TWISTED JACQUET MODULES: A CONJECTURE OF D. PRASAD

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Abstract. In this note, we study the twisted Jacquet modules of sub-quotients of principal series representations of $\mathrm{GL}_2(D)$ where D is a division algebra over a non-archimedean local field F. We begin with a proof of a conjecture of D. Prasad on twisted Jacquet modules of Speh representations of $\mathrm{GL}_2(D)$ when D is the quaternionic division algebra. For arbitrary division algebras D over F, we focus on depth-zero principal series. We compute the dimensions of twisted Jacquet modules of generalized Speh representations and investigate their structure explicitly in the depth-zero situation.

1. Introduction

The multiplicity one result on Whittaker models has been a central result in the representation theory of quasi-split reductive groups over local fields. The dimension of the space of generalised Whittaker linear functions on an irreducible smooth representation of a reductive group over a local field can be greater than one. The dimensions of the space of generalised Whittaker models for reductive p-adic groups is a useful invariant to describe the measure of the representation. It is also intricately related with some well known branching problems, for instance, Prasad conjectured that the space of generalised Whittaker linear functions of a Speh representation of $\mathrm{GL}_2(\mathbb{H})$, where $\mathbb{H}$ is quaternionic division algebra over F is one dimensional; this is related to the Gan–Gross–Prasad conjectures. However, the space of Whittaker linear functionals seems to be far from being well understood. In this note, we focus on the group $\mathrm{GL}_2(D)$ where D is an arbitrary division algebra over a non-archimedean local field F and study non-degenerate Whittaker models, also known as the twisted Jacquet modules, of sub-quotients of principal series representations of $\mathrm{GL}_2(D)$.

To fix some notations, let τ be an irreducible smooth representation of $D^\times$. Let ν_τ be an unramified character of $D^\times$ such that the normalised induction $\tau\nu_\tau^{-1/2} \times \tau\nu_\tau^{1/2}$ is reducible and the generalised Steinberg representation $\mathrm{St}(\tau)$ occurs as the quotient. The irreducible sub-representation of $\tau\nu_\tau^{-1/2} \times \tau\nu_\tau^{1/2}$, denoted by $\mathrm{Sp}(\tau)$, is the Speh representation associated with τ. Let B be the minimal parabolic subgroup of $\mathrm{GL}_2(D)$ consisting of upper triangular matrices with unipotent radical N. Let $\psi : F \to \mathbb{C}^\times$ be a non-trivial additive character on F, viewed as the character on N via

$$\psi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = \psi(\mathrm{Tr}_{D/F}(x)),$$

where $\mathrm{Tr}_{D/F}$ is the reduced trace. The twisted Jacquet module of a smooth representation (π, V) of $\mathrm{GL}_2(D)$ is the space $\pi_{N,\psi}$ of ψ-coinvariants of N in V, and is naturally a representation of $D^\times$.

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Based on a multiplicity one result of Rallis, D. Prasad made a conjecture for the quaternionic division algebra D which predicts that $\mathrm{Sp}(\tau)_{N,\psi}$ is a character and precisely describes this character as a $D^\times$-representation. We first prove this conjecture (see Theorem 3.1):

Theorem 1.1. *Let D be the quaternionic division algebra and τ be a smooth irreducible representation of $D^\times$ of dimension > 1. Then the $D^\times$-representation $\mathrm{Sp}(\tau)_{N,\psi}$ is isomorphic to $\omega_\tau \circ \mathrm{Nr}_{D/F}$ where ω_τ is the central character of τ and $\mathrm{Nr}_{D/F}$ is the reduced norm map of D.*

The above theorem is proved by comparing the germ expansions of representations of $\mathrm{GL}_2(D)$ and $\mathrm{GL}_4(F)$ which correspond to each other under the Jacquet–Langlands correspondence. This gives that the twisted Jacquet module $\mathrm{Sp}(\tau)_{N,\psi}$ is one-dimensional. The explicit action of $D^\times$ on this one-dimensional space is given by a result of Gan and Takeda on Shalika models of $\mathrm{GL}_2(D)$. The idea of comparing local character expansions of representations to study twisted Jacquet modules for non-quasi-split groups has been recently used by Y. Cai to construct a family of Speh representations having unique models of degenerate type [Cai23].

For an arbitrary division algebra D, we focus on depth-zero principal series of $\mathrm{GL}_2(D)$. Let I be the standard Iwahori subgroup and $I(1)$ be the pro-p radical of I. Assume that ψ restricts to a non-trivial character ψ_0 on $I(1) \cap N$. For any smooth representation (σ, W) of I, let $W^{I(1),\psi_0}$ denote the space $\{w \in W : \sigma(g)w = \psi_0(g)w, \forall g \in I(1)\}$. We prove the following theorem which can be considered as the analogue of the result of Moy and Prasad on the compatibility of $I(1)$-invariants with Jacquet modules (see Theorem 4.1).

Theorem 1.2. *Let τ_1 and τ_2 be two irreducible depth-zero representations of $D^\times$. Then the natural map*

$$(\tau_1 \times \tau_2)^{I(1),\psi_0} \to (\tau_1 \times \tau_2)_{N,\psi}$$

is an isomorphism.

