$$\tau \xrightarrow{r_{N_{n-rm,rm}(E)}} L(\Delta'_E) \otimes L(\Delta''_E) \xrightarrow{\mathrm{id} \otimes r_{N_{rm}(E),\Theta_E}} \tau^{(rm)},$$

where $r_{N_{n-rm,rm}(E)}$ and $\mathrm{id} \otimes r_{N_{rm}(E),\Theta_E}$ are the quotient maps of corresponding Jacquet module and twisted Jacquet module. Note that the maps $r_{N_{n-rm,rm}(E)}$ and $r_{N_{rm}(E),\Theta_E}$ preserve integral structures (see [Dat05, Proposition 1.4(i)] and [Vig04, Theorem III.2]). Thus, we get that $\mathcal{L}^{(rm)}$, defined as

$$\mathcal{L}^{(rm)} = \Psi^- \circ (\Phi^-)^{rm-1}(\mathrm{res}_{P_n(E)}(\mathcal{L})),$$

is a $G_{n-rm}(E) \rtimes \Gamma$-stable lattice in $\tau^{(rm)}$. Let $\mathcal{L}_i \subset \tau_i$ be the $P_n(E) \rtimes \Gamma$-invariant $\overline{\mathbb{Z}}_l$-lattice

$$\mathcal{L}_i = (\Phi^+)^{i-1} \circ (\Phi^-)^{i-1}(\mathrm{res}_{P_n(E)}(\mathcal{L})),$$

for all $1 \leq i \leq n$. For $1 \leq r \leq k-1$, we have the short exact sequence of $P_n(E) \rtimes \Gamma$-modules

$$0 \longrightarrow \mathcal{L}_{(r+1)m} \longrightarrow \mathcal{L}_{rm} \longrightarrow (\Phi^+)^{rm-1} \circ \Psi^+(\mathcal{L}^{(rm)}) \longrightarrow 0 \tag{4.21}$$

By induction hypothesis, we have $\widehat{H}^1(\mathcal{L}^{(rm)}) = 0$. Then the long exact sequence of Tate cohomology corresponding to (4.21) gives

$$\cdots \longrightarrow \widehat{H}^1(\mathcal{L}_{(r+1)m}) \longrightarrow \widehat{H}^1(\mathcal{L}_{rm}) \longrightarrow 0 \longrightarrow \widehat{H}^0(\mathcal{L}_{(r+1)m}) \longrightarrow \cdots$$

For $s = k-1$, we have the representation τ_n, which equals $\mathrm{ind}_{N_n(E)}^{G_n(E)}(\Theta_E)$. Using Proposition 2.8.2, we have $\widehat{H}^1(\mathcal{L}_n) = 0$. Then, from the above long exact sequence, we get that $\widehat{H}^1(\mathcal{L}_{(k-1)m}) = 0$. Again, using the above long exact sequence for $s = k-2$, we get $\widehat{H}^1(\mathcal{L}_{(k-2)m}) = 0$. Thus, an inductive process gives

$$\widehat{H}^1(\mathcal{L}) = \widehat{H}^1(\mathcal{L}_m) = 0.$$

This completes the proof. □

4.5.2 Tate cohomology of $L(\Delta)$

Let π_E be an integral l-adic generic representation of $G_n(E)$. Then π_E is of the form

$$\mathcal{L}(\Delta_1) \times \mathcal{L}(\Delta_2) \times \cdots \times \mathcal{L}(\Delta_t),$$

where for each $1 \leq j \leq t$, the representation $\mathcal{L}(\Delta_j)$ is integral. Let $\mathcal{L}_j$ be a lattice in $L(\Delta_j)$, defined as in Subsection 4.5.1. Let $T_{\gamma,j}$ be the isomorphism between $L(\Delta_j)$ and $L(\Delta_j)^\gamma$ such that

$T_{\gamma,j}(\mathcal{L}_j) = \mathcal{L}_j$. Now consider the $\overline{\mathbb{Z}}_l$-module $\mathcal{L} = \mathcal{L}_1 \times \cdots \times \mathcal{L}_t$. Then $\mathcal{L}$ is a lattice in π_E that is stable under the action of $G_n(E)$. Moreover, we have an isomorphism $T_\gamma : \pi_E \to \pi_E^\gamma$, induced by $\{T_{\gamma,j}\}_{j=1}^t$, such that $T_\gamma(\mathcal{L}) = \mathcal{L}$. Note that the prime l is banal for $G_n(E)$. Since the mod-l reduction of π_E is irreducible, any lattice in π_E is homothetic to $\mathcal{L}$.

Corollary 4.5.2. *Assume that l does not divide $|G_n(\kappa_F)|$. Let π_E be an integral, l-adic generic representation of $G_n(E)$ as above. Let $\mathcal{L}$ be a $\overline{\mathbb{Z}}_l$-lattice in π_E that is stable under the action of $G_n(E)$ and T_γ. Then $\widehat{H}^1(\mathcal{L}) = 0$.*

Proof. Using Proposition 2.8.2, we have

$$\widehat{H}^1(\mathcal{L}) = \widehat{H}^1(\mathcal{L}_1) \times \cdots \times \widehat{H}^1(\mathcal{L}_t).$$

Now, applying Proposition 4.5.1, we get that $\widehat{H}^1(\mathcal{L}_i) = 0$ for each i. Hence, the theorem. $\square$

Next, we aim to prove the following

Theorem 4.5.3. *Let E/F be a finite Galois extension with $[E : F] = l$, where l and p are distinct primes such that l does not divide $|G_n(\kappa_F)|$. Let σ_F be an integral, l-adic cuspidal representation of $G_m(F)$, and let σ_E be an integral, l-adic cuspidal representation of $G_m(E)$ obtained as a base change lifting of σ_F. Let $\Delta_F = \{\sigma_F, \sigma_F\nu_F, \dots, \sigma_F\nu_F^{k-1}\}$ and $\Delta_E = \{\sigma_E, \sigma_E\nu_E, \dots, \sigma_E\nu_E^{k-1}\}$ be two segments (Here $n = km$). Then*

$$\widehat{H}^0(r_l(L(\Delta_E))) \simeq r_l(L(\Delta_F))^{(l)}.$$

Proof. We prove the theorem using induction on $\ell(\Delta_F)$. Since l does not divide $|G_n(\kappa_F)|$, the mod-l reductions of the irreducible integral representations $L(\Delta_F)$ and $L(\Delta_E)$ are also irreducible, and we have

$$r_l(L(\Delta_F)) = L(r_l(\Delta_F))$$

and

$$r_l(L(\Delta_E)) = L(r_l(\Delta_E))$$

where $r_l(\Delta)$ is defined as in Subsection 4.4.3. Using the same short exact sequence (4.21) for $r_l(L(\Delta_E))$, the corresponding long exact sequence in Tate cohomology gives a filtration

$$\mathrm{res}_{P_n(F)}\widehat{H}^0(r_l(L(\Delta_E))) = \eta_1 \supseteq \eta_2 \supseteq \cdots \supseteq \eta_n,$$

such that the quotient $\eta_i/\eta_{i+1} \neq 0$ if and only if i is a multiple of m. By induction hypothesis, we get that $\eta_{ms}/\eta_{m(s+1)}$ is an irreducible representation of $P_n(F)$. Theorem 4.3.6 says that the Frobenius twist $r_l(L(\Delta_F))^{(l)}$ is the generic sub-quotient of $\widehat{H}^0(r_l(L(\Delta_E)))$. Since the lengths of

$P_n(F)$-representations $\widehat{H}^0(r_l(L(\Delta_E)))$ and $r_l(L(\Delta_F))$ are the same, we get that $r_l(L(\Delta_F))^{(l)}$ is isomorphic to $\widehat{H}^0(r_l(L(\Delta_E)))$. □

Let us continue with the hypothesis as in Theorem 4.5.3. Let π be an integral l-adic generic smooth representation of $G_n(E)$. Since l does not divide $|G_n(\kappa_F)|$, the mod-l-reduction $r_l(\pi)$ is irreducible and hence generic. Then, we have the following immediate corollary.

Corollary 4.5.4. *Let E/F be a finite Galois extension of degree l, where l and p are distinct primes and l does not divide $|G_n(\kappa_F)|$. Let π_F be an integral, l-adic generic representation of $G_n(F)$, and let π_E be the base change lifting of π_F to $G_n(E)$. Then*

$$\widehat{H}^0(r_l(\pi_E)) \simeq r_l(\pi_F)^{(l)}.$$

Proof. This follows from Proposition 2.8.2 and Theorem 4.5.3. □

Chapter 5

Compatibility of Tate cohomology with local base change for cuspidal representations of GL_3

5.1 Introduction

The main result of the previous chapter (i.e., Theorem 4.3.6) assumed that l does not divide the pro-order of $\mathrm{GL}_{n-1}(F)$, for $n \geq 3$. The main objective of this chapter is to prove the theorem without the hypothesis on l. The proof of Theorem 4.3.6 was in the spirit of the proof of local converse theorem over $\overline{\mathbb{F}}_l$, and the hypothesis on l is required in the proof of a vanishing result (Lemma 4.3.5), which in general fails for arbitrary prime l. This vanishing result is essential for proving the local converse theorem over $\overline{\mathbb{F}}_l$. The hypothesis on l may be removed using local γ-factors over Artin local $\overline{\mathbb{F}}_l$-algebras as defined in the work of Matringe–Moss ([MM22]). In this chapter, using the local converse theorem over Artin local $\overline{\mathbb{F}}_l$-algebras (see Theorem 5.2.4 for the precise statement), we prove Theorem 4.3.6 for the base change liftings of l-adic cuspidal representations of $\mathrm{GL}_3(F)$.

5.2 Rankin–Selberg gamma factors in families

In this part, we recall the Rankin–Selberg theory over arbitrary Noetherian $W(\overline{\mathbb{F}}_l)$ for smooth representations of $\mathrm{GL}_3(K) \times \mathrm{GL}_1(K)$, where K is a p-adic field and $W(\overline{\mathbb{F}}_l)$ is the ring of Witt vectors of $\overline{\mathbb{F}}_l$ (see Subsection 2.4.1 for general $\mathrm{GL}_n(K)$). We start by recalling the definition of co-Whittaker modules. Then we discuss the notion of nilpotent lifts for l-modular generic representations of $\mathrm{GL}_n(K)$. For precise references, we refer to [HM18] and [Mos21]. We also put up some initial results on local γ-factors over Noetherian $W(\overline{\mathbb{F}}_l)$-algebras, which are essential in the proof of the main theorem.

5.2.1 Co-Whittaker modules

We fix a non-trivial additive character $\psi_K : K \to W(\overline{\mathbb{F}}_l)^\times$. Let R be an arbitrary Noetherian $W(\overline{\mathbb{F}}_l)$-algebra, and let (π, V) be a smooth $R[G_n(K)]$ module. The space of co-invariants of V with respect to the character ψ_K is denoted by $V^{(n)}$ (see Section 2.3).

Definition 5.2.1. A smooth admissible $R[G_n(K)]$ module (π, V) is called *co-Whittaker* if

1. $V^{(n)}$ is a free R-module of rank one,
2. for any quotient W of V with $W^{(n)} = 0$, we have $W = 0$.

In particular, a co-Whittaker module (π, V) is also of Whittaker type (see Section 2.3 for the definition) and hence admits a Whittaker model. It follows from [Hel16b, Proposition 6.2] that (π, V) admits a central character, denoted by $\omega_\pi : F^\times \to R^\times$.

5.2.2 Nilpotent lifts of Generic representations

Let R be an Artin local $\overline{\mathbb{F}}_l$-algebra. By definition, R is a finite dimensional local $\overline{\mathbb{F}}_l$-algebra with unique non-zero prime ideal, say $\mathfrak{m}_R$, such that the quotient $R/\mathfrak{m}_R$ is isomorphic to $\overline{\mathbb{F}}_l$ and the composition $\overline{\mathbb{F}}_l \hookrightarrow R \to \overline{\mathbb{F}}_l$ is the identity map.

Definition 5.2.2. Given an l-modular generic representation π of $G_n(K)$. A *nilpotent lift* of π to an Artin local $\overline{\mathbb{F}}_l$-algebra R is a smooth admissible $R[G_n(K)]$ submodule $\widetilde{\pi} \subseteq \mathrm{Ind}_{N_n(K)}^{G_n(K)}(\Theta_{K,R})$, such that

$$\widetilde{\pi} \otimes_R \overline{\mathbb{F}}_l \simeq \mathbb{W}(\pi, \overline{\psi}_K).$$

For an Artin local $\overline{\mathbb{F}}_l$-algebra R, we denote by $\mathcal{A}_R^{nil}(n, K)$ the set of isomorphism classes of all nilpotent lifts of objects in $\mathcal{A}_{\overline{\mathbb{F}}_l}^{gen}(n, K)$. For $n = 1$, the objects in $\mathcal{A}_{\overline{\mathbb{F}}_l}^{gen}(1, K)$ are the $\overline{\mathbb{F}}_l$-valued characters of $K^\times$, and the nilpotent lift of one such character $\overline{\chi}$ to an Artin local $\overline{\mathbb{F}}_l$-algebra R is given by a free R-module $\mathcal{M}$ on which $K^\times$ acts via a character $\chi : K^\times \to R^\times$ such that the reduction of χ modulo the maximal ideal $\mathfrak{m}_R$ is equal to $\overline{\chi}$. An Artin local $\overline{\mathbb{F}}_l$-algebra R is an l-adically seperated (i.e., $\bigcap_{i \geq 0} l^i R$ is zero ideal) Noetherian $W(\overline{\mathbb{F}}_l)$-algebra. So, the theory of co-Whittaker modules and local γ-factors is also valid over R. It follows from the work of [EH14, Lemma 6.3.2, Proposition 6.3.4] that a nilpotent lift $\widetilde{\pi} \in \mathcal{A}_R^{nil}(n, K)$ is a co-Whittaker $R[G_n(K)]$ module.

5.2.3 Rankin–Selberg functional equation for $\mathrm{GL}_3 \times \mathrm{GL}_1$

In this part, we recall the Rankin–Selberg theory over arbitrary Noetherian $W(\overline{\mathbb{F}}_l)$-algebra for $\mathrm{GL}_3 \times \mathrm{GL}_1$. We follow the notations of the subsection 2.4.1. Let A and B be two Noetherian $W(\overline{\mathbb{F}}_l)$-algebras, and let $R = A \otimes_{W(\overline{\mathbb{F}}_l)} B$. Let π be a co-Whittaker $A[G_3(K)]$ module, and let χ be a B-valued character of $K^\times$. For $W \in \mathbb{W}(\pi, \psi_K)$, the integrals

$$c_r^K(W, \chi; 0) = \int_{\varpi_K^r \mathfrak{o}_K^\times} W \begin{pmatrix} g & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes \chi(g)\, dg$$

and

$$c_r^K(W, \chi; 1) = \int_K \int_{\varpi_K^r \mathfrak{o}_K^\times} W \begin{pmatrix} g & 0 & 0 \\ x & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes \chi(g)\, dg\, dx$$

are well defined. For each $0 \le j \le 1$, the following Laurent series

$$\sum_{r \in \mathbb{Z}} c_r^K(W, \chi; j) X^r$$

lies in $S^{-1}(R[X, X^{-1}])$ ([Mos16, Theorem 3.2]), where S is the multiplicative subset of the ring $R[X, X^{-1}]$ consisting of all polynomials whose leading and trailing coefficients are units. For any function W on $G_3(K)$, let $\widetilde{W}$ be the function on $G_3(K)$, defined by $\widetilde{W}(g) = W(w_3(g^{-1})^t)$, for $g \in G_3(K)$. The following result is due to [Mos16, Theorem 5.3]–which gives the existence of γ-factors for the pair (π, χ).

Theorem 5.2.3. *Let π be a co-Whittaker $A[G_3(K)]$ module and let η be a B-valued character of $K^\times$. Then, there exists an unique element $\gamma_R(X, \pi, \eta, \psi_K) \in S^{-1}(R[X, X^{-1}])$ such that*

$$\sum_{r \in \mathbb{Z}} c_r^K(w_{3,1}\widetilde{W}, \eta^{-1}; 1) q_K^r X^{-r} = \gamma_R(X, \pi, \eta, \psi_K) \sum_{r \in \mathbb{Z}} c_r^K(W, \eta; 0) X^r \tag{5.1}$$

and

$$\sum_{r \in \mathbb{Z}} c_r^K(w_{3,1}\widetilde{W}, \eta^{-1}; 0) q_K^r X^{-r} = \gamma_R(X, \pi, \eta, \psi_K) \sum_{r \in \mathbb{Z}} c_r^K(W, \eta; 1) X^r, \tag{5.2}$$

for all $W \in \mathbb{W}(\pi, \psi_K)$.

We now state the local converse theorem over Artin local $\overline{\mathbb{F}}_l$-algebras. Over algebraically closed fields of characteristic zero, the converse theorem was proved by several authors (see [Hen93], [JPPS79]). In general, the local converse theorem fails over the field $\overline{\mathbb{F}}_l$. To be more precise, say π_1 and π_2 are two l-modular generic representations for $G_n(K)$. Suppose we have the following equality of γ-factors:

$$\gamma(X,\pi_1,\tau,\overline{\psi}_K) = \gamma(X,\pi_2,\tau,\overline{\psi}_K),$$

for all $\tau \in \mathcal{A}^{gen}_{\overline{\mathbb{F}}_l}(t,K)$ with $1 \leq t \leq n-1$, then it may happen that π_1 is not isomorphic to π_2 (see [Mos21, Section 2]). In [Mos21, Theorem 1.2], the author proved that the local converse theorem over $\overline{\mathbb{F}}_l$ can be recovered if we vary the representations τ over $\mathcal{A}^{nil}_R(t,K)$, for all Artin local $\overline{\mathbb{F}}_l$-algebras R and all $1 \leq t \leq [\frac{n}{2}]$. We now state the theorem in our context.

Theorem 5.2.4. *Let π_1 and π_2 be two l-modular generic representations of $G_3(K)$. Suppose*

$$\gamma_R(X,\pi_1,\eta,\overline{\psi}_K) = \gamma_R(X,\pi_2,\eta,\overline{\psi}_K),$$

for all nilpotent lifts $\eta \in \mathcal{A}^{nil}_R(1,K)$ and for all Artin local $\overline{\mathbb{F}}_l$-algebras R. Then π_1 is isomorphic to π_2.

We end this section with the following lemma, which is crucial for the arguments of the main result. It is analogous to Lemma 2.4.5 in Chapter 2, and the proof is exactly similar.

Lemma 5.2.5. *Let (π,V) be an l-modular generic representation of $G_3(K)$. Let R be an Artin local $\overline{\mathbb{F}}_l$-algebra, and let η be an R-valued character of $K^\times$. Then we have*

$$\gamma_R(X,\pi,\eta,\overline{\psi}_K)^l = \gamma_R(X^l,\pi^{(l)},\eta^l,\overline{\psi}^l_K).$$

Proof. Let $\mathcal{W}_\pi$ be a Whittaker functional on (π,V) and let $\mathcal{W}_{\pi^{(l)}}$ be the composite map

$$V \xrightarrow{\mathcal{W}_\pi} \overline{\mathbb{F}}_l \xrightarrow{x\mapsto x^l} \overline{\mathbb{F}}_l.$$

Then $\mathcal{W}_{\pi^{(l)}}$ is a Whittaker functional on $\pi^{(l)}$ with respect to the character $\overline{\psi}^l_K$, and the Whittaker model $\mathbb{W}(\pi^{(l)},\overline{\psi}^l_K)$ consists of the functions W^l_v, where W_v varies in $\mathbb{W}(\pi,\overline{\psi}_K)$. Using functional equations (5.1) and (5.2), we have

$$\sum_{r\in\mathbb{Z}} c^K_r(\widetilde{W}_v,\eta^{-1};1-j)^l q_K^{lr} X^{-lr} = \gamma_R(X,\pi,\eta,\overline{\psi}_K)^l \sum_{r\in\mathbb{Z}} c^K_r(W_v,\eta;j)^l X^{lr} \tag{5.3}$$

and

$$\sum_{r\in\mathbb{Z}} c^K_r(\widetilde{W^l_v},\eta^{-l};1-j) q_K^{r} X^{-r} = \gamma_R(X,\pi^{(l)},\eta^l,\overline{\psi}^l_K) \sum_{r\in\mathbb{Z}} c^K_r(W^l_v,\eta^l;j) X^{r}, \tag{5.4}$$

for each $j \in \{0,1\}$. Replacing X by X^l to (5.4) and comparing the resulting relation with the identity (5.3), we get

$$\gamma_R(X,\pi,\eta,\overline{\psi}_K)^l = \gamma_R(X^l,\pi^{(l)},\eta^l,\overline{\psi}^l_K).$$

Hence, the lemma. $\square$

5.3 Characterization of mod-l local Langlands correspondence via nilpotent lifts

Recall that the mod-l local Langlands correspondence is a sequence of bijections

$$\left\{\overline{\Pi}_{K,n} : \mathcal{A}^{gen}_{\overline{\mathbb{F}}_l}(n,K) \to \mathcal{G}^{ss}_{\overline{\mathbb{F}}_l}(n,K)\right\}_{n\geq 1}$$

and it is uniquely characterized by its compatibility with l-adic local Langlands correspondence under reduction mod-l. Moss in his work [Mos21, Theorem 1.4] gave a characterization of the sequence $\{\overline{\Pi}_{K,n}\}_{n\geq 1}$ directly, by extending $\{\overline{\Pi}_{K,n}\}_{n\geq 1}$ to a correspondence on nilpotent lifts. To state it more precisely, we need to recall the notion of nilpotent lifts for mod-l Weil–Deligne representations.

5.3.1 Nilpotent lifts of Weil–Deligne representations

Let R be an Artin local $\overline{\mathbb{F}}_l$-algebra. Let ρ be a smooth n-dimensional semisimple l-modular representation of $\mathcal{W}_K$. A *nilpotent lift* of ρ to R is a smooth $R[\mathcal{W}_K]$-module $\widetilde{\rho}$ such that $\widetilde{\rho}$ is free R-module of rank n, and there is an isomorphism

$$\widetilde{\rho}\otimes_R \overline{\mathbb{F}}_l \simeq \rho.$$

The set of all isomorphism classes of nilpotent lifts of objects in $\mathcal{G}^{ss}_{\overline{\mathbb{F}}_l}(n,K)$ to the Artin local $\overline{\mathbb{F}}_l$-algebra R is denoted by $\mathcal{G}_R(n,K)$. For $n=1$, the objects in $\mathcal{G}^{ss}_{\overline{\mathbb{F}}_l}(1,K)$ are precisely the $\overline{\mathbb{F}}_l$-valued characters of $\mathcal{W}_K$. Say η is a mod-l character of $\mathcal{W}_K$. A nilpotent lift of η to R is a character $\widetilde{\eta} : \mathcal{W}_K \to R^\times$ such that the mod-l reduction $\widetilde{\eta}\otimes_R \overline{\mathbb{F}}_l$ is equal to η.

5.3.2 Local Langlands correspondence on nilpotent lifts

Let R be an Artin local $\overline{\mathbb{F}}_l$-algebra. In particular, R is a Noetherian $W(\overline{\mathbb{F}}_l)$-algebra. The seminal work of [HM18, Section 7] gives the existence of a sequence of maps

$$\Pi^R_{K,n} : \mathcal{G}_R(n,K) \longrightarrow \mathcal{A}^{nil}_R(n,K).$$

For $n=1$, the map $\Pi^R_{K,1}$ is given by the local class field theory, and is a bijection. We now recall the characterization of the sequence of bijections $\{\overline{\Pi}_{K,n}\}_{n\geq 1}$ (see Subsection 2.6.2) via the sequence of maps $\{\Pi^R_{K,n}\}_{n\geq 1}$.

Theorem 5.3.1. *For each $n\geq 1$, there is a unique bijection*

$$\overline{\Pi}_{K,n} : \mathcal{G}^{ss}_{\overline{\mathbb{F}}_l}(n,K) \longrightarrow \mathcal{A}^{gen}_{\overline{\mathbb{F}}_l}(n,K)$$

such that, for every Artin local $\overline{\mathbb{F}}_l$-algebras R and for all $\rho \in \mathcal{G}_{R,K}(n)$, $\rho' \in \mathcal{G}_{R',K}(m)$ with $m < n$, the following equality holds

$$\gamma\big(X, \Pi^R_{K,n}(\rho), \Pi^{R'}_{K,m}(\rho'), \overline{\psi}_K\big) = \gamma(X, \rho \otimes \rho', \overline{\psi}_K) \tag{5.5}$$

in the ring of Laurent series $(R \otimes_{\overline{\mathbb{F}}_l} R')[[X]][X^{-1}]$.

5.4 Tate cohomology of cuspidal representations of GL_3

Let F be a finite extension of $\mathbb{Q}_p$, and let E be a finite Galois extension of F of degree l, where l and p are distinct primes. In this section, we compute the Tate cohomology group of base change liftings of integral l-adic cuspidal representations of $G_3(F)$ to $G_3(E)$. We first discuss an important consequence of the base change liftings of integral l-adic cuspidal representations of $G_n(F)$ to $G_n(E)$.

Let us assume that l does not divide n. Let π_F be an integral l-adic cuspidal representation of $G_n(F)$, and let π_E be the base change lifting of π_F. Note that π_E is also cuspidal. From the definition of base change lifting (see Section 2.7), we have

$$\mathrm{res}_{\mathcal{W}_E}(\Pi_{F,n}(\pi_F)) \simeq \Pi_{E,n}(\pi_E).$$

Note that the l-adic representations $\Pi_{F,n}(\pi_F)$ and $\Pi_{E,n}(\pi_E)$ are both integral. Taking mod-l reduction to the above identity, we get

$$\mathrm{res}_{\mathcal{W}_E}\big(r_l(\Pi_{F,n}(\pi_F))\big) \simeq r_l(\Pi_{E,n}(\pi_E)). \tag{5.6}$$

Let $\{\tau_F^1, \tau_F^2, \ldots, \tau_F^r\}$ and $\{\tau_E^1, \tau_E^2, \ldots, \tau_E^s\}$ be the supercuspidal supports of $r_l(\pi_F)$ and $r_l(\pi_E)$ respectively, where $r, s \in \{1, 2, \ldots, n\}$. Let ρ_F^i (resp. ρ_E^j) be the n-dimensional semisimple l-modular representation of $\mathcal{W}_F$ (resp. $\mathcal{W}_E$) that corresponds to τ_F^i (resp. τ_F^j) under the mod-l local Langlands correspondence $\overline{\Pi}_{F,n}$ (resp. $\overline{\Pi}_{E,n}$). Then, it follows from (5.6) and property–(3) of Subsection 2.6.2 that

$$\mathrm{res}_{\mathcal{W}_E}(\rho_F^1 \oplus \cdots \oplus \rho_F^r) \simeq \rho_E^1 \oplus \cdots \oplus \rho_E^s.$$

Thus, we get that the restriction $\mathrm{res}_{\mathcal{W}_E}\big(\overline{\Pi}_{F,n}(r_l(\pi_F))\big)$ is isomorphic to $\overline{\Pi}_{E,n}(r_l(\pi_E))$. Now, we prove our main theorem.

Theorem 5.4.1. *Let F be a finite extension of $\mathbb{Q}_p$, and let E be a finite Galois extension of F with $[E : F] = l$, where p and l are distinct odd primes. Let π_F be an integral l-adic cuspidal representation of $G_3(F)$, and let π_E be the l-adic cuspidal representation of $G_3(E)$, obtained as a* *base change lift of π_F. Then the Frobenius twist $r_l(\pi_F)^{(l)}$ is isomorphic to $\widehat{H}^0(r_l(\pi_E))$.*

Proof. We rely on the local converse theorem over local Artinian $\overline{\mathbb{F}}_l$-algebras (Theorem 5.2.4) to prove the theorem. Say R is an Artin local $\overline{\mathbb{F}}_l$-algebra and let η_F be an element in $\mathcal{A}_R^{\mathrm{nil}}(1,F)$. Then, we show that

$$\gamma_R(X, \widehat{H}^0(r_l(\pi_E)), \eta_F, \overline{\psi}_F^l) = \gamma_R(X, r_l(\pi_F)^{(l)}, \eta_F, \overline{\psi}_F^l). \tag{5.7}$$

Since η_F is co-Whittaker $R[F^\times]$ module, it is enough to prove the above identity for all R-valued characters of $F^\times$. The proof is divided into three parts. In the first part, we show that (5.7) is equivalent to the following identity of γ-factors:

$$\gamma_R(X, \widehat{H}^0(r_l(\pi_E)), \eta_0^l, \overline{\psi}_F^l) = \gamma_R(X, r_l(\pi_F)^{(l)}, \eta_0^l, \overline{\psi}_F^l),$$

where η_0 is a character of $F^\times$, trivial on the l-torsion part of $\kappa_F^\times$. In the second part, using lifting of integrals on homogeneous space over F to that on homogeneous space over E, we obtain the following identity of γ-factors

$$\gamma_R(X, \widehat{H}^0(r_l(\pi_E)), \eta_0^l, \overline{\psi}_F^l) = \gamma_R(X, r_l(\pi_E), \eta_0 \circ \mathrm{Nr}_{E/F}, \overline{\psi}_E).$$

Thus, the assertion (5.7) reduces to prove the following equality:

$$\gamma_R(X, r_l(\pi_E), \eta_0 \circ \mathrm{Nr}_{E/F}, \overline{\psi}_E) = \gamma_R(X, r_l(\pi_F)^{(l)}, \eta_0^l, \overline{\psi}_F^l),$$

In the last part, we prove the above identity using congruence properties of local γ-factors.

Part 1.