Using the above theorem, we prove that the natural maps

$$\mathrm{Sp}(\tau)^{I(1),\psi_0} \to \mathrm{Sp}(\tau)_{N,\psi}$$

and

$$\mathrm{St}(\tau)^{I(1),\psi_0} \to \mathrm{St}(\tau)_{N,\psi}$$

are isomorphisms. The results of Minguez and Secherre [MS14] on the k_{max} functor describe a crucial part of the space of invariants for the first principal congruence subgroup $K(1)$. Putting together the results of Minguez and Secherre with the above isomorphisms, we obtain for depth zero τ that the dimension of $\mathrm{Sp}(\tau)_{N,\psi}$ is equal to $d(d-1)/2$ where d is the dimension of τ (see Corollary 4.5). Note that the dimension of $\mathrm{Sp}(\tau)_{N,\psi}$ does *not* depend on the index of the division algebra D and the Speh representations *no* longer support unique Whittaker models when $d > 2$. In the case where d is odd, we show that the $D^\times$-representation $\mathrm{Sp}(\tau)_{N,\psi}$ is isomorphic to the exterior square representation (see Proposition 4.8). The case where $d = 2$ is arithmetically more involved and we use some computations on Gauss sums to determine the explicit structure of the twisted Jacquet module of $\mathrm{Sp}(\tau)$. In this context, we generalize Theorem 1.1 to arbitrary division algebras and obtain a different proof of it in the situation of the quaternionic division algebra (see Theorem 4.9). The key here is to carefully analyze the action of $D^\times$ (sitting diagonally in the subgroup of diagonal matrices) on $V^{K(1)}$ for a depth-zero irreducible non-cuspidal representation V of $\mathrm{GL}_2(D)$.

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2. Preliminaries

We fix some notation and recall some facts.

2.1. Let F be a non-archimedean local field of residue characteristic p, $\mathfrak{o}_F$ be the ring of integers in F, $\mathfrak{p}_F \subseteq \mathfrak{o}_F$ be the maximal ideal, and $\mathbb{F}_q$ be the residue field of F of cardinality q. Let D be a central division algebra over F index n. The maximal order of D is denoted by $\mathfrak{o}_D$ and the maximal ideal of $\mathfrak{o}_D$ is denoted by $\mathfrak{p}_D$. For a central simple algebra A over a field k, the reduced norm map (resp. the reduced trace map) is denoted by $\mathrm{Nr}_{A/k}$ (resp. $\mathrm{Tr}_{A/k}$). Similarly, for a finite field extension l/k, the field norm map (resp. the field trace map) is denoted by $\mathrm{Nr}_{l/k}$ (resp. $\mathrm{Tr}_{l/k}$). For $z \in \mathbb{C}^\times$ we denote by μ_z the unramified character of $D^\times$ which sends a uniformizer ϖ_D of D to z. Let $\varpi_F = \varpi_D^n$. Then $\mathrm{Nr}_{D/F}(\varpi_D) = (-1)^{n+1}\varpi_F$.

2.2. For a divisor d of n with $n = md$, let F_d denote the unramified extension of F of degree d viewed as a subfield of D, and D_m denote the centralizer of F_d in D. The algebra D_m is a central division algebra over F_d of index m. Let $\theta : F_d^\times \to \mathbb{C}^\times$ be a tamely ramified character all whose Galois conjugates are distinct. Composing it with the reduced norm $\mathrm{Nr}_{D_m/F_d} : D_m^\times \to F_d^\times$ and extending it to $D_m^\times D(1)$ by declaring it to be trivial on $D(1) = 1 + \varpi_D \mathcal{O}_D$, we have a character $\tilde{\theta} : D_m^\times D(1) \to \mathbb{C}^\times$. Note that $D_m^\times D(1) = \mathfrak{o}_D^\times \rtimes \varpi_D^{d\mathbb{Z}}$. Inducing $\tilde{\theta}$ to $D^\times$, we obtain a smooth tamely ramified irreducible d-dimensional representation $\mathrm{Ind}_{D_m^\times D(1)}^{D^\times} \tilde{\theta}$ of $D^\times$. All smooth tamely ramified irreducible representations of $D^\times$ are obtained in this fashion [SZ05].

2.3. Let G be the group $\mathrm{GL}_2(D)$. Let $B \subseteq G$ be the subgroup of upper triangular matrices (the standard minimal parabolic subgroup), $N \subseteq B$ be the subgroup of upper triangular unipotent matrices (the unipotent radical of B), and $T \subseteq B$ be the subgroup of diagonal matrices (the Levi quotient of B). The group $D^\times$ is viewed as a subgroup of T sitting diagonally in it. We denote by K the maximal compact subgroup $\mathrm{GL}_2(\mathfrak{o}_D)$ of G. Let I denote the standard Iwahori subgroup of G and $K(1)$ and $I(1)$ be the pro-p radicals of K and I respectively. Let $T_0 = T \cap K$. A non-trivial additive (smooth) character $\psi_F : F \to \mathbb{C}^\times$ gives rise to a non-trivial additive character $\psi = \psi_F \circ \mathrm{Tr}_{D/F}$ on D which is to be considered as a character of N. For a smooth representation (π, V) of G, the space spanned by the set of vectors $\{\pi(n)v - \psi(n)v : v \in V, n \in N\}$ is denoted by $V(N,\psi)$. The twisted Jacquet module $V_{N,\psi}$ of V is the quotient $V/V(N,\psi)$ considered as a representation of $\mathrm{stab}_T(\psi) = D^\times$. Recall that the Jacquet-Langlands lemma says that a vector $v \in V(N,\psi)$ if and only if

$$\int_{\mathcal{N}} \psi^{-1}(n)\pi(n)v\,dn = 0,$$

for some compact open subgroup $\mathcal{N}$ of N (see [BZ76, Lemma 2.33]).

2.4. For an irreducible smooth representation τ of $D^\times$, there exists an unramified character ν_τ such that the normalized principal series representation $\tau\nu_\tau^{-1/2} \times \tau\nu_\tau^{1/2}$ of G is reducible of length 2 and has a unique square-integrable quotient, the Steinberg representation, denoted by $\mathrm{St}(\tau)$. The subrepresentation $\mathrm{Sp}(\tau)$ of $\tau\nu_\tau^{-1/2}\times\tau\nu_\tau^{1/2}$ is called the Speh representation. We have the following short exact sequences of G-representations:

$$0 \longrightarrow \mathrm{Sp}(\tau) \longrightarrow \tau\nu_\tau^{-1/2}\times\tau\nu_\tau^{1/2} \longrightarrow \mathrm{St}(\tau) \to 0$$

and

$$0 \longrightarrow \mathrm{St}(\tau) \longrightarrow \tau\nu_\tau^{1/2}\times\tau\nu_\tau^{-1/2} \longrightarrow \mathrm{Sp}(\tau) \to 0.$$

We refer to Tadic for the above results [Tad90]. For a principal series $\tau_1\times\tau_2$ of G, there is a natural isomorphism of $D^\times$-representations

$$(\tau_1\times\tau_2)_{N,\psi} \simeq \tau_1\otimes\tau_2,$$

see [PR00, Theorem 2.1]. We note that $\mathrm{Sp}(\tau)_{N,\psi}\neq 0$ if and only if τ has dimension >1.

3. Proof of the conjecture of D. Prasad

In a note [Pra], D. Prasad conjectured that

$$\mathrm{Sp}(\tau)_{N,\psi} \simeq \omega_\tau\circ \mathrm{Nr}_{D/F}$$

as $D^\times$-representations when D is the quaternionic division algebra and τ is a smooth irreducible representation of $D^\times$ of dimension >1. We first prove this conjecture:

Theorem 3.1. *Let D be the quaternionic division algebra over F and let τ be a smooth irreducible representation of $D^\times$ of dimension >1. Then*

$$\mathrm{Sp}(\tau)_{N,\psi} \simeq \omega_\tau\circ \mathrm{Nr}_{D/F}$$

as $D^\times$-representations.

Proof. It is enough to show that $\mathrm{Sp}(\tau)_{N,\psi}$ is one-dimensional because a result [GT10, Theorem 8.6] of Gan and Takeda on the Shalika models of Speh representations then implies that the $D^\times$-representation $\mathrm{Sp}(\tau)_{N,\psi}$ is isomorphic to $\omega_\tau\circ\mathrm{Nr}_{D/F}$.

Denote by σ the Jacquet-Langlands lift of τ. Let Δ be the segment $[\sigma\nu^{-1/2},\sigma\nu^{1/2}]$ and let $\langle\Delta\rangle$ be the irreducible subrepresentation of $\mathrm{GL}_4(F)$ associated with the segment Δ in [Zel80, Section 3]. The Jacquet-Langlands correspondence between $G=\mathrm{GL}_2(D)$ and $\mathrm{GL}_4(F)$, and its extension to the Grothendieck groups of irreducible smooth representations, takes the representation $\mathrm{Sp}(\tau)$ to $\langle\Delta\rangle$. Note that the coefficient of the leading term in the germ expansion of $\mathrm{Sp}(\tau)$, denoted by $c_{\mathcal{O}}(\mathrm{Sp}(\tau))$, is the dimension of $\mathrm{Sp}(\tau)_{N,\psi}$. Now, comparing the germ expansions of $\mathrm{Sp}(\tau)$ and $\langle\Delta\rangle$, we get that

$$c_{\mathcal{O}}(\mathrm{Sp}(\tau)) = c_{\mathcal{O}'}(\langle\Delta\rangle),$$

where $\mathcal{O}'$ is the nilpotent orbit of $\mathfrak{gl}_4(F)$ corresponding to the partition $(2,2)$ (see [Pra00, Theorem 2]). Using [Zel80, Proposition 3.4]), we get that $(2,2)$ is the maximal element in the Whittaker support of $\langle\Delta\rangle$. Then using [MgW87, Theorem I.16], the nilpotent orbit $(2,2)$ is the maximal nilpotent orbit in the germ expansion of $\langle\Delta\rangle$ and thus $c_{\mathcal{O}'}(\langle\Delta\rangle)$ is 1. □