Let R be an Artin local $\overline{\mathbb{F}}_l$-algebra and let η_F be an R-valued character of $F^\times$. For $W \in \mathbb{W}(\widehat{H}^0(r_l(\pi_E)), \overline{\psi}_F^l)$ and $U \in \mathbb{W}(r_l(\pi_F)^{(l)}, \overline{\psi}_F^l)$, we have the functional equations

$$\sum_{m\in\mathbb{Z}} c_m^F(\widetilde{W}, \eta_F^{-1}; 1) q_F^m X^{-m} = \gamma_R(X, \widehat{H}^0(r_l(\pi_E)), \eta_F, \overline{\psi}_F^l) \sum_{m\in\mathbb{Z}} c_m^F(W, \eta_F; 0) X^m \tag{5.8}$$

and

$$\sum_{m\in\mathbb{Z}} c_m^F(\widetilde{U}, \eta_F^{-1}; 1) q_F^m X^{-m} = \gamma_R(X, r_l(\pi_F)^{(l)}, \eta_F, \overline{\psi}_F^l) \sum_{m\in\mathbb{Z}} c_m^F(U, \eta_F; 0) X^m. \tag{5.9}$$

Let $k_F^\times[l]$ be the l-torsion part of $k_F^\times$ of order, say l^a, and let Y be the subgroup of $\mathfrak{o}_F^\times$ whose pro-order is coprime to l. Then $\mathfrak{o}_F^\times$ is the direct product of $k_F^\times[l]$ and Y. Let x_0 be a generator of the cyclic group $k_F^\times[l]$. Since $(\eta_F(x_0) - 1)^{l^a} = 0$, the element $\eta_F(x_0) - 1$ belongs to the maximal ideal $\mathfrak{m}_R \subseteq R$. Let u be the element of $\mathfrak{m}_R$ such that $\eta_F(x_0) = 1 + u$. Let η_0 be an R-valued character of $F^\times$ such that η_0 is trivial on the l-torsion part $k_F^\times[l]$ and η_0^l coincides with η_F on the subgroup generated by Y and the uniformizer ϖ_F. Then

$$c_m^F(\widetilde{W}, \eta_F^{-1}; 1) = c_m^F(\widetilde{W}, \eta_0^{-l}; 1) + \sum_{i=1}^{l^a}\sum_{k=1}^{i}(-1)^k\binom{i}{k}\int_F\int_{\varpi_F^m x_0^i Y}\widetilde{W}\begin{pmatrix} g & 0 & 0\\ x & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\otimes\eta_0^l(g^{-1})u^k\,dg\,dx. \tag{5.10}$$

and

$$c_m^F(\widetilde{U}, \eta_F^{-1}; 1) = c_m^F(\widetilde{U}, \eta_0^{-l}; 1) + \sum_{i=1}^{l^a}\sum_{k=1}^{i}(-1)^k\binom{i}{k}\int_F\int_{\varpi_F^m x_0^i Y}\widetilde{U}\begin{pmatrix} g & 0 & 0\\ x & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\otimes\eta_0^l(g)u^k\,dg\,dx. \tag{5.11}$$

Take φ to be the characteristic function of U_F^1. Since the Tate cohomology group $\widehat{H}^0(r_l(\pi_E))$ is isomorphic to $r_l(\pi_F)^{(l)}$, as a representation of $P_3(F)$ (Proposition 3.3.1), there exists $W_0 \in \mathbb{W}(\widehat{H}^0(r_l(\pi_E)), \overline{\psi}_F^l)$ and $U_0 \in \mathbb{W}(r_l(\pi_F)^{(l)}, \overline{\psi}_F^l)$ such that

$$W_0\begin{pmatrix} g & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = U_0\begin{pmatrix} g & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = \varphi(g).$$

From the functional equations (5.8) and (5.9), we get the following relation

$$\begin{aligned}&\sum_{m\in\mathbb{Z}} c_m^F(\widetilde{W_0}, \eta_F^{-1}; 1)q_F^m X^{-m} - \sum_{m\in\mathbb{Z}} c_m^F(\widetilde{U_0}, \eta_F^{-1}; 1)q_F^m X^{-m}\\ &\quad= \big(\gamma_R(X, \widehat{H}^0(r_l(\pi_E)), \eta_F, \overline{\psi}_F^l) - \gamma_R(X, r_l(\pi_F)^{(l)}, \eta_F, \overline{\psi}_F^l)\big)\sum_{m\in\mathbb{Z}}\Big(\int_{\varpi_F^m \mathfrak{o}_F^\times}\varphi(g)\otimes\eta_F(g)\,dg\Big)X^m\end{aligned} \tag{5.12}$$

Again, using functional equations, we have

$$\begin{aligned}&\sum_{m\in\mathbb{Z}} c_m^F(\widetilde{W_0}, \eta_0^{-l}; 1)q_F^m X^{-m} - \sum_{m\in\mathbb{Z}} c_m^F(\widetilde{U_0}, \eta_0^{-l}; 1)q_F^m X^{-m}\\ &\quad= \big(\gamma_R(X, \widehat{H}^0(r_l(\pi_E)), \eta_0^l, \overline{\psi}_F^l) - \gamma_R(X, r_l(\pi_F)^{(l)}, \eta_0^l, \overline{\psi}_F^l)\big)\sum_{m\in\mathbb{Z}}\Big(\int_{\varpi_F^m \mathfrak{o}_F^\times}\varphi(g)\otimes\eta_0^l(g)\,dg\Big)X^m\end{aligned} \tag{5.13}$$

Since the restrictions $\mathrm{res}_{P_3(F)}(W_0)$ and $\mathrm{res}_{P_3(F)}(U_0)$ are compactly supported, then for each $i \in$

$\{1, 2, \ldots, l^a\}$, we have the following relations:

$$\sum_{m\in\mathbb{Z}}\Big(\int_F\int_{\varpi_F^m x_0^i Y}\widetilde{W}_0\begin{pmatrix} g & 0 & 0\\ x & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\otimes\eta_0^l(g^{-1})\,dg\,dx\Big)q_F^m X^{-m}$$
$$=\gamma_R\big(X, \widehat{H}^0(r_l(\pi_E)), \eta_0^l, \overline{\psi}_F^l\big)\sum_{m\in\mathbb{Z}}\Big(\int_{\varpi_F^m x_0^i Y}\varphi(g)\otimes\eta_0^l(g)\,dg\Big)X^m$$

and

$$\sum_{m\in\mathbb{Z}}\Big(\int_F\int_{\varpi_F^m x_0^i Y}\widetilde{U}_0\begin{pmatrix} g & 0 & 0\\ x & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\otimes\eta_0^l(g^{-1})\,dg\,dx\Big)q_F^m X^{-m}$$
$$=\gamma_R(X, r_l(\pi_F)^{(l)}, \eta_0^l, \overline{\psi}_F^l)\sum_{m\in\mathbb{Z}}\Big(\int_{\varpi_F^m x_0^i Y}\varphi(g)\otimes\eta_0^l(g)\,dg\Big)X^m.$$

Using relations (5.10) and (5.11) together with (5.13) and the preceding two equations, it follows from (5.12) that the assertion (5.7) is equivalent to the following identity:

$$\gamma_R\big(X, \widehat{H}^0(r_l(\pi_E)), \eta_0^l, \overline{\psi}_F^l\big) = \gamma_R(X, r_l(\pi_F)^{(l)}, \eta_0^l, \overline{\psi}_F^l).$$

Part 2.

Let $W \in \mathbb{W}(\widehat{H}^0(r_l(\pi_E)), \overline{\psi}_F^l)$. Using functional equation for the pair $(\widehat{H}^0(r_l(\pi_E)), \eta_0^l)$, we get

$$\sum_{m\in\mathbb{Z}} c_m^F(\widetilde{W}, \eta_0^{-l}; 1)q_F^m X^{-m} = \gamma_R\big(X, \widehat{H}^0(r_l(\pi_E)), \eta_0^l, \overline{\psi}_F^l\big)\sum_{m\in\mathbb{Z}} c_m^F(W, \eta_0^l; 0)X^m, \tag{5.14}$$

Let V be an element in $\mathbb{W}(r_l(\pi_E), \overline{\psi}_E)^\Gamma$ such that V is mapped to $\mathrm{res}_{P_3(F)}(W)$ under the map

$$\Phi : \mathbb{W}(r_l(\pi_E), \overline{\psi}_E)^\Gamma \xrightarrow{\mathrm{res}_{P_3(F)}} \mathrm{Ind}_{N_3(F)}^{P_3(F)}(\overline{\Theta}_F^l).$$

Let η_E be the composition $\eta_0 \circ \mathrm{Nr}_{E/F}$, where $\mathrm{Nr}_{E/F} : E^\times \to F^\times$ is the norm map. Now, the functional equation for the pair $(r_l(\pi_E), \eta_E)$ gives

$$\sum_{m\in\mathbb{Z}} c_m^E(\widetilde{V}, \eta_E^{-1}; 1)q_F^{fm} X^{-fm} = \gamma_R(X, r_l(\pi_E), \eta_E, \overline{\psi}_E)\sum_{m\in\mathbb{Z}} c_m^E(V, \eta_E; 0)X^{fm}.$$

Using Remark 2.8.5, followed by the map Φ, we get

$$\sum_{m\in\mathbb{Z}} c_m^F(\widetilde{W}, \eta_0^{-l}; 1)q_F^{lm} X^{-lm} = \gamma_R(X, r_l(\pi_E), \eta_E, \overline{\psi}_E)\sum_{m\in\mathbb{Z}} c_m^F(W, \eta_0^l; 0)X^{lm}.$$

Hence, it follows from (5.14) and the above relation that

$$\gamma_R(X, r_l(\pi_E), \eta_E, \overline{\psi}_E) = \gamma_R(X^l, \widehat{H}^0(r_l(\pi_E)), \eta_0^l, \overline{\psi}_F^l).$$

Part 3. In this part, we aim to show that

$$\gamma_R\big(X, r_l(\pi_E), \eta_E, \overline{\psi}_E\big) = \gamma_R\big(X^l, r_l(\pi_F)^{(l)}, \eta_0^l, \overline{\psi}_F^l\big).$$

For each $1 \leq i \leq s$ and $1 \leq j \leq r$ with $r, s \leq 3$, let ρ_F^i and ρ_E^j be the irreducible mod-l representations of $\mathcal{W}_F$ and $\mathcal{W}_E$ respectively, such that the representation $r_l(\pi_E)$ (resp. $r_l(\pi_F)$) is associated with the semisimple representation $\bigoplus_{i=1}^s \rho_F^i$ (resp. $\bigoplus_{j=1}^r \rho_E^j$), under mod-$l$ local Langlands correspondence. Since π_E is the base change of π_F, Remark 5.6 gives the following isomorphism:

$$\mathrm{res}_{\mathcal{W}_E}\big(\bigoplus_{i=1}^s \rho_F^i\big) \simeq \bigoplus_{j=1}^r \rho_E^j. \tag{5.15}$$

Now, using the identity (5.5), we get

$$\begin{aligned}\gamma_R(X, r_l(\pi_E), \eta_E, \overline{\psi}_E) &= \gamma_R\big(X, \big(\bigoplus_{j=1}^r \rho_E^j\big) \otimes \eta_E, \overline{\psi}_E\big)\\ &= \gamma_R\big(X, \mathrm{ind}_{\mathcal{W}_E}^{\mathcal{W}_F}\big(\big(\bigoplus_{j=1}^r \rho_E^j\big) \otimes \eta_E\big), \overline{\psi}_F\big).\end{aligned}$$

In view of the isomorphism (5.15), we have the following relation

$$\gamma_R\big(X, r_l(\pi_E), \eta_E, \overline{\psi}_E\big) = \prod_{i=1}^s \gamma_R\big(X, (\rho_F^i \otimes \eta_0) \otimes \mathrm{ind}_{\mathcal{W}_E}^{\mathcal{W}_F}(R), \overline{\psi}_F\big). \tag{5.16}$$

For each $1 \leq i \leq s$, we consider the R-linear map

$$(\rho_F^i \otimes \eta_0) \otimes_R \mathrm{ind}_{\mathcal{W}_E}^{\mathcal{W}_F}(R) \xrightarrow{\varphi} (\rho_F^i \otimes \eta_0) \otimes_R \mathrm{ind}_{\mathcal{W}_E}^{\mathcal{W}_F}(\overline{\mathbb{F}}_l),$$

$$(v \otimes f) \longmapsto (v \otimes \overline{f}),$$

where $\overline{f}$ is induced by f via the natural quotient map $R \to \overline{\mathbb{F}}_l$. Then:

1. Take $v \in \rho_F^i \otimes \eta_0$ and $f \in \mathrm{ind}_{\mathcal{W}_E}^{\mathcal{W}_F}(\overline{\mathbb{F}}_l)$. Since the composition $\overline{\mathbb{F}}_l \hookrightarrow R \to \overline{\mathbb{F}}_l$ is the identity map, the element $f \in \mathrm{ind}_{\mathcal{W}_E}^{\mathcal{W}_F}(R)$ and

$$\varphi(v \otimes f) = v \otimes f.$$

This implies that φ is surjective.

2. $\dim_R\big((\rho_F^i \otimes \eta_0) \otimes_R \mathrm{ind}_{\mathcal{W}_E}^{\mathcal{W}_F}(R)\big) = \dim_R\big((\rho_F^i \otimes \eta_0) \otimes_R \mathrm{ind}_{\mathcal{W}_E}^{\mathcal{W}_F}(\overline{\mathbb{F}}_l)\big)$.

3. φ is compatible with the action of $\mathcal{W}_F$.

Thus, (1) − (3) shows that φ is an $\mathcal{W}_F$ equivariant isomorphism. Now, the Jordan–Holder series of $\overline{\mathbb{F}}_l[\Gamma]$ gives the relation

$$\gamma_R\big(X, \rho_F^i \otimes \eta_0 \otimes \mathrm{ind}_{\mathcal{W}_E}^{\mathcal{W}_F}(\overline{\mathbb{F}}_l), \overline{\psi}_F\big) = \gamma_R(X, \rho_F^i \otimes \eta_0, \overline{\psi}_F)^l.$$

Using above relation and the identity (5.5), it follows from (5.16) that

$$\gamma_R(X, r_l(\pi_E), \eta_E, \overline{\psi}_E) = \gamma_R(X, r_l(\pi_F), \eta_0, \overline{\psi}_F)^l.$$

Finally, Lemma 5.2.5 gives

$$\gamma_R\big(X, r_l(\pi_E), \eta_E, \overline{\psi}_E\big) = \gamma_R(X^l, r_l(\pi_F)^{(l)}, \eta_0^l, \overline{\psi}_F^l).$$

This completes the proof. $\square$

Chapter 6

Compatibility of Kazhdan and Brauer homomorphism

6.1 Introduction

The local Langlands correspondence relates the set of all irreducible smooth complex representations of the group of rational points of a connected reductive group over a local field–with the representation theory of its Weil–Deligne group. For a connected split reductive group $\mathbf{G}$ defined over $\mathbb{Z}$, Kazhdan conjectured a link between the local Langlands correspondences for $\mathbf{G}(F)$ and $\mathbf{G}(F')$, where F is a p-adic field and F' is a non-Archimedean positive characteristic local field, which are sufficiently close. Kazhdan's approach ([Kaz86, Theorem A]) is via isomorphisms between the respective Hecke algebras with complex coefficients, $\mathcal{H}(\mathbf{G}(F), K)$ and $\mathcal{H}(\mathbf{G}(F'), K')$, for some specific choice of compact open subgroups K and K' of $\mathbf{G}(F)$ and $\mathbf{G}(F')$ respectively. Kazhdan isomorphism and its various properties have been further studied by several others, notably by R. Ganapathy (see [Gan15], [Gan22] for instance).