Remark 3.2. *Note that the above argument does not work when D is not the quaternionic division algebra because the non-trivial nilpotent orbit of $\mathfrak{gl}_2(D)$ corresponds to the nilpotent orbit of $\mathfrak{gl}_{2n}(F)$ associated with the partition (n,n) of $2n$, where as the maximal element in the Whittaker support of $\langle\Delta\rangle$ corresponds to the partition $(2,2,\ldots,2)$ of $2n$.*

4. Further results in the tame case

To understand the structure of twisted Jacquet modules of Speh representations for arbitrary division algebra, we restrict ourselves from now on to tamely ramified (depth-0) representations. A generalization of Theorem 3.1 is obtained for an arbitrary division algebra in the tame case.

4.1. **Dimension formulae.** Fix an additive character $\psi_F : F \to \mathbb{C}^\times$ such that ψ_F is non-trivial on $\mathfrak{o}_F$ but trivial on $\mathfrak{p}_F$. Then $\psi = \psi_F \circ \mathrm{Tr}_{D/F}$ is non-trivial on $\mathfrak{o}_D$ and trivial on $\mathfrak{p}_D$. The map

$$\begin{pmatrix} a & b \\ \varpi_D c & d \end{pmatrix} \mapsto \psi(b)$$

defines a non-trivial character ψ_0 on the group $I(1)$. For any smooth representation V of $I(1)$, the space of ψ_0-semi-invariants is

$$V^{I(1),\psi_0} = \{v \in V : g.v = \psi_0(g)v \text{ for all } g \in I(1)\}.$$

If V is a smooth G-representation, then we note that $V^{I(1),\psi_0}$ is stable under the action of $D^\times$. This is because ψ is trivial on $\mathfrak{p}_F$ and factors through $\mathrm{Tr}_{D/F}$.

Let τ_1 and τ_2 be two irreducible smooth depth-zero representations of $D^\times$ of dimensions d_1 and d_2 respectively. In this subsection, we prove the following theorem.

Theorem 4.1. *The restriction of the natural map $\tau_1 \times \tau_2 \to (\tau_1 \times \tau_2)_{N,\psi}$ to the subspace $(\tau_1 \times \tau_2)^{I(1),\psi_0}$:*

$$(\tau_1 \times \tau_2)^{I(1),\psi_0} \to (\tau_1 \times \tau_2)_{N,\psi} \tag{4.1}$$

is an isomorphism of $D^\times$-representations.

Note that

$$(\tau_1 \times \tau_2)^{I(1),\psi_0} = \mathrm{Hom}_{I(1)}(\psi_0, \tau_1 \times \tau_2) = \mathrm{Hom}_{I(1)}(\psi_0, (\tau_1 \times \tau_2)^{K(1)}) = \mathrm{Hom}_{I(1)}(\psi_0, \mathrm{Ind}_I^K(\tau_1 \otimes \tau_2)).$$

Thus the space $(\tau_1 \times \tau_2)^{I(1),\psi_0}$ has dimension $d_1 d_2$.

Before we begin the proof of the theorem, we prove some lemmas. For an integer r, let $N(r) = \begin{pmatrix} 1 & \mathfrak{p}_D^r \\ 0 & 1 \end{pmatrix}$. Note $\psi|_{N(0)} = \psi_0|_{N(0)}$.

Lemma 4.2. *Let f be a non-zero element of $(\tau_1 \times \tau_2)^{I(1),\psi_0}$, then we have*

$$\int_{N(0)} \psi^{-1}(n) f(sn) dn = \mathrm{vol}(N(0)) f(s) \neq 0.$$

Proof. The function f is nonzero if and only if $f|_K$ is so. We have $K = I \sqcup IsI$. Observe that $f(1) = 0$ because $\psi|_{N(0)}$ is non-trivial. From this, we get that $f(i) = 0$, for all $i \in I = (k_D^\times \times k_D^\times)I(1)$. The double coset IsI is equal to the set $(I \cap B)sI(1)$. If $f(s) = 0$, then the function f is identically zero on the double coset IsI, and hence on K. Thus $f(s) \neq 0$. □

Lemma 4.3. *For any smooth representation V of N, and $v \in V$, the image of v in $V_{N,\psi}$ is non-zero if and only if*

$$\int_{N(-r)} \psi^{-1}(n) \pi(n) v dn$$

is non-zero for all $r >> 0$.