The method of close local fields is crucial in obtaining analogous results for reductive groups over non-Archimedean local fields of positive characteristic, which are known for reductive groups over p-adic fields. Using the close local fields approach, A.I. Badulescu ([Bad02]) proved the Jacquet-Langlands correspondence for $\mathrm{GL}_n(F')$. A similar technique was also used by Badulescu–Henniart–Lemaire–Sécherre ([BHLS10]) to classify the smooth unitary dual of $\mathrm{GL}_n(D)$, where D is a central division algebra over F'.

Let L be a non-Archimedean local field with residue characteristic p and assume that K is a nice compact open subgroup of $\mathbf{G}(L)$–for instance, a congruence subgroup of positive level. When

$\mathbf{G}(L)$ has an automorphism σ of prime order l with $l \neq p$ such that $\sigma(K) = K$, Treumann–Venkatesh ([TV16, Section 4]) define an $\overline{\mathbb{F}}_l$-algebra homomorphism between the Hecke algebras $\mathcal{H}(\mathbf{G}(L), K) \otimes \overline{\mathbb{F}}_l$ and $\mathcal{H}(\mathbf{G}^\sigma(L), K^\sigma) \otimes \overline{\mathbb{F}}_l$, which is called the Brauer homomorphism. In this chapter, we formulate Kazhdan isomorphism over $\overline{\mathbb{F}}_l$–which was originally constructed over $\mathbb{C}$ (see Section 6.4). Then we prove that the Kazhdan isomorphism is compatible with the Brauer homomorphism (Theorem 6.5.1) in the local base change setting. In [TV16], the authors defined the notion of linkage (see Subsection 6.6.1 for the definition), which is the representation theoretic analogue of Brauer homomorphism. Using Theorem 6.5.1, we prove the compatibility of linkage under the close local fields F and F' (see Theorem 6.6.2).

6.2 Hecke Algebra

In this section, we review the notion of Hecke algebra on any locally profinite group. We refer to [TV16, Subsection 2.10] for more details.

Let k be an algebraically closed field of positive characteristic l. Let G be a locally profinite (i.e., locally compact and totally disconnected) group. Let K be a compact open subgroup of G. We denote by G/K the set of all distinct left cosets of K in G, and it is a discrete set with a left action of G. For any $g \in G$, we sometimes use the notation $[g]$ to denote the left coset gK. There is a natural action of $G \times G$, via left translation, on the space of all k-valued functions on the discrete set $G/K \times G/K$. We denote by $\mathcal{F}(G//K)$ the space of all such k-valued functions on $G/K \times G/K$ that are invariant under the action of the diagonal subgroup $\Delta G = \{(g, g) : g \in G\}$, and whose support is a union of finitely many G-orbits. There is a multiplicative structure in the space $\mathcal{F}(G//K)$, given by

$$(f_1 * f_2)([x], [z]) = \sum_{y \in G/K} f_1([x], [y]) f_2([y], [z]),$$

for all $[x], [z] \in G/K$ and $f_1, f_2 \in \mathcal{F}(G//K)$. The space $\mathcal{F}(G//K)$ with the above multiplicative structure is a k-algebra, and it is called the *Hecke* algebra of G associated with K.

We now describe another equivalent formulation of the Hecke algebra relevant to our context. Let us first consider the equivalence relation $\sim$ on the set $G/K \times G/K$, where

$$([x], [y]) \sim ([x'], [y'])$$

if and only if there exists $g \in G$ such that $[gx] = [gx']$ and $[gy] = [gy']$. Then, for any elements $[g], [h] \in G/K$, the map $([g], [h]) \mapsto Kg^{-1}hK$ induces a bijection from the set of equivalence classes $(G/K \times G/K)/\sim$ onto the double coset space $K \setminus G/K$. Let $\mathcal{H}(G, K)$ be the space of all k-

valued functions which are bi-K-invariant, i.e., $f(k_1 g k_2) = f(g)$, for all $k_1, k_2 \in K$, $g \in G$ and $f \in \mathcal{H}(G, K)$. Then, the preceding bijection gives the following map

$$\mathcal{F}(G//K) \longrightarrow \mathcal{H}(G, K)$$

$$\phi \longmapsto \widetilde{\phi}, \tag{6.1}$$

where the function $\widetilde{\phi} : G \to k$ is defined as $\widetilde{\phi}(g) := \phi([1], [g])$, for all $g \in G$. The map (6.1) is a bijection, and it induces a k-algebra structure on the space $\mathcal{H}(G, K)$. In the literature, the k-algebra $\mathcal{H}(G, K)$ is referred as the Hecke algebra of G associated with K.

6.3 Brauer Homomorphism

Let G be a locally profinite group, and let σ be an automorphism of G of order l. Let K be a compact open subgroup of G with $\sigma(K) = K$. Let G^σ (resp. K^σ) be the the fixed subgroup of G (resp. K) under the action of σ. Moreover, the compact open subgroup K is called *σ-plain* if

1. The inclusion $G^\sigma / K^\sigma \hookrightarrow (G/K)^\sigma$, sending gK^σ to gK, is a bijection.

2. There exists a subgroup K' of K of finite index such that K' is an inverse limit of finite subgroups, each of which has order coprime to l.

For any σ-plain compact open subgroup K of G, the discrete set $G/K \times G/K$ is equipped with an action of the group $\langle \sigma \rangle$, defined as

$$\sigma.([g], [h]) := \big([\sigma(g)], [\sigma(h)]\big),$$

for all $[g], [h] \in G/K$. This action further factorizes through the quotient space $G/K \times G/K/\sim$, and it is given by

$$\sigma.\overline{([g], [h])} = \overline{([\sigma(g)], [\sigma(h)])},$$

where $\overline{([g], [h])}$ denotes the equivalence class of the element $([g], [h])$. The space $\mathcal{F}(G//K)$ carries a natural action of $\langle \sigma \rangle$, defined by

$$(\sigma.\phi)([x], [y]) = \phi\big([\sigma^{-1}(x)], [\sigma^{-1}(y)]\big),$$

for all $x, y \in G$ and $\phi \in \mathcal{F}(G//K)$. In [TV16, Section 4], the authors proved that the *Brauer homomorphism* Br is a k-algebra homomorphism from $\mathcal{F}(G//K)^\sigma$ to $\mathcal{F}(G^\sigma//K^\sigma)$, and it is defined by the following the restriction map

$$\mathrm{Br}(\phi) := \mathrm{res}_{(G^\sigma/K^\sigma \times G^\sigma/K^\sigma)}(\phi). \tag{6.2}$$

6.3.1 Equivalent formulation of Brauer homomorphism

Now, we give an equivalent formulation of the map (6.2) for the k-algebra $\mathcal{H}(G,K)$, defined as in the preceding section. First, let us consider the action of the group $\langle\sigma\rangle$ on the double coset space $K \setminus G/K$:

$$\sigma.(KgK) := K\sigma(g)K,$$

for all $g \in G$. Then, we have a natural induced action of $\langle\sigma\rangle$ on the space $\mathcal{H}(G,K)$, defined as

$$(\sigma.f)(g) := f(\sigma^{-1}(g)),$$

for all $g \in G$. Note that the bijection (6.1) preserves the action of the group $\langle\sigma\rangle$. Then, we have a commutative diagram

$$\begin{tikzcd} \mathcal{F}(G//K)^\sigma \arrow[r] \arrow[d, "\mathrm{Br}"'] & \mathcal{H}(G,K)^\sigma \arrow[d, "\overline{\mathrm{Br}}"] \\ \mathcal{F}(G^\sigma//K^\sigma) \arrow[r] & \mathcal{H}(G^\sigma,K^\sigma) \end{tikzcd}$$

The horizontal arrows in the diagram are the bijections induced by (6.1). We observe that the k-algebra map $\overline{\mathrm{Br}} : \mathcal{H}(G,K)^\sigma \to \mathcal{H}(G^\sigma,K^\sigma)$, induced by the above diagram, is the restriction map to subgroup G^σ, i.e., for all $f \in \mathcal{H}(G,K)^\sigma$, we have

$$\overline{\mathrm{Br}}(f) := \mathrm{res}_{G^\sigma}(f).$$

6.4 Kazhdan isomorphism

In this section, we review Kazhdan isomorphism over complex numbers and establish a mod-l version of Kazhdan isomorphism. We also set up some initial results. Let F and F' be two non-Archimedean local fields with the same residue characteristic, say p. Let $\mathfrak{o}_F$ be the ring of integers of F and let $\mathfrak{p}_F$ be its unique maximal ideal. Similar notations follow for F'. The fields F and F' are called *m-close* if there exists a ring isomorphism $\Lambda : \mathfrak{o}_F/\mathfrak{p}_F^m \to \mathfrak{o}_{F'}/\mathfrak{p}_{F'}^m$. We choose uniformizer ϖ (resp. ϖ') of the field F (resp. F') such that the class of ϖ corresponds to the class of ϖ' via the map Λ.

Let $\mathbf{G}$ be a connected split reductive group defined over $\mathbb{Z}$. We fix a Borel subgroup $\mathbf{B}$ of $\mathbf{G}$, defined over $\mathbb{Z}$. Let $\mathbf{B} = \mathbf{TU}$ be a Levi decomposition over $\mathbb{Z}$ with split maximal torus $\mathbf{T}$ and unipotent radical $\mathbf{U}$. We denote by $X^*(\mathbf{T})$ and $X_*(\mathbf{T})$ the character lattice and the cocharacter lattice of $\mathbf{T}$ respectively. There is a natural $\mathbb{Z}$-bilinear pairing

$$X^*(\mathbf{T}) \times X_*(\mathbf{T}) \longrightarrow \mathbb{Z}$$

$$(\alpha, \lambda) \longmapsto \langle \alpha, \lambda \rangle,$$

where $(\lambda \circ \alpha)(x) = x^{\langle \alpha, \lambda \rangle}$, for all $x \in \mathbb{G}_m$. Let $\Phi \subseteq X^*(\mathbf{T})$ be the set of roots of $\mathbf{T}$ in $\mathbf{G}$, and let Φ^+ be the set of positive roots of $\mathbf{T}$ in $\mathbf{B}$. For any $\mathbb{Z}$-algebra R, we denote by G_R the group of R-points $\mathbf{G}(R)$. Consider the subgroup

$$K_F := \mathrm{Ker}\big(\mathbf{G}(\mathfrak{o}_F) \longrightarrow \mathbf{G}(\mathfrak{o}_F/\mathfrak{p}_F^m)\big).$$

Then K_F is a compact open subgroup of G_F. We fix a Haar measure μ_F on the group G_F such that $\mu_F(K_F) = 1$. For each $g \in G_F$, let t_g be the characteristic function on the double coset $K_F g K_F$. Then, the Hecke algebra $\mathcal{H}(G_F, K_F)$ is generated as a k-vector space by $\{t_g : g \in G_F\}$. We denote by $\mathcal{H}_{\mathbb{Z}}(G_F, K_F)$ the space generated by $\{t_g : g \in G_F\}$ over $\mathbb{Z}$. Set

$$X_*(\mathbf{T})^- = \big\{\lambda \in X_*(\mathbf{T}) : \langle \alpha, \lambda \rangle \leq 0, \forall \alpha \in \Phi^+\big\}.$$

We have the Cartan decomposition

$$G_F = \coprod_{\mu \in X_*(\mathbf{T})^-} \mathbf{G}(\mathfrak{o}_F)\varpi_\mu \mathbf{G}(\mathfrak{o}_F), \tag{6.3}$$

where $\varpi_\mu = \mu(\varpi)$. For each $\mu \in X_*(\mathbf{T})^-$, the double coset $G_F(\mu) := \mathbf{G}(\mathfrak{o}_F)\varpi_\mu \mathbf{G}(\mathfrak{o}_F)$ is a homogeneous space under the action of the group $\mathbf{G}(\mathfrak{o}_F) \times \mathbf{G}(\mathfrak{o}_F)$, defined as

$$(x, y).g := xgy^{-1},$$

for all $x, y \in \mathbf{G}(\mathfrak{o}_F)$ and $g \in G_F(\mu)$. Then, the finite group $H_m := \mathbf{G}(\mathfrak{o}_F/\mathfrak{p}_F^m) \times \mathbf{G}(\mathfrak{o}_F/\mathfrak{p}_F^m)$ acts transitively on the set of double cosets $\big\{K_F x K_F : x \in G_F(\mu)\big\}$. Let $\Gamma_\mu(F)$ be the stabilizer of $K_F \varpi_\mu K_F$ in H_m. Let $T_\mu(F) \subseteq \mathbf{G}(\mathfrak{o}_F) \times \mathbf{G}(\mathfrak{o}_F)$ be the set of representatives of the coset space $H_m/\Gamma_\mu(F)$ under the composition

$$\mathbf{G}(\mathfrak{o}_F) \times \mathbf{G}(\mathfrak{o}_F) \xrightarrow{\mathrm{mod}-\mathfrak{p}_F^m} H_m \longrightarrow H_m/\Gamma_\mu(F).$$

Then the decomposition (6.3) can be re-written as

$$G_F = \coprod_{\mu \in X_*(\mathbf{T})^-} \coprod_{(b_i,b_j)\in T_\mu(F)} K_F b_i \varpi_\mu b_j^{-1} K_F. \tag{6.4}$$

For each $\mu \in X_*(\mathbf{T})^-$, the map Λ induces a bijection $T_\mu(F) \to T_\mu(F')$. Under this bijection, there is an isomorphism ([Kaz86, Section 1]) of $\mathbb{Z}$-modules:

$$\mathcal{H}_{\mathbb{Z}}(G_F, K_F) \longrightarrow \mathcal{H}_{\mathbb{Z}}(G_{F'}, K_{F'})$$

$$t_{a_i \varpi_\lambda a_j^{-1}} \longmapsto t_{a'_i \varpi'_\lambda a'^{-1}_j},$$

and this induces a $\mathbb{C}$-vector space isomorphism

$$\mathrm{Kaz}^F_{m,\mathbb{C}} : \mathcal{H}_{\mathbb{C}}(G_F, K_F) \longrightarrow \mathcal{H}_{\mathbb{C}}(G_{F'}, K_{F'}),$$

where $\lambda \in X_*(\mathbf{T})^-$, $(a_i, a_j) \in T_\lambda(F)$ and (a'_i, a'_j) is the corresponding element in $T_\lambda(F')$. For F and F', sufficiently close fields, the following result is due to Kazhdan ([Kaz86, Theorem A]).