Proof. Assume that the image of a vector $v \in V$ in $V_{N,\psi}$ is zero. Then there exists a compact open subgroup $\mathcal{N}$ such that

$$\int_{\mathcal{N}} \psi^{-1}(n)\pi(n)v dn = 0.$$

Since $\{N(-r) : r > 0\}$ is an increasing filtration of N, there exists an r such that $\mathcal{N} \subset N(-r)$. Thus,

$$\int_{N(-r)} \psi^{-1}(n)\pi(n)v dn = \sum_{g \in N(-r)/\mathcal{N}} \psi^{-1}(g)\pi(g) \int_{\mathcal{N}} \psi^{-1}(n)\pi(n)v dn = 0,$$

for all r such that $\mathcal{N} \subset N(-r)$. Conversely, if the above integral is zero for any $r > 0$, then the image of v in $V_{N,\psi}$ is zero. □

Proof of Theorem 4.1. For any positive integer r and $u \in \mathfrak{o}_D^\times$, we have the following matrix identity:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \varpi_D^{-r}u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & \varpi_D^{-r}u \end{pmatrix} = \begin{pmatrix} -\varpi_D^{-r}u & 1 \\ 0 & \varpi_D^r \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ \varpi_D^r & u \end{pmatrix}. \tag{4.2}$$

Let $f \in (\tau_1 \times \tau_2)^{I(1),\psi_0}$ be a non-zero function and $f_r := \int_{N(-r)} \psi^{-1}(n)\pi(n)f dn$. Then

$$\begin{aligned} f_r(s) = \int_{N(-r)} \psi^{-1}(n)f(sn)dn = & \int_{\mathfrak{p}_D^{-r}} \psi^{-1}(y)f\left(s\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\right) dy \\ = & \sum_{\overline{a} \in \mathfrak{p}_D^{-r}/\mathfrak{p}_D^{-r+1}} \int_{\mathfrak{p}_D^{-r+1}} \psi^{-1}(a+y)f\left(s\begin{pmatrix} 1 & (a+y) \\ 0 & 1 \end{pmatrix}\right) dy \\ = & f_{r-1}(s) + \sum_{\overline{a} \neq \overline{0}} \int_{\mathfrak{p}_D^{-r+1}} \psi^{-1}(a+y)f\left(s\begin{pmatrix} 1 & (a+y) \\ 0 & 1 \end{pmatrix}\right) dy. \end{aligned}$$

If $r > 0$, then using the identity (4.2) and that $f(i) = 0$ for $i \in I$ (cf. the proof of Lemma 4.2), we get that

$$\int_{\mathfrak{p}_D^{-r+1}} \psi^{-1}(a+y)f\left(s\begin{pmatrix} 1 & (a+y) \\ 0 & 1 \end{pmatrix}\right) dy = 0 \text{ for } \overline{a} \neq \overline{0}.$$

Thus, we obtain $f_r(s) = f_{r-1}(s)$ for all $r > 0$. By Lemma 4.2, we get that f_r is non-zero for all $r \geq 0$. Hence, by Lemma 4.3, the natural map

$$(\tau_1 \times \tau_2)^{I(1),\psi_0} \to (\tau_1 \times \tau_2)_{N,\psi} \tag{4.3}$$

is injective. However, $(\tau_1 \times \tau_2)_{N,\psi} \simeq \tau_1 \otimes \tau_2$ [PR00, Theorem 2.1]. So, $\dim_{\mathbb{C}}(\tau_1 \times \tau_2)^{I(1),\psi_0} = \dim_{\mathbb{C}}(\tau_1 \times \tau_2)_{N,\psi} = d_1 d_2$. Thus, the map in (4.3) is an isomorphism. □

Proposition 4.4. *Let τ be a tamely ramified irreducible representation of $D^\times$. The natural maps*

$$\mathrm{Sp}(\tau)^{I(1),\psi_0} \to \mathrm{Sp}(\tau)_{N,\psi}$$

and

$$\mathrm{St}(\tau)^{I(1),\psi_0} \to \mathrm{St}(\tau)_{N,\psi}$$

are isomorphisms.

Proof. We have the following commutative diagrams:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Sp}(\tau)_{N,\psi} & \longrightarrow & (\tau\nu_\tau^{-1/2} \times \tau\nu_\tau^{1/2})_{N,\psi} & \longrightarrow & \mathrm{St}(\tau)_{N,\psi} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle g} & & \uparrow{\scriptstyle h} & & \\
0 & \longrightarrow & \mathrm{Sp}(\tau)^{I(1),\psi_0} & \longrightarrow & (\tau\nu_\tau^{-1/2} \times \tau\nu_\tau^{1/2})^{I(1),\psi_0} & \longrightarrow & \mathrm{St}(\tau)^{I(1),\psi_0} & \longrightarrow & 0
\end{array}$$

and

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{St}(\tau)_{N,\psi} & \longrightarrow & (\tau\nu_\tau^{1/2} \times \tau\nu_\tau^{-1/2})_{N,\psi} & \longrightarrow & \mathrm{Sp}(\tau)_{N,\psi} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle h} & & \uparrow{\scriptstyle g'} & & \uparrow{\scriptstyle f} & & \\
0 & \longrightarrow & \mathrm{St}(\tau)^{I(1),\psi_0} & \longrightarrow & (\tau\nu_\tau^{1/2} \times \tau\nu_\tau^{-1/2})^{I(1),\psi_0} & \longrightarrow & \mathrm{Sp}(\tau)^{I(1),\psi_0} & \longrightarrow & 0
\end{array}$$

where f, g, g' and h are the natural maps. Since g and g' are isomorphisms from Theorem 4.1, we get that f is injective and h is surjective from the first diagram and f is surjective and h is injective from the second diagram. $\square$

Corollary 4.5. *Let $\tau = \mathrm{Ind}_{D_m^\times D(1)}^{D^\times}\tilde{\theta}$ be a d-dimensional tamely ramified irreducible representation of $D^\times$. We then have*

$$\dim_{\mathbb{C}} \mathrm{St}(\tau)_{N,\psi} = \frac{d(d-1)}{2} + d \quad \textit{and} \quad \dim_{\mathbb{C}} \mathrm{Sp}(\tau)_{N,\psi} = \frac{d(d-1)}{2}.$$

Proof. From the work of Minguez and Secherre [MS14], we find that as K-representations

$$\mathrm{St}(\tau)^{K(1)} \simeq \bigoplus_{\substack{i,j\in\mathbb{Z}/d\mathbb{Z}\\ i\neq j}} \mathrm{Ind}_I^K(\tilde{\theta}^{q^i} \otimes \tilde{\theta}^{q^j}) \oplus \bigoplus_{i\in\mathbb{Z}/d\mathbb{Z}} \mathrm{st}(\tilde{\theta}^{q^i}) \text{ and}$$

$$\mathrm{Sp}(\tau)^{K(1)} \simeq \bigoplus_{\substack{i,j\in\mathbb{Z}/d\mathbb{Z}\\ i\neq j}} \mathrm{Ind}_I^K(\tilde{\theta}^{q^i} \otimes \tilde{\theta}^{q^j}) \oplus \bigoplus_{i\in\mathbb{Z}/d\mathbb{Z}} \tilde{\theta}^{q^i} \circ \det(\bar{\cdot}),$$

where $\det(\bar{\cdot})$ is the composition of the determinant character of $\mathrm{GL}_2(\mathbb{F}_{q^n})$ and the natural surjection $K \twoheadrightarrow \mathrm{GL}_2(\mathbb{F}_{q^n})$, and $\tilde{\theta}^{q^i} \circ \det(\bar{\cdot})$ and $\mathrm{st}(\tilde{\theta}^{q^i})$ are the two simple factors of the reducible induction $\mathrm{Ind}_I^K(\tilde{\theta}^{q^i} \otimes \tilde{\theta}^{q^i})$ (see [NS24, Lemma 4.5]). Hence,

$$\dim_{\mathbb{C}} \mathrm{St}(\tau)^{I(1),\psi_0} = \dim_{\mathbb{C}} \mathrm{Hom}_{I(1)}(\psi_0, \mathrm{St}(\tau)^{K(1)}) = \frac{d(d-1)}{2} + d$$

and

$$\dim_{\mathbb{C}} \mathrm{Sp}(\tau)^{I(1),\psi_0} = \dim_{\mathbb{C}} \mathrm{Hom}_{I(1)}(\psi_0, \mathrm{Sp}(\tau)^{K(1)}) = \frac{d(d-1)}{2}.$$

The corollary now follows from Proposition 4.4. $\square$

Remark 4.6. *Twisted jacquet modules for irreducible principal series representation $\tau_1 \times \tau_2$ of arbitrary depth is clearly isomorphic to $\tau_1 \otimes \tau_2$ as $D^\times$ representation. The dimensions of tiwsted Jacquet modules for cuspidal can be related to the inducing datum (see...).*

4.2. **The $D^\times$-action on the twisted Jacquet module.** Let $\tau = \mathrm{Ind}_{D_m^\times D(1)}^{D^\times}\tilde{\theta}$ be a d-dimensional tamely ramified irreducible representation of $D^\times$. Using Proposition 4.4, we now find the explicit structure of the $D^\times$-representation $\mathrm{Sp}(\tau)_{N,\psi}$. The analysis depends on the parity of d.

The space of $K(1)$-invariants of the principal series $\tau\nu_\tau^{-1/2}\times\tau\nu_\tau^{1/2}$ as a $KD^\times$-representation is isomorphic to

$$\mathrm{Ind}_{ID^\times}^{KD^\times}(\tau\otimes\tau) = \mathrm{Ind}_{ID^\times}^{KD^\times}\left(\oplus_{y\in\mathbb{Z}/d\mathbb{Z}}W_y\right),$$

where W_y is an irreducible representation of $ID^\times$ such that

$$\mathrm{Res}_I W_y = \bigoplus_{x\in\mathbb{Z}/d\mathbb{Z}} \tilde{\theta}^{q^x}\otimes\tilde{\theta}^{q^{x+y}}.$$

Lemma 4.7. *Let $y\in\mathbb{Z}/d\mathbb{Z}$. If $2y\neq 0$, then the representation $\mathrm{Ind}_{ID^\times}^{KD^\times}W_y$ is irreducible; otherwise it has two distinct irreducible subrepresentations ρ_1 and ρ_2. When $y\neq 0$ (and $2y=0$), we have $\mathrm{Res}_K\rho_1\simeq\mathrm{Res}_K\rho_2$. Moreover,*

$$\mathrm{Ind}_{ID^\times}^{KD^\times}W_y\simeq\mathrm{Ind}_{ID^\times}^{KD^\times}W_{-y}$$

for all $2y\neq 0$.