Proposition 6.4.1. *Given a positive integer m. There exists an integer $r \geq m$ such that, if F and F' are r-close, the vector space isomorphism $\mathrm{Kaz}^F_{m,\mathbb{C}}$ is a $\mathbb{C}$-algebra isomorphism.*

In order to establish that the vector space isomorphism $\mathrm{Kaz}^F_{m,\mathbb{C}}$ is compatible with algebra structures, the fields need to be a few levels closer. Kazhdan conjectured that one could take $r = m$ in Proposition 6.4.1– which was proved by R. Ganapathy ([Gan15, Conjecture 3.15]), using the finite presentation of the Iwahori Hecke algebras of level m.

6.4.1 Kazhdan isomorphism over positive characteristic

In this section, we formulate a mod-l version of Proposition 6.4.1. Recall that $\mathcal{H}(G_F, K_F)$ denotes the Hecke algebra, consisting of all k-valued smooth, compactly supported and bi-K_F-invariant functions on G_F. Suppose F and F' are m-close. We denote by $X_m(F)$ the set of all K_F double cosets in G_F. Similar notations are followed for F'. The bijection $\phi : X_m(F) \to X_m(F')$ gives the $\mathbb{Z}$-module isomorphism (see [Kaz86, Section 1]) of integral Hecke algebras

$$\mathcal{H}_{\mathbb{Z}}(G_F, K_F) \longrightarrow \mathcal{H}_{\mathbb{Z}}(G_{F'}, K_{F'})$$

$$t_x \longmapsto t_{\phi(x)},$$

which induces an isomorphism of k-vector spaces

$$\mathrm{Kaz}^F_m : \mathcal{H}(G_F, K_F) \longrightarrow \mathcal{H}(G_{F'}, K_{F'}).$$

Kazhdan's approach in proving the Hecke algebra isomorphism over $\mathbb{C}$, relies on the fact that $\mathcal{H}_{\mathbb{C}}(G_F, K_F)$ is finitely presented–which follows from the work of [Ber84]. So, in order to establish that Kaz_m^F is an algebra map, we need the finiteness of Hecke algebras over k–which is made available due to seminal work of Dat–Helm–Kurinczuk–Moss ([DHMK24]).

Lemma 6.4.2. *For any compact open subgroup K of G_F, the Hecke algebra $\mathcal{H}(G_F, K)$ is finitely presented.*

Proof. Note that an algebra $\mathcal{A}$ is finitely presented if $\mathcal{A}$ is finitely generated and Noetherian. Then the above lemma follows from [DHMK24, Theorem 1.1 and Corollary 1.4]. □

Remark 6.4.3. We now recall some results from [Kaz86]. Let C be a finite subset of $X_*(\mathbf{T})^-$ and we set

$$G_C = \bigcup_{\mu \in C} G_F(\mu).$$

By [Kaz86, Lemma 1.3], there exists a positive integer $n_C \geq m$, such that for all $g \in G_C$, we have

$$gK_{n_C}(F)g^{-1} \subseteq K_F,$$

where $K_{n_C}(F)$ is the n_C-th usual congruence subgroup of G_F. Moreover, if F and F' are n_C-close, then

$$\mathrm{Kaz}_m^F(f_1 * f_2) = \mathrm{Kaz}_m^F(f_1) * \mathrm{Kaz}_m^F(f_2), \tag{6.5}$$

for all $f_1, f_2 \in \mathcal{H}(G_F, K_F)$ supported on G_C. The next lemma, which was essential for [Kaz86, Theorem A], is valid over k as well.

Lemma 6.4.4. *For all $\lambda, \mu \in X_*(\boldsymbol{T})^-$, we have*

$$t_{\lambda(\varpi)} * t_{\mu(\varpi)} = t_{(\lambda+\mu)(\varpi)}$$

Moreover, if C is a finite subset of $X_*(\boldsymbol{T})^-$, then for all $x_1, x_2 \in G_C$, we have

$$t_{x_1\lambda(\varpi)x_2} = t_{x_1} * t_{\lambda(\varpi)} * t_{x_2}.$$

Proof. This follows from [All72, Theorem 2] by extension of scalars. □

From these, we obtain Kazhdan isomorphism of mod-l Hecke algebras. The proof follows the same line of arguments of [Kaz86, Theorem A].

Theorem 6.4.5. *Given an integer $m \geq 1$. There exists a positive integer $r \geq m$ such that if F and F' are r-close, the map Kaz_m^F is an isomorphism of k-algebras.*

Proof. We will show that Kaz_m^F preserves the ring structures. Fix a finite subset $C \subseteq X_*(\mathbf{T})^-$ containing 0, such that C generates $X_*(\mathbf{T})^-$ as a semigroup. Consider the finite set

$$X_0 = \bigcup_{\lambda \in C} X_\lambda,$$

where $X_\lambda = \big\{K_F g K_F \mid g \in \mathbf{G}(\mathfrak{o}_F)\lambda(\varpi)\mathbf{G}(\mathfrak{o}_F)\big\}$. It follows from Lemma 6.4.4 that the algebra $\mathcal{H}(G_F, K_F)$ is generated by the set $\big\{t_g : g \in X_0\big\}$, where t_g denotes the characteristic function on the double coset $K_F g K_F$. Assume that $X_0 = \{x_1, \ldots, x_s\}$. Let $k\langle X_1, \ldots, X_s\rangle$ denote the non-commutative polynomial algebra (free algebra) with generators $X_1, \cdots, X_s$. Since the Hecke algebra $\mathcal{H}(G_F, K_F)$ is finitely presented, there exists a surjective k-algebra homomorphism

$$\varphi : k\langle X_1 \ldots, X_s\rangle \to \mathcal{H}(G_F, K_F)$$

$$X_i \longmapsto t_{x_i}$$

such that $\mathrm{Ker}(\varphi)$ is a two-sided ideal generated by non-commutative polynomials $f_1, f_2, \ldots, f_d \in k\langle X_1, \ldots, X_s\rangle$. Let

$$M = \max\big\{a_i : 1 \leq i \leq d\big\},$$

where a_i is the degree of the polynomial f_i. Then there exists a finite set $B \subseteq \Phi^+$ such that

$$X_B = \bigcup_{\lambda \in B} X_\lambda$$

contains a compact set of the form $\big\{g_1 g_2 \ldots g_M : g_i \in X_0\big\}$. By Remark 6.4.3, there is an integer $r = r_B \geq m$, such that if F and F' are r-close, we have

$$\mathrm{Kaz}_m^F(f_1 * f_2) = \mathrm{Kaz}_m^F(f_1) * \mathrm{Kaz}_m^F(f_2),$$

for all functions f_1, f_2 supported on G_B. Now, for each $1 \leq i \leq d$, the above identity gives

$$f_i\big(\mathrm{Kaz}_m^F(t_{x_1}), \ldots, \mathrm{Kaz}_m^F(t_{x_s})\big) = 0$$

or, equivalently

$$f_i\big(t_{\phi(x_1)}, \ldots, t_{\phi(x_s)}\big) = 0.$$

Consider the homomorphism $\varphi' : k\langle X_1, \ldots, X_s\rangle \longrightarrow \mathcal{H}(G_{F'}, K_{F'})$, sending X_j to the function $t_{\phi(x_j)}$. Note that φ' factors through $\mathrm{Ker}(\varphi)$, and hence, we get a unique algebra morphism

$$\widetilde{\varphi}' : \mathcal{H}(G_F, K_F) \longrightarrow \mathcal{H}(G_{F'}, K_{F'})$$

such that $\widetilde{\varphi}'(t_{x_j}) = t_{\phi(x_j)}$, for each j. Since the finite set C generates $X_*(\mathbf{T})^-$ as a semigroup, using the identities in Lemma 6.4.4, we get that

$$\widetilde{\varphi}'(t_x) = t_{\phi(x)} = \mathrm{Kaz}_m^F(t_x),$$

for all $x \in X_m(F)$. This shows that $\widetilde{\varphi}' = \mathrm{Kaz}_m^F$, which is an algebra morphism. Hence, the theorem. □

6.4.2 Galois extension of close local fields

Let F and F' be two non-Archimedean m-close local fields of residue characteristic p. In the seminal work [Del84, Section 3.5], Deligne proved that there is an isomorphism

$$\mathrm{Gal}(\overline{F}/F)/\mathcal{I}_F^m \xrightarrow{\mathrm{Del}_m} \mathrm{Gal}(\overline{F'}/F')/\mathcal{I}_{F'}^m,$$

where $\overline{F}$ (resp. $\overline{F'}$) denotes the seperable closure of F (resp. F'), and $\mathcal{I}_F^m$ (resp. $\mathcal{I}_{F'}^m$) denotes the m-th higher ramification subgroup of the inertia group $\mathcal{I}_F$ (resp. $\mathcal{I}_{F'}$). Let E be a finite Galois extension of F of prime degree l with $l \neq p$. Following the description of the map Del_m (see [Del84, Section 1.3] and [Gan15, Section 2.1]), we have a finite Galois extension E' of F' of the same degree l. Moreover, the extension E'/F' can be so constructed that the fields E and E' are also sufficiently close.

Lemma 6.4.6. *Let F and F' be two m-close non-Archimedean local fields. Let E be a finite Galois extension of F of prime degree l. Then, there exists a finite Galois extension E' of F' of degree l such that E' is m-close to E if E/F is unramified and E' is lm-close to E if E/F is totally ramified.*

Proof. The lemma follows from [Del84]. For completeness, we give a sketch of the proof. The argument is divided into two parts. In the first part, we consider the case where E is unramified over F. The second part deals with totally ramified case. Note that the extension E/F is tamely ramified as $l \neq p$. Let A_m and A'_m denote the quotient rings $\mathfrak{o}_F/\mathfrak{p}_F^m$ and $\mathfrak{o}_{F'}/\mathfrak{p}_{F'}^m$, respectively. Recall that the isomorphism $\Lambda : A_m \xrightarrow{\sim} A'_m$ takes the class of ϖ to the class of ϖ'.

Part 1 : Unramified case

Suppose E/F is unramified. In this case, E is of the form $F[T]/(f)$, where f is an irreducible monic polynomial with coefficients in $\mathfrak{o}_F$ such that its mod-$\mathfrak{p}_F$ reduction, say $\overline{f}$, is irreducible. Moreover, the quotient ring $\mathfrak{o}_E/\mathfrak{p}_E^m$ is isomorphic to $A_m[T]/(\overline{f})$. Let $\overline{f}'$ be the image of $\overline{f}$ under the ring isomorphism Λ. Then we have the ring isomorphism $\mathfrak{o}_E/\mathfrak{p}_E^m \xrightarrow{\sim} A'_m[T]/(\overline{f}')$. Let f' be an irreducible polynomial in $\mathfrak{o}_{F'}[T]$ with its mod-$\mathfrak{p}_F^m$ reduction $\overline{f}'$. Let E' be the field extension

$F'[T]/(f')$. Then E' is unramified over F' and there is a ring isomorphism

$$\mathfrak{o}_{E'}/\mathfrak{p}_{E'}^m \simeq A'_m[T]/(\overline{f}').$$

Thus, we have a ring isomorphism $\mathfrak{o}_E/\mathfrak{p}_E^m \xrightarrow{\sim} \mathfrak{o}_{E'}/\mathfrak{p}_{E'}^m$. Hence, E and E' are m-close.

Part 2 : **Totally ramified case**

Suppose E/F is totally ramified. Since $l \neq p$, the field E is of the form $F[T]/(T^l - \varpi)$ and the ring of integers $\mathfrak{o}_E$ is given by $\mathfrak{o}_F[T]/(T^l - \varpi)$. Recall that the class of ϖ corresponds to that of ϖ' under the isomorphism Λ. Consider the field extension

$$E' := F[T]/(T^l - \varpi').$$

Then E' is also totally ramified over F' with the ring of integers $\mathfrak{o}_{E'} = \mathfrak{o}_{F'}[T]/(T^l - \varpi')$. Let $\overline{\varpi}$ be the image of ϖ under mod-$\mathfrak{p}_F^m$. Then, the natural surjection

$$\mathfrak{o}_F[T]/(T^l - \varpi) \longrightarrow A_m[T]/(T^l - \overline{\varpi})$$

factorizes through the quotient $\mathfrak{p}_F^m[T]/(T^l - \varpi)$, and it gives the ring isomorphism

$$\mathfrak{o}_E/\mathfrak{p}_E^{lm} \xrightarrow{\sim} A_m[T]/(T^l - \overline{\varpi})$$

Similarly, there exists a ring isomorphism $\mathfrak{o}_{E'}/\mathfrak{p}_{E'}^{lm} \xrightarrow{\sim} A'_m[T]/(T^l - \overline{\varpi'})$, where $\overline{\varpi'}$ is the image of the uniformizer ϖ' under the mod-$\mathfrak{p}_{F'}^m$ reduction map. On the other hand, Λ induces the ring isomorphism

$$A_m[T]/(T^l - \overline{\varpi}) \xrightarrow{\sim} A'_m[T]/(T^l - \overline{\varpi'}).$$

This shows that the ring $\mathfrak{o}_E/\mathfrak{p}_E^{lm}$ is isomorphic to $\mathfrak{o}_{E'}/\mathfrak{p}_{E'}^{lm}$, and hence the fields E and E' are lm-close. □

Remark 6.4.7. Let F, F', E and E' be as in Lemma 6.4.6. For simplicity, we write E and E' are em-close, where we define $e = 1$ if E/F is unramified and $e = l$ if E/F is totally ramified. We fix a ring isomorphism $\Pi : \mathfrak{o}_E/\mathfrak{p}_E^{em} \xrightarrow{\sim} \mathfrak{o}_{E'}/\mathfrak{p}_{E'}^{em}$ as in the above lemma– induced by Λ, which makes commutative the following diagram:

$$\begin{tikzcd}
\mathfrak{o}_F/\mathfrak{p}_F^m \arrow[r, "\Lambda"] \arrow[d] & \mathfrak{o}_{F'}/\mathfrak{p}_{F'}^m \arrow[d] \\
\mathfrak{o}_E/\mathfrak{p}_E^{em} \arrow[r, "\Pi"'] & \mathfrak{o}_{E'}/\mathfrak{p}_{E'}^{em}
\end{tikzcd}$$

Here the vertical arrows are the natural inclusion maps. We choose uniformizer π (resp. π') of the field E (resp. E') such that the class of π is mapped to the class of π' under the isomorphism Π. For any $\overline{a} \in \mathfrak{o}_F/\mathfrak{p}_F^m$ (or $\mathfrak{o}_E/\mathfrak{p}_E^{em}$), we write $\overline{a}'$ for the image $\Lambda(\overline{a})$ (or $\Pi(\overline{a})$), whenever no confusion arises.