Proof. Applying the Mackey decomposition, we get that

$$\mathrm{Hom}_{KD^\times}(\mathrm{Ind}_{ID^\times}^{KD^\times}W_y,\mathrm{Ind}_{ID^\times}^{KD^\times}W_{y'}) = \mathrm{Hom}_{T_0D^\times}(W_y,W_{y'})\oplus\mathrm{Hom}_{T_0D^\times}(W_y,W_{y'}^s),$$

where W_y^s is a $T_0D^\times$-representation on the space W_y equipped with the $T_0D^\times$-action conjugated by $s=\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$. For $y\in\mathbb{Z}/d\mathbb{Z}$, the representations W_y of $T_0D^\times$ are distinct and irreducible. As W_y^s is equal to W_{-y}, the lemma follows. □

Let $\{e_i : i\in\mathbb{Z}/d\mathbb{Z}\}$ be a basis of τ consisting of functions $e_i : D^\times\to\mathbb{C}$ such that $\mathrm{supp}(e_i) = \mathfrak{o}_D^\times\rtimes\varpi_D^{d\mathbb{Z}}\varpi_D^i$ and $e_i(\varpi_D^i)=1$. The (diagonal) character of $\mathfrak{o}_D^\times$ on the 1-dimensional space spanned by the vector $e_i\otimes e_{i+y}$ is $(\tilde{\theta}^{1+q^y})^{q^i}$. For $y\in\mathbb{Z}/d\mathbb{Z}$, $\mathrm{Res}_{D^\times}W_y$ is a representation of $D^\times$ such that

$$\mathrm{Res}_{\mathfrak{o}_D^\times}W_y\simeq\bigoplus_{i\in\mathbb{Z}/d\mathbb{Z}}(\tilde{\theta}^{1+q^y})^{q^i}$$

with ϖ_D mapping $e_i\otimes e_{i+y}$ to $e_{i-1}\otimes e_{i-1+y}$ for $i\neq 0$ and $e_0\otimes e_y$ to $\tilde{\theta}(\varpi_D^{2d})e_{-1}\otimes e_{-1+y}$.

4.2.1. *$d=\dim_{\mathbb{C}}(\tau)$ is odd.*

Proposition 4.8. *If $d=\dim_{\mathbb{C}}(\tau)$ is odd, then the $D^\times$-representation $\mathrm{Sp}(\tau)_{N,\psi}$ is isomorphic to $\bigwedge^2\tau$.*

Proof. Let $S\subseteq\mathbb{Z}/d\mathbb{Z}$ be the subset consisting of elements y's defined by the condition that $2y\neq 0$ and $y\in S$ if and only if $-y\notin S$. By the proof of Corollary 4.5 and Lemma 4.7, we have as $KD^\times$-representations,

$$\mathrm{Sp}(\tau)^{K(1)}\simeq V'\oplus\bigoplus_{y\in S}\mathrm{Ind}_{ID^\times}^{KD^\times}W_y,$$

where $\mathrm{Res}_K V' = \bigoplus_{i\in\mathbb{Z}/d\mathbb{Z}}\tilde{\theta}^{q^i}\circ\det(\bar{\cdot})$. Considering the ψ_0-semi-invariants for the action of $I(1)$, we get that

$$\mathrm{Sp}(\tau)^{I(1),\psi_0}\simeq\bigoplus_{y\in S}\mathrm{Res}_{D^\times}W_y.$$

The set $\{e_i \otimes e_j - e_j \otimes e_i : i, j \in \mathbb{Z}/d\mathbb{Z} \text{ and } i \neq j\}$ is a basis for $\bigwedge^2(\tau)$, whereas the space W_y is spanned by vectors $e_i \otimes e_{i+y}, i \in \mathbb{Z}/d\mathbb{Z}$. The map

$$e_i \otimes e_{i+y} \mapsto e_i \otimes e_{i+y} - e_{i+y} \otimes e_i$$

defines an isomorphism of $\bigoplus_{y \in S} \mathrm{Res}_{D^\times} W_y$ with $\bigwedge^2 \tau$. □

4.2.2. $d = \dim_{\mathbb{C}}(\tau)$ *is even.*

As in the proof of Proposition 4.8, we have

$$\mathrm{Sp}(\tau)_{N,\psi} \simeq \mathrm{Sp}(\tau)^{I(1),\psi_0} \simeq V \oplus \bigoplus_{y \in S} \mathrm{Res}_{D^\times} W_y \text{ where } \mathrm{Res}_{\mathfrak{o}_D^\times} V \simeq \bigoplus_{i=0}^{\frac{d}{2}-1} (\tilde{\theta}^{1+q^{\frac{d}{2}}})^{q^i}. \tag{4.4}$$

Suppose $(\tilde{\theta}^{1+q^{\frac{d}{2}}})^{q^k} = \tilde{\theta}^{1+q^{\frac{d}{2}}}$ with $kk' = \frac{d}{2}$. By Frobenius reciprocity, the $D^\times$-representation V is a sum of k' copies of the induction $\mathrm{Ind}_{\mathfrak{o}_D^\times \rtimes \varpi_D^{k\mathbb{Z}}}^{D^\times}(\tilde{\theta}^{1+q^{\frac{d}{2}}})$. When $k = 1$, the induction is the character $\tilde{\theta}^{1+q^{\frac{d}{2}}}$ and one needs to analyze the action of ϖ_D on the underlying space of $\tilde{\theta}^{1+q^{\frac{d}{2}}}$. We do this for $d = 2$ in the remaining part of this subsection.