Remark 6.4.8. Let Γ and Γ' denote the cyclic groups $\mathrm{Gal}(E/F)$ and $\mathrm{Gal}(E'/F')$ respectively. We choose a generator σ (resp. σ') of Γ (resp. Γ') such that the following diagram

$$\begin{tikzcd}
\mathfrak{o}_E/\mathfrak{p}_E^{em} \arrow[r, "\sigma"] \arrow[d, "\Pi"'] & \mathfrak{o}_E/\mathfrak{p}_E^{em} \arrow[d, "\Pi"] \\
\mathfrak{o}_{E'}/\mathfrak{p}_{E'}^{em} \arrow[r, "\sigma'"'] & \mathfrak{o}_{E'}/\mathfrak{p}_{E'}^{em}
\end{tikzcd}$$

commutes. We now describe σ and σ' in the totally ramified case. One can similarly describe σ and σ' in the unramified case. So, assume that E/F is totally ramified. From Lemma 6.4.6, we get that E'/F' is totally ramified and E' is lm-close to E. Then

$$E = \frac{F[T]}{(T^l - \varpi)} \text{ and } E' = \frac{F'[T]}{(T^l - \varpi')}.$$

We have the isomorphism (see Part 2 of Lemma 6.4.6)

$$\Pi : \frac{A_m[T]}{(T^l - \overline{\varpi})} \longrightarrow \frac{A'_m[T]}{(T^l - \overline{\varpi'})},$$

induced by the ring isomorphism $\Lambda : A_m \to A'_m$ and it is given by

$$\Pi\Big(\sum_{i=0}^{l-1} \overline{x_i} T^i + (T^l - \overline{\varpi})\Big) = \sum_{i=0}^{l-1} \Lambda(\overline{x_i}) T^i + (T^l - \overline{\varpi'}),$$

where $\overline{x_i} \in A_m$, for each i. Let σ and σ' to be the automorphisms of E and E' respectively, that sends the class of T to the class of T^2, i.e.,

$$\sigma(T + (T^l - \varpi)) = T^2 + (T^l - \varpi),$$

$$\sigma'(T + (T^l - \varpi')) = T^2 + (T^l - \varpi'),$$

Then σ (resp. σ') is a generator of Γ (resp. Γ'). Moreover, the automorphism σ induces a ring isomorphism $\mathfrak{o}_E/\mathfrak{p}_E^{em} \xrightarrow{\sim} \mathfrak{o}_E/\mathfrak{p}_E^{em}$, and the induced map is again denoted by σ. Similarly for σ'. Then we have the identity

$$\Pi \circ \sigma = \sigma' \circ \Pi.$$

6.4.3 Compatibility of Kazhdan isomorphism with Galois action

We follow the notations of the preceding subsection. Fix two generators σ and σ' of the cyclic groups $\mathrm{Gal}(E/F)$ and $\mathrm{Gal}(E'/F')$ respectively, as discussed in Remark 6.4.8. Then we have the commutative diagram

$$\begin{tikzcd} \mathfrak{o}_E/\mathfrak{p}_E^{em} \arrow[r, "\sigma"] \arrow[d, "\Pi"'] & \mathfrak{o}_E/\mathfrak{p}_E^{em} \arrow[d, "\Pi"] \\ \mathfrak{o}_{E'}/\mathfrak{p}_{E'}^{em} \arrow[r, "\sigma'"'] & \mathfrak{o}_{E'}/\mathfrak{p}_{E'}^{em} \end{tikzcd}$$

where $e = 1$ if E/F is unramified and $e = l$ if E/F is totally ramified. Here the horizontal maps are the ring isomorphisms induced by $\sigma : E \to E$ and $\sigma' : E' \to E'$. To be more precise, for any $a \in \mathfrak{o}_E$, we have the relation

$$\overline{\sigma(a)}' = \sigma(\overline{a})' = \sigma'(\overline{a}') = \overline{\sigma'(a')}, \tag{6.6}$$

where $a' \in \mathfrak{o}_{E'}$ is the preimage of $\overline{a}'$ under the reduction mod-$\mathfrak{p}_{E'}^{em}$. Note that σ can be regarded as an automorphism of G_E via its natural action on G_E. Consider the compact open subgroup

$$K_E = \mathrm{Ker}\big(\mathbf{G}(\mathfrak{o}_E) \to \mathbf{G}(\mathfrak{o}_E/\mathfrak{p}_E^{em})\big).$$

Since K_E is a pro-p subgroup of G_E, the arguments of [Fen24, Lemma 6.6] shows that K_E is σ-plain. There is an induced action of $\langle\sigma\rangle$ on the Hecke algebra $\mathcal{H}(G_E, K_E)$, given by

$$(\sigma.f)(x) = f(\sigma^{-1}(x)),$$

for all $x \in G_E$ and $f \in \mathcal{H}(G_E, K_E)$. Similarly, there is an action of $\mathrm{Gal}(E'/F')$ on the Hecke algebra $\mathcal{H}(G_{E'}, K_{E'})$. We now prove that the Kazhdan isomorphism is compatible with these Galois actions.

Proposition 6.4.9. *For all $f \in \mathcal{H}(G_E, K_E)$, we have*

$$\mathrm{Kaz}_{em}^E(\sigma.f) = \sigma'.\mathrm{Kaz}_{em}^E(f).$$

Proof. In view of the Cartan decomposition (6.4), it is enough to show that

$$\mathrm{Kaz}_{em}^E(\sigma.t_{a_i\pi_\mu a_j^{-1}}) = \sigma'.\mathrm{Kaz}_{em}^E(t_{a_i\pi_\mu a_j^{-1}}),$$

for all $(a_i, a_j) \in T_\mu(E)$ and for all $\mu \in X_*(\mathbf{T})^-$. By definition, we have

$$\sigma.t_{a_i\pi_\mu a_j^{-1}} = t_{\sigma(a_i)\sigma(\pi_\mu)\sigma(a_j^{-1})}.$$

Using Cartan decomposition (6.4), we get an element $(c_i, c_j) \in T_\mu(E)$ such that the double coset $K_E\sigma(a_i)\sigma(\pi_\mu)\sigma(a_j^{-1})K_E$ is of the form $K_E c_i \pi_\mu c_j^{-1} K_E$, and we have the following relations

$$\mathrm{Kaz}^E_{em}(\sigma.t_{a_i\pi_\mu a_j^{-1}}) = \mathrm{Kaz}^E_{em}\big(t_{\sigma(a_i)\sigma(\pi_\mu)\sigma(a_j)^{-1}}\big) = \mathrm{Kaz}^E_{em}\big(t_{c_i\pi_\mu c_j^{-1}}\big) = t_{c_i'\pi_\mu' c_j'^{-1}}. \tag{6.7}$$

Consider the Galois conjugate $\sigma(\pi) = \pi t = \pi\sigma(s)$, where $t, s \in \mathfrak{o}_E^\times$. Using the identity $\sigma(\mu(x)) = \mu(\sigma(x))$, for $x \in E^\times$, we observe that

$$\begin{aligned} K_E c_i \pi_\mu c_j^{-1} K_E &= K_E\sigma(a_i)\sigma(\pi_\mu)\sigma(a_j^{-1})K_E \\ &= K_E\sigma(a_i)\pi_\mu\sigma(s_\mu a_j^{-1})K_E \\ &= K_E\sigma(a_i)\pi_\mu\sigma(d_j^{-1})K_E, \end{aligned}$$

where $s_\mu = \mu(s)$ and $d_j = a_j s_\mu^{-1} \in \mathbf{G}(\mathfrak{o}_E)$. Then $\big(\bar{c_i}^{-1}\overline{\sigma(a_i)}, \overline{\sigma(d_j^{-1})}\bar{c_j}\big) \in \Gamma_\mu(E)$, and from the bijection $\Gamma_\mu(E) \to \Gamma_\mu(E')$, we get

$$\big(\bar{c_i}'^{-1}\overline{\sigma(a_i)}', \overline{\sigma(d_j^{-1})}'\bar{c_j}'\big) \in \Gamma_\mu(E').$$

Recall that the image of d_j under the reduction map $\mathbf{G}(\mathfrak{o}_E) \to \mathbf{G}(\mathfrak{o}_E/\mathfrak{p}_E^{em})$ is denoted by $\overline{d_j}$, and $\overline{d_j}'$ is the image of $\overline{d_j}$ under the isomorphism $\mathbf{G}(\mathfrak{o}_E/\mathfrak{p}_E^{em}) \xrightarrow{\sim} \mathbf{G}(\mathfrak{o}_{E'}/\mathfrak{p}_{E'}^{em})$. Let $d_j' \in \mathbf{G}(\mathfrak{o}_{E'})$ be a preimage of $\overline{d_j}'$ under reduction mod-$\mathfrak{p}_{E'}^{em}$. Then, the commutative diagram

$$\begin{tikzcd} \mathbb{G}_m(\mathfrak{o}_E/\mathfrak{p}_E^{em}) \arrow[r, "\mu"] \arrow[d, "\Pi"'] & \mathbf{T}(\mathfrak{o}_E/\mathfrak{p}_E^{em}) \arrow[d, "\Pi"] \\ \mathbb{G}_m(\mathfrak{o}_{E'}/\mathfrak{p}_{E'}^{em}) \arrow[r, "\mu"'] & \mathbf{T}(\mathfrak{o}_{E'}/\mathfrak{p}_{E'}^{em}) \end{tikzcd}$$

gives the relation

$$\overline{d_j'} = \overline{a_j'}\,\overline{s_\mu}'^{-1} = \overline{a_j'}(\overline{s_\mu'})^{-1},$$

where $s' \in \mathfrak{o}_{E'}$ is a preimage of $\overline{s}'$ under reduction mod-$\mathfrak{p}_{E'}^{em}$. Now, the relation $\sigma(\pi) = \pi\sigma(s)$ together with the identity (6.6) implies that

$$\overline{\sigma'(\pi')} = \sigma'(\overline{\pi}') = \overline{\pi'}\,\overline{\sigma'(s')}.$$

For any $\mu \in X_*(\mathbf{T})^-$, the above equality gives

$$\overline{(\sigma'(\pi'))_\mu} = \overline{(\pi'\sigma'(s'))_\mu}.$$

This implies that $(\sigma'(\pi'))_\mu^{-1}(\pi'\sigma'(s'))_\mu \in K_{E'}$. Using the normality of $K_{E'}$ in $\mathbf{G}(\mathfrak{o}_{E'})$ and the relation $\sigma(\mu(x)) = \mu(\sigma(x))$, for $x \in E^\times$, we get

$$\begin{aligned} K_{E'}c'_i\pi'_\mu c_j'^{-1}K_{E'} &= K_{E'}\sigma'(a'_i)\pi'_\mu\sigma'(d_j'^{-1})K_{E'} \\ &= K_{E'}\sigma'(a'_i)\sigma'(\pi'_\mu)\sigma'(a_j'^{-1})K_{E'}. \end{aligned}$$

This implies that

$$t_{c'_i\pi'_\mu c_j'^{-1}} = t_{\sigma'(a'_i)\sigma'(\pi'_\mu)\sigma'(a_j'^{-1})} = \sigma'.t_{a'_i\pi'_\mu a_j'^{-1}}.$$

Thus, it follows from the above relation and the identity (6.7) that

$$\mathrm{Kaz}_{em}^E(\sigma.t_{a_i\pi_\mu a_j^{-1}}) = \sigma'.\mathrm{Kaz}_{em}^E(t_{a_i\pi_\mu a_j^{-1}}).$$

This completes the proof. □

6.5 Compatibility of Brauer homomorphism and Kazhdan isomorphism

In this section, we prove Theorem 1.4.6. We follow the same notations of the preceding sections. Fix a positive integer m. Let F and F' be two m-close non-Archimedean local fields with the same residue characteristic p. Let E be a finite Galois extension of F of prime degree l with $l \neq p$. According to Lemma 6.4.6, there exists a finite Galois extension E'/F' of degree l and E' is em-close to E, where $e = 1$ if E/F is unramified and $e = l$ if E/F is totally ramified.

Let $\mathbf{G}$ be a connected split reductive group defined over $\mathbb{Z}$. It follows from [KP23, Proposition 12.9.4] that the fixed point subgroups K_E^σ and $K_{E'}^{\sigma'}$ are the m-th congruence subgroups of $\mathbf{G}(\mathfrak{o}_F)$ and $\mathbf{G}(\mathfrak{o}_{F'})$ respectively, i.e, $K_E^\sigma = K_F$ and $K_{E'}^{\sigma'} = K_{F'}$, where

$$K_F = \mathrm{Ker}\big(\mathbf{G}(\mathfrak{o}_F) \to \mathbf{G}(\mathfrak{o}_F/\mathfrak{p}_F^m)\big),$$

$$K_{F'} = \mathrm{Ker}\big(\mathbf{G}(\mathfrak{o}_{F'}) \to \mathbf{G}(\mathfrak{o}_{F'}/\mathfrak{p}_{F'}^m)\big).$$

Proposition 6.4.9 gives an isomorphism between $\mathcal{H}(G_E, K_E)^\sigma$ and $\mathcal{H}(G_{E'}, K_{E'})^{\sigma'}$, induced by Kaz_{em}^E. On the other hand, we have the Brauer homomorphism $\overline{\mathrm{Br}}$ (resp. $\overline{\mathrm{Br}}'$) from the space $\mathcal{H}(G_E, K_E)^\sigma$ (resp. $\mathcal{H}(G_{E'}, K_{E'})^{\sigma'}$) to the Hecke algebra $\mathcal{H}(G_F, K_F)$ (resp. $\mathcal{H}(G_{F'}, K_{F'})$). With these setup, we now prove

Theorem 6.5.1. *Let F, F', E and E' be as above. Then, for any split connected reductive group*

$\boldsymbol{G}$ defined over $\mathbb{Z}$, we have

$$\mathrm{Kaz}_m^F \circ \overline{\mathrm{Br}} = \overline{\mathrm{Br}}' \circ \mathrm{Kaz}_{em}^E,$$

where $e = 1$ if E/F is unramified and $e = l$ if E/F is totally ramified.

Proof. The proof is divided into two parts. In the first part, we consider the case where the extension E/F is unramified. The second part deals with the case where E/F is totally ramified. To begin with, let us recall that an arbitrary $h \in \mathcal{H}(G_E, K_E)^\sigma$ can be written as

$$h = \sum_{\mu \in X_*(\mathbf{T})^-} \sum_{(a_i,a_j) \in T_\mu(E)} \alpha_\mu(a_i, a_j) t_{a_i \pi_\mu a_j^{-1}},$$

where each $\alpha_\mu(a_i, a_j) \in k$. Then

$$(\mathrm{Kaz}_m^F \circ \overline{\mathrm{Br}})(h) = \sum_{(a_i,a_j) \in T_\mu(E)} \alpha_\mu(a_i, a_j)(\mathrm{Kaz}_m^F \circ \overline{\mathrm{Br}})(t_{a_i \pi_\mu a_j^{-1}}) \tag{6.8}$$

and

$$(\overline{\mathrm{Br}}' \circ \mathrm{Kaz}_{em}^E)(h) = \sum_{\mu \in X_*(\mathbf{T})^-} \sum_{(a_i,a_j) \in T_\mu(E')} \alpha_\mu(a_i, a_j) \overline{\mathrm{Br}}'(t_{a'_i \pi'_\mu a'^{-1}_j}). \tag{6.9}$$

For each $\mu \in X_*(\mathbf{T})^-$ and $(a_i, a_j) \in T_\mu(E)$, we have $\overline{\mathrm{Br}}(t_{a_i \pi_\mu a_j^{-1}}) = 1_{G_F \cap K_E a_i \pi_\mu a_j^{-1} K_E}$. Then Cartan decomposition (6.4) gives

$$K_E a_i \pi_\mu a_j^{-1} K_E \cap G_F = \coprod_{\lambda \in X_*(\mathbf{T})^-} \coprod_{(b_i,b_j) \in T_\lambda(F)} K_E a_i \pi_\mu a_j^{-1} K_E \cap K_F b_i \varpi_\lambda b_j^{-1} K_F. \tag{6.10}$$

Similarly, we have

$$K_{E'} a'_i \pi'_\mu a'^{-1}_j K_{E'} \cap G_{F'} = \coprod_{\nu \in X_*(\mathbf{T})^-} \coprod_{(d'_i,d'_j) \in T_\nu(F')} K_{E'} a'_i \pi'_\mu a'^{-1}_j K_{E'} \cap K_{F'} d'_i \varpi'_\nu d'^{-1}_j K_{F'}. \tag{6.11}$$

<u>Part 1</u> : Unramified case

Suppose E is unramified over F. Then, the field E' is also unramified over F' and it is m-close to E (Lemma 6.4.6). In this case, $\pi = \varpi$ and $\pi' = \varpi'$. For any $\lambda \in X_*(\mathbf{T})^-$ and $(b_i, b_j) \in T_\lambda(F)$, it follows from (6.10) and (6.11) that $b_i \varpi_\lambda b_j^{-1} \in K_E a_i \varpi_\mu a_j^{-1} K_E$ if and only if $\varpi_\lambda \in K_E b_i^{-1} a_i \varpi_\mu a_j^{-1} b_j K_E$, and this will happen only if $\lambda = \mu$. Then

$$\varpi_\mu \in K_E b_i^{-1} a_i \varpi_\mu a_j^{-1} b_j K_E$$

if and only if

$$(b_i^{-1} a_i, b_j^{-1} a_j) \in \Gamma_\mu(E).$$

Using the bijection $\Gamma_\mu(E) \to \Gamma_\mu(E')$ $(\overline{a} \mapsto \overline{a}')$, we have

$$(\overline{b_i}^{-1}\overline{a_i}, \overline{b_j}^{-1}\overline{a_j}) \in \Gamma_\mu(E)$$

if and only if

$$(\overline{b_i}'^{-1}\overline{a_i}', \overline{b_j}'^{-1}\overline{a_j}') = (\overline{b_i'}^{-1}\overline{a_i'}, \overline{b_j'}^{-1}\overline{a_j'}) \in \Gamma_\mu(E'),$$

where $(b_i', b_j') \in T_\mu(F')$. Therefore, we have

$$b_i\varpi_\mu b_j^{-1} \in K_E a_i \varpi_\mu a_j^{-1} K_E \iff b_i'\varpi_\mu' b_j'^{-1} \in K_{E'} a_i'\varpi_\mu' a_j'^{-1} K_{E'} \tag{6.12}$$

In view of (6.12), it follows from (6.8) and (6.9) that

$$(\mathrm{Kaz}_m^F \circ \overline{\mathrm{Br}})(h) = (\overline{\mathrm{Br}}' \circ \mathrm{Kaz}_m^E)(h).$$

<u>Part 2</u> : Totally ramified case

Here, we assume that the extension E/F is totally ramified. Then Lemma 6.4.6 says that the extension E'/F' is also totally ramified and E' is lm-close to E, where $l = [E : F] = [E' : F']$. Note that

$$K_E = \mathrm{Ker}\big(\mathbf{G}(\mathfrak{o}_E) \xrightarrow{\mathrm{mod}-\mathfrak{p}_E^{lm}} \mathbf{G}(\mathfrak{o}_E/\mathfrak{p}_E^{lm})\big)$$

and

$$K_F = \mathrm{Ker}\big(\mathbf{G}(\mathfrak{o}_F) \xrightarrow{\mathrm{mod}-\mathfrak{p}_F^{m}} \mathbf{G}(\mathfrak{o}_F/\mathfrak{p}_F^{m})\big).$$

Let $\xi \in X_*(\mathbf{T})^-$ and $(c_i, c_j) \in T_\xi(F)$. From the decompositions (6.10) and (6.11), we get that $c_i\varpi_\xi c_j^{-1} \in K_E a_i \pi_\mu a_j^{-1} K_E$ if and only if $\varpi_\xi \in K_E c_i^{-1} a_i \pi_\mu a_j^{-1} c_j K_E$, and it is possible only if $l\xi = \mu$. Therefore, it follows that

$$\pi_{l\xi} \in K_E c_i^{-1} a_i \pi_{l\xi} a_j^{-1} c_j K_E$$

if and only if

$$(\overline{c_i}^{-1}\overline{a_i}, \overline{a_j}^{-1}\overline{c_j}) \in \Gamma_{l\xi}(E).$$

Under the bijection $\Gamma_{l\xi}(E) \to \Gamma_{l\xi}(E')$, we have

$$(\overline{c_i}^{-1}\overline{a_i}, \overline{a_j}^{-1}\overline{c_j}) \in \Gamma_{l\xi}(E')$$

if and only if

$$(\overline{c_i}'^{-1}\overline{a_i}', \overline{c_j}'^{-1}\overline{a_j}') = (\overline{c_i'}^{-1}\overline{a_i'}, \overline{c_j'}^{-1}\overline{a_j'}) \in \Gamma_{l\xi}(E'),$$

where $(c'_i, c'_j) \in T_{l\xi}(F')$. Thus, we have

$$c_i \varpi_\xi c_j^{-1} \in K_E a_i \pi_\mu a_j^{-1} K_E \iff c'_i \varpi'_\xi c'^{-1}_j \in K_{E'} a'_i \pi'_\mu a'^{-1}_j K_{E'}. \tag{6.13}$$

From the relations (6.8), (6.9) and (6.13), we get

$$(\mathrm{Kaz}_m^F \circ \overline{\mathrm{Br}})(h) = (\overline{\mathrm{Br}}' \circ \mathrm{Kaz}_{lm}^E)(h).$$

This completes the proof. □

6.6 Linkage over close local fields

This section gives an application of Theorem 6.5.1 in the context of representation theory. In the article [TV16], Treumann–Venkatesh defined the notion of linkage, which is conjectured to be compatible with the Langlands functorial transfer. Using Theorem 6.5.1, we show that linkage is compatible under close local fields. To prove this, we first formulate the compatibility of linkage with Brauer homomorphism, as described in [TV16, Section 6.2]. To begin with, we recall the notion of linkage and Tate cohomology. For reference, see [TV16, Section 3 and 6].

6.6.1 Linkage

Let F be a non-Archimedean local field and G be the F-points of a connected reductive group defined over F. Recall that a representation (ρ, V) of G is said to be *smooth* if, for each vector $v \in V$, the G-stabilizer of v is a compact open subgroup of G. Moreover, the representation ρ is called *admissible* if for each compact open subgroup K of G, the space of K-fixed vectors V^K (sometimes denoted by ρ^K) is finite-dimensional over k. In [Vig96b, Chpater II, Section 2.8], Vignéras proved that any smooth irreducible k-representation of G is admissible. In this section, the underlying vector spaces of all representations are defined over k.

Let σ be an automorphism of G of prime order l. An irreducible representation (Ξ, W) of G is called *σ-fixed* if Ξ is isomorphic to $\Xi \circ \sigma$. Using irreducibility and admissibility of Ξ, we get an unique action of $\langle\sigma\rangle$ on W–which is compatible with the $\langle\sigma\rangle$-action on G ([TV16, Proposition 6.1]), and we have the Tate cohomology spaces $\widehat{H}^i(W)$, $i \in \{0, 1\}$, defined as k-representations of G^σ. We denote it by $\widehat{H}^i(\Xi)$. In [TV16, Section 6.3], the authors conjectured that each Tate cohomology group $\widehat{H}^i(\Xi)$ has finite length as a representation of G^σ. A proof of this conjecture is given for the group $\mathrm{GL}_n(F)$, where σ is an automorphism of a finite Galois extension E/F of degree l (see Proposition 4.2.2). So, it is reasonable to expect the validity of the conjecture for arbitrary reductive groups defined over F.

Definition 6.6.1. An irreducible representation ρ of G^σ is said to be linked with the σ-fixed representation Ξ if the Frobenius twist $\rho^{(l)} := \rho \otimes_{k,\mathrm{Frob}_l} k$ occurs as a sub-quotient of the Tate cohomology group $\widehat{H}^0(\Xi)$ or $\widehat{H}^1(\Xi)$.

In the next section, we give a formulation of the compatibility of Brauer homomorphism with linkage–which enables us to study linkage from the module theoretic point of view via Brauer homomorphism. Then, using Theorem 6.5.1, we show that the formulation is compatible with the Kazhdan isomorphism.

6.6.2 Compatibility of Brauer homomorphism with linkage

Let Ξ be an irreducible σ-fixed representation of G, and let $T : \Xi \to \Xi \circ \sigma$ be an isomorphism such that $T^l = \mathrm{id}$. For any σ-plain compact open subgroup K of G, the inclusion map $\Xi^K \hookrightarrow \Xi$ is σ-invariant and it induces a map $\widehat{H}^i(\Xi^K) \to \widehat{H}^i(\Xi)$, taking values in the $\mathcal{H}(G^\sigma, K^\sigma)$-module $\widehat{H}^i(\Xi)^{K^\sigma}$. In [TV16, Section 6.2], the authors proved that for all $h \in \mathcal{H}(G,K)^\sigma$, the Brauer homomorphism $\overline{\mathrm{Br}}(h)$ induces the following commutative diagram

$$\begin{array}{ccc}
\widehat{H}^i(\Xi^K) & \longrightarrow & \widehat{H}^i(\Xi)^{K^\sigma} \\
\Big\downarrow{\scriptstyle \widehat{H}^i(h)} & & \Big\downarrow{\scriptstyle \overline{\mathrm{Br}}(h)} \\
\widehat{H}^i(\Xi^K) & \longrightarrow & \widehat{H}^i(\Xi)^{K^\sigma}
\end{array}$$

and it is compatible with linkage in the following sense: An irreducible representation ρ of G^σ is linked with Ξ if and only if there exists a composition factor $\mathcal{N}$ of the $\mathcal{H}(G,K)^\sigma$-module $\widehat{H}^i(\Xi^K)$ such that

$$\mathcal{H}(G^\sigma, K^\sigma) \otimes_{\mathcal{H}(G,K)^\sigma} \mathcal{N} \simeq (\rho^{(l)})^{K^\sigma}. \tag{6.14}$$

6.6.3 Compatibility of Kazhdan isomorphism with linkage

Let F and F' be two m-close non-Archimedean local fields with the same residue characteristic p. Let E be a finite Galois extension of F with $[E : F] = l$, where l and p are distinct primes. By Lemma 6.4.6, there exists a finite Galois extension E' of F' with $[E' : F'] = l$ such that E' is em-close to E, where $e = 1$ if E/F is unramified and $e = l$ if E/F is totally ramified. We fix generators σ and σ' of $\mathrm{Gal}(E/F)$ and $\mathrm{Gal}(E'/F')$ respectively, as discussed in Remark 6.4.8.

Let $\mathbf{G}$ be a connected split reductive group over $\mathbb{Z}$. Let K_F (resp. K_E) be the congruence subgroups of G_F (resp. G_E) of level m (resp. em) (see Section 6.5). Then we have

$$\mathrm{Kaz}_m^F \circ \overline{\mathrm{Br}} = \overline{\mathrm{Br}}' \circ \mathrm{Kaz}_{em}^E, \tag{6.15}$$

Moreover, we have the following bijections ([Gan15, Section 2.3])

$$\Big\{(\rho_E, V_E) \in \mathrm{Irr}(G_E) : V_E^{K_E} \neq 0\Big\} \longleftrightarrow \Big\{(\rho_{E'}, V_{E'}) \in \mathrm{Irr}(G_{E'}) : V_{E'}^{K_{E'}} \neq 0\Big\} \tag{6.16}$$

and

$$\Big\{(\rho_F, V_F) \in \mathrm{Irr}(G_F) : V_F^{K_F} \neq 0\Big\} \longleftrightarrow \Big\{(\rho_{F'}, V_{F'}) \in \mathrm{Irr}(G_{F'}) : V_{F'}^{K_{F'}} \neq 0\Big\}, \tag{6.17}$$

induced by the Kazhdan isomorphisms Kaz_{em}^E and Kaz_m^F, respectively. With these notations and terminologies, we now prove

Theorem 6.6.2. *Let (ρ_E, V_E) be an irreducible σ-fixed representation of G_E with $V_E^{K_E} \neq 0$, and let (ρ_F, V_F) be an irreducible representation of G_F with $V_F^{K_F} \neq 0$. Let $(\rho_{E'}, V_{E'})$ and $(\rho_{F'}, V_{F'})$ be the corresponding objects under the bijections (6.16) and (6.17). Then, the representation ρ_F is linked with ρ_E if and only if $\rho_{F'}$ is linked with $\rho_{E'}$.*

Proof. We divide the proof into two parts. In the first part, we show that the irreducible representation $\rho_{E'}$ is σ'-fixed. The second part proves that the linkage between ρ_F and ρ_E implies the linkage between $\rho_{F'}$ and $\rho_{E'}$, and vice versa.

Part 1.

Recall that the space of K_E-fixed vectors $V_E^{K_E}$ is a simple $\mathcal{H}(G_E, K_E)$-module and it can be regarded as $\mathcal{H}(G_{E'}, K_{E'})$-module via Kazhdan isomorphism. Consider the $\mathcal{H}(G_{E'})$-module

$$\mathcal{M}' = \mathcal{H}(G_{E'}) \otimes_{\mathcal{H}(G_{E'}, K_{E'})} V_E^{K_E}.$$

Note that the action of $\langle \sigma' \rangle$ on $\mathcal{H}(G_{E'})$ induces an action of $\langle \sigma' \rangle$ on $\mathcal{M}'$, by setting

$$\sigma'.(f' \otimes v) = (\sigma'.f') \otimes v,$$

for all $f' \in \mathcal{H}(G_{E'})$ and $v \in V_E^{K_E}$. Following the construction of [Vig96b, Chapter I, Subsection 4.4], the $\mathcal{H}(G_{E'})$-module $\mathcal{M}'$ gives the irreducible representation $\rho_{E'}$, defined as

$$\rho_{E'}(g')(x') := 1_{g' K_{E'}} * x',$$

for $g' \in G_{E'}$ and $x' \in V_{E'}$. Here $*$ denotes the action of the Hecke algebra $\mathcal{H}(G_{E'})$ on $V_{E'}$ and $1_{g' K_{E'}}$ is the characteristic function of the left coset $g' K_{E'}$. Since ρ_E is σ-fixed, there exists an isomorphism $T_\sigma : \rho_E \to \rho_E \circ \sigma$. Then the map $T_{\sigma'} : \mathcal{M}' \to \mathcal{M}'$, defined by

$$T_{\sigma'}(f' \otimes v) := (\sigma'.f') \otimes T_\sigma(v),$$

induces an isomorphism $\rho_{E'} \xrightarrow{\sim} \rho_{E'} \circ \sigma'$. Hence, the representation $\rho_{E'}$ is σ'-fixed.

Part 2.

Suppose the representation ρ_F is linked with ρ_E. Then, the $\mathcal{H}(G_F, K_F)$-module $(V_F^{(l)})^{K_F}$ is a composition factor of $\widehat{H}^i(V_E)^{K_F}$ for some $i \in \{0, 1\}$. If we regard these as $\mathcal{H}(G_E, K_E)^\sigma$-modules via the Brauer homomorphism $\overline{\mathrm{Br}}$, then the formulation (6.14) gives a composition factor $\mathcal{N}$ of $\widehat{H}^i(V_E^{K_E})$ such that

$$\mathcal{H}(G_F, K_F) \otimes_{\mathcal{H}(G_E,K_E)^\sigma} \mathcal{N} \simeq (V_F^{(l)})^{K_F}. \tag{6.18}$$

Note that the isomorphism $V_E^{K_E} \to V_{E'}^{K_{E'}}$ of $\mathcal{H}(G_E, K_E)$-modules induces a $\mathcal{H}(G_E, K_E)^\sigma$-module isomorphism $\Phi : \widehat{H}^i(V_E^{K_E}) \to \widehat{H}^i(V_{E'}^{K_{E'}})$, which is also a $\mathcal{H}(G_{E'}, K_{E'})^{\sigma'}$-module isomorphism via Kaz_{em}^E.

Let $\mathcal{N}'$ be the image of $\mathcal{N}$ under Φ. Then $\mathcal{N}'$ is a composition factor of $\widehat{H}^i(V_{E'}^{K_{E'}})$, and the $\mathcal{H}(G_{F'}, K_{F'})$-module

$$\mathcal{H}(G_{F'}, K_{F'}) \otimes_{\mathcal{H}(G_{E'},K_{E'})^{\sigma'}} \mathcal{N}'$$

is a composition factor of $\widehat{H}^i(V_{E'})^{K_{F'}}$. We also have an isomorphism $\varphi : V_F^{K_F} \to V_{F'}^{K_{F'}}$, which is equivariant under the action of $\mathcal{H}(G_F, K_F)$ and hence of $\mathcal{H}(G_{F'}, K_{F'})$, via the Kazhdan isomorphism Kaz_m^F. Using the isomorphisms Φ, φ and the relation (6.15), it follows from (6.18) that

$$\mathcal{H}(G_{F,}, K_{F'}) \otimes_{\mathcal{H}(G_{E'},K_{E'})^{\sigma'}} \mathcal{N}' \simeq (V_{F'}^{(l)})^{K_{F'}}.$$

This implies that $\rho_{F'}$ is linked with $\rho_{E'}$. A similar argument also shows that if $\rho_{F'}$ is linked with $\rho_{E'}$, then ρ_F is linked with ρ_E. Hence the theorem. $\square$

Bibliography

[AC89] James Arthur and Laurent Clozel, *Simple algebras, base change, and the advanced theory of the trace formula*, Annals of Mathematics Studies, vol. 120, Princeton University Press, Princeton, NJ, 1989. MR 1007299 3, 35, 43, 47, 62

[Bad02] Alexandru Ioan Badulescu, *Correspondance de Jacquet-Langlands pour les corps locaux de caractéristique non nulle*, Ann. Sci. École Norm. Sup. (4) **35** (2002), no. 5, 695–747. MR 1951441 85

[BH06] Colin J. Bushnell and Guy Henniart, *The local Langlands conjecture for* GL(2), Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 335, Springer-Verlag, Berlin, 2006. MR 2234120 16, 29, 32, 34, 35, 43, 44

[BHLS10] A. I. Badulescu, G. Henniart, B. Lemaire, and V. Sécherre, *Sur le dual unitaire de* $\mathrm{GL}_r(D)$, Amer. J. Math. **132** (2010), no. 5, 1365–1396. MR 2732351 85

[BZ77] I. N. Bernstein and A. V. Zelevinsky, *Induced representations of reductive p-adic groups. I*, Ann. Sci. École Norm. Sup. (4) **10** (1977), no. 4, 441–472. MR 579172 4, 17, 22, 50, 65

[Clo90] Laurent Clozel, *Motifs et formes automorphes: applications du principe de fonctorialité*, Automorphic forms, Shimura varieties, and L-functions, Vol. I (Ann Arbor, MI, 1988), Perspect. Math., vol. 10, Academic Press, Boston, MA, 1990, pp. 77–159. MR 1044819 2, 34

[Del73] P. Deligne, *Les constantes des équations fonctionnelles des fonctions L*, Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), 1973, pp. 501–597. Lecture Notes in Math., Vol. 349. MR 0349635 29, 31, 33

[Del84] ______, *Les corps locaux de caractéristique p, limites de corps locaux de caractéristique* 0, Representations of reductive groups over a local field, Travaux en Cours, Hermann, Paris, 1984, pp. 119–157. MR 771673 12, 93

[EH14] Matthew Emerton and David Helm, *The local Langlands correspondence for* GL_n *in families*, Ann. Sci. Éc. Norm. Supér. (4) **47** (2014), no. 4, 655–722. MR 3250061 19, 74

[Gan15] Radhika Ganapathy, *The local langlands correspondence for gsp 4 over local function fields*, American Journal of Mathematics (2015), 1441–1534. 85, 90, 93, 103

[Gan22] ______, *A Hecke algebra isomorphism over close local fields*, Pacific J. Math. **319** (2022), no. 2, 307–332. MR 4482718 85

[Hel16b] David Helm, *Whittaker models and the integral Bernstein center for* GL_n, Duke Math. J. **165** (2016), no. 9, 1597–1628. MR 3513570 58

[Hen00] Guy Henniart, *Une preuve simple des conjectures de Langlands pour* $\mathrm{GL}(n)$ *sur un corps p-adique*, Invent. Math. **139** (2000), no. 2, 439–455. MR 1738446 2, 34

[Hen01] ______, *Sur la conjecture de Langlands locale pour* GL_n, vol. 13, 2001, 21st Journées Arithmétiques (Rome, 2001), pp. 167–187. MR 1838079 2, 34, 36

[HM15] David Helm and Gilbert Moss, *Deligne–langlands gamma factors in families*, arXiv preprint arXiv:1510.08743 (2015). 11, 32, 33

[HT01] Michael Harris and Richard Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies, vol. 151, Princeton University Press, Princeton, NJ, 2001, With an appendix by Vladimir G. Berkovich. MR 1876802 2, 34

[JPSS83] H. Jacquet, I. I. Piatetskii-Shapiro, and J. A. Shalika, *Rankin-Selberg convolutions*, Amer. J. Math. **105** (1983), no. 2, 367–464. MR 701565 23, 25

[Kaz86] D. Kazhdan, *Representations of groups over close local fields*, J. Analyse Math. **47** (1986), 175–179. MR 874049 ix, 11, 85, 90, 91

[KM17] Robert Kurinczuk and Nadir Matringe, *Rankin-Selberg local factors modulo* ℓ, Selecta Math. (N.S.) **23** (2017), no. 1, 767–811. MR 3595906 6, 9, 21, 23, 25, 26, 27

[KM21] R. Kurinczuk and N. Matringe, *The* ℓ*-modular local Langlands correspondence and local constants*, J. Inst. Math. Jussieu **20** (2021), no. 5, 1585–1635. MR 4311563 31

[KP23] Tasho Kaletha and Gopal Prasad, *Bruhat-Tits theory—a new approach*, New Mathematical Monographs, vol. 44, Cambridge University Press, Cambridge, 2023. MR 4520154 98

[MM22] Nadir Matringe and Gilbert Moss, *The kirillov model in families*, Monatshefte für Mathematik (2022), 1–18. 20, 45, 59, 73

[Mos16] Gilbert Moss, *Gamma factors of pairs and a local converse theorem in families*, Int. Math. Res. Not. IMRN (2016), no. 16, 4903–4936. MR 3556429 6, 11, 23, 24, 25, 75

[MS14] Alberto Mínguez and Vincent Sécherre, *Représentations lisses modulo* l *de* $\mathrm{GL}_m(D)$, Duke Math. J. **163** (2014), no. 4, 795–887. MR 3178433 21, 22, 26, 67

[Sch13] Peter Scholze, *The local Langlands correspondence for* GL_n *over* p*-adic fields*, Invent. Math. **192** (2013), no. 3, 663–715. MR 3049932 34

[Ser] Jean-Pierre Serre, *Cohomologie galoisienne*, fifth ed., Lecture Notes in Mathematics, vol. 5. 38, 39, 52

[TV16] David Treumann and Akshay Venkatesh, *Functoriality, Smith theory, and the Brauer homomorphism*, Ann. of Math. (2) **183** (2016), no. 1, 177–228. MR 3432583 vii, ix, 1, 3, 12, 36, 38, 86, 87, 101, 102

[Vig96b] ______, *Représentations* l*-modulaires d'un groupe réductif* p*-adique avec* $l \neq p$, Progress in Mathematics, vol. 137, Birkhäuser Boston, Inc., Boston, MA, 1996. MR 1395151 4, 13, 17, 18, 19, 23, 40, 45, 50, 51, 53, 65, 68, 101, 103

[Vig98] ______, *Induced* R*-representations of* p*-adic reductive groups*, Selecta Math. (N.S.) **4** (1998), no. 4, 549–623. MR 1668044 22, 34, 36, 68

[Vig01] ______, *Correspondance de Langlands semi-simple pour* $\mathrm{GL}(n, F)$ *modulo* $l \neq p$, Invent. Math. **144** (2001), no. 1, 177–223. MR 1821157 2, 4, 8, 27, 34, 36

[Vig04] ______, *On highest Whittaker models and integral structures*, Contributions to automorphic forms, geometry, and number theory, Johns Hopkins Univ. Press, Baltimore, MD, 2004, pp. 773–801. MR 2058628 9, 23, 45, 63, 67, 70

[Zel80] A. V. Zelevinsky, *Induced representations of reductive p-adic groups. II. On irreducible representations of* $\mathrm{GL}(n)$, Ann. Sci. École Norm. Sup. (4) **13** (1980), no. 2, 165–210. MR 584084 21, 22, 50, 65, 69

[Ron16] Ronchetti, Niccolò, *Local base change via Tate cohomology*, Representation Theory. An Electronic Journal of the American Mathematical Society, 2016, pp. 263–294. MR 3551160 viii, 4, 5, 45, 66, 69

[Hel16a] Helm, David, *The Bernstein center of the category of smooth* $W(k)[\mathrm{GL}_n(F)]$*-modules*, Forum Math. Sigma, 2016, MR 3508741 58

[Mos21] Moss, Gilbert, *Characterizing the mod-ℓ local Langlands correspondence by nilpotent gamma factors*, Nagoya Math. J., 2021, pp. 119–135. MR 4335904 ix, 10, 11, 58, 74, 76, 77

[Dat05] Dat, J.-F., *v-tempered representations of p-adic groups. I. l-adic case*, Duke Math. J., 2005, pp. 397–469. MR 2120114 67, 70

[Hel20] Helm, David, *Curtis homomorphisms and the integral Bernstein center for* GL_n, Algebra Number Theory, 2020, pp. 2607–2645. MR 4190413

[HM18] Helm, David and Moss, Gilbert, *Converse theorems and the local Langlands correspondence in families*, Invent. Math., 2018, pp. 999–1022. MR 3867634 6, 20, 25, 74, 77

[LM20] Liu, Baiying and Moss, Gilbert, *On the local converse theorem and the descent theorem in families*, Math. Z., 2020, pp. 463–483. MR 4100015

[Hel16b] Helm, David, *Whittaker models and the integral Bernstein center for* GL_n, Duke Math. J., 2016, pp. 1597–1628. MR 3513570 74

[DN22] Dhar, Sabyasachi and Nadimpalli, Santosh, *Tate cohomology of Whittaker lattices and Base change of cuspidal representations of* GL_n, arXiv preprint arXiv:2204.02131, 2022.

[GL17] Genestier, Alain and Lafforgue, Vincent, *Restricted chtoucas for groups r\'eductive and param\'etrization of local Langlands*, arXiv preprint arXiv:1709.00978, 2017. 5

[Fen24] Feng, Tony, *Smith theory and cyclic base change functoriality*, Forum of Mathematics, Pi, 2024 viii, 5, 96

[Fen23] Feng, Tony, *Modular functoriality in the local Langlands correspondence*, arXiv preprint arXiv:2312.12542, 2023. viii, 3, 4, 5

[FS21] Fargues, Laurent and Scholze, Peter, *Geometrization of the local Langlands correspondence*, arXiv preprint arXiv:2102.13459, 2021. 3, 5

[KM21] Kurinczuk, R. and Matringe, N., *The ℓ-modular local Langlands correspondence and local constants*, J. Inst. Math. Jussieu, 2021, pp. 1585–1635. MR 4311563 6

[Vig96a] Vignéras, Marie-France, *Erratum: "Modular representations of* $\mathrm{GL}(2,F)$ *in characteristic l, with F a p-adic field, $p \neq l$" [Compositio Math.* **72** *(1989), no. 1, 33–66; MR1026328 (90k:22029)]*, Compositio Math., 1996, pp. 109–113. MR 1390835 43

[JPSS81] Jacquet, H. and Piatetski-Shapiro, I. I. and Shalika, J., *Conducteur des représentations du groupe linéaire* Math. Ann., 1981, pp. 199–214. MR 620708 8

[BH03] Bushnell, Colin J. and Henniart, Guy, *Generalized Whittaker models and the Bernstein center* Amer. J. Math., 2003, pp. 513–547. MR 1981032 8

[Ber84] Bernstein, J. N., *Le "centre" de Bernstein* Representations of reductive groups over a local field, Hermann, Paris, 1984. MR 771671 91

[DHMK24] Dat, Jean-François and Helm, David and Kurinczuk, Robert and Moss, Gilbert, *Finiteness for Hecke algebras of p-adic groups* Journal of the American Mathematical Society, 2024, pp. 929–949. 12, 36, 91

[All72] Allan, Nelo D., *Hecke rings of congruence subgroups* Bull. Amer. Math. Soc., 1972, pp. 541–545. MR 304563 91

[Hen93] Henniart, Guy, *Caractérisation de la correspondance de Langlands locale par les facteurs ϵ de paires* Invent. Math., 1993, pp. 339–350. MR 1228128 viii, 75

[JPPS79] Jacquet, Hervé and Piatetski-Shapiro, Ilja Iosifovitch and Shalika, Joseph, *Automorphic forms on* $\mathrm{GL}(3)$*. I* Ann. of Math. (2), 1979, pp. 169–212. MR 519356 viii, 75

[Vig97] Vignéras, Marie-France, *À propos d'une conjecture de Langlands modulaire*, Finite reductive groups (Luminy, 1994), Progr. Math., Birkhäuser Boston, Boston, MA, 1997, pp. 415–452. MR 1429882 36