Let $d = \dim_{\mathbb{C}}(\tau) = 2$ from now on. We also assume that $p > 2$. As $d = 2$, the index of D is $n = 2m$ and the set S is empty. By (4.4) or by Corollary 4.5, we know that $\mathrm{Sp}(\tau)_{N,\psi}$ is a character of $D^\times$. The following theorem precisely describes this character generalizing Theorem 3.1.

Theorem 4.9. *The $D^\times$-representation $\mathrm{Sp}(\tau)_{N,\psi}$ is the character $(\theta \circ \mathrm{Nr}_{D/F})\mu_{(-1)^{m+1}}$.*

Proof. Let f be a non-zero function in $\mathrm{Sp}(\tau)^{K(1)}$ such that $k.f = \tilde{\theta}(\det(\overline{k}))f$ for all $k \in K$ and let

$$t := \begin{pmatrix} \varpi_D & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that $tit^{-1} \in K$ for $i \in I$ and thus, $it^{-1}f = \tilde{\theta}(\det(\overline{tit^{-1}}))t^{-1}f$, which implies that $t^{-1}f \in \mathrm{Sp}(\tau)^{I,\tilde{\theta}^q \otimes \tilde{\theta}}$. From the decomposition of $\mathrm{Sp}(\tau)^{K(1)}$ given in the proof of Corollary 4.5, we find that the K-representation $\langle K \cdot t^{-1}f\rangle$ is stable under the action of $D^\times$. We are interested in the $D^\times$-representation on the space $\langle K \cdot t^{-1}f\rangle^{I(1),\psi_0}$.

The Frobenius reciprocity induces an isomorphism of K-representations

$$\Phi : \mathrm{Ind}_I^K(\tilde{\theta}^q \otimes \tilde{\theta}) \to \langle K \cdot t^{-1}f\rangle$$

such that $\Phi(\varphi) = \sum_{k \in \{1, sn_x\}} \varphi(k^{-1})kt^{-1}f$. Here, $\{1, sn_x = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & [x] \\ 0 & 1 \end{smallmatrix}\right) : x \in \mathbb{F}_{q^{2m}}\}$ is a set of representatives for $I\backslash K$. Let $\mathbb{1}_I \in \mathrm{Ind}_I^K(\tilde{\theta}^q \otimes \tilde{\theta})$ be the function such that $\Phi(\mathbb{1}_I) = t^{-1}f$. It is the function supported on I mapping 1 to 1. Using the operator

$$T(\varphi)(k) = \frac{\theta(-1)^{m+1}\theta(\varpi_F)}{q^m} \sum_{y \in \mathbb{F}_{q^{2m}}} \varphi(sn_y \varpi_D k \varpi_D^{-1})$$

we make $\mathrm{Ind}_I^K(\tilde{\theta}^q \otimes \tilde{\theta})$ into a representation of $KD^\times$ such that ϖ_D acts by T. We claim that Φ is then an isomorphism of $KD^\times$-representations. Indeed, we note that $T^{2m} = \theta(\varpi_F)^2\mathrm{Id}$ and T corresponds to an intertwining operator in $\mathrm{Hom}_K(\mathrm{Ind}_I^K(\tilde{\theta}^q \otimes \tilde{\theta}), \mathrm{Ind}_I^K(\tilde{\theta} \otimes \tilde{\theta}^q)) \simeq \mathbb{C}$. As

ϖ_D^{2m} acts on $\langle K\cdot t^{-1}f\rangle$ by the scalar-multiplication by $\theta(\varpi_F)^2$, there exists a scalar (in fact, an m-th root of unity) ϵ such that

$$\Phi(T(\mathbb{1}_I)) = \epsilon\varpi_D t^{-1}f.$$

Expanding the left-hand side of the above, we find that

$$\begin{aligned}\Phi(T(\mathbb{1}_I)) &= \sum_{k\in\{1,sn_x\}} T(\mathbb{1}_I)(k^{-1})kt^{-1}f\\ &= \frac{\theta(-1)^{m+1}\theta(\varpi_F)}{q^m}\sum_{k\in\{1,sn_x\}}\sum_{y\in\mathbb{F}_{q^{2m}}}\mathbb{1}_I(sn_y\varpi_D k^{-1}\varpi_D^{-1})kt^{-1}f\\ &= \frac{\theta(-1)^{m+1}\theta(\varpi_F)}{q^m}\sum_{x\in\mathbb{F}_{q^{2m}}}\sum_{y\in\mathbb{F}_{q^{2m}}}\mathbb{1}_I(sn_y\varpi_D n_x s\varpi_D^{-1})sn_{-x}t^{-1}f\\ &= \frac{\theta(-1)^{m+1}\theta(\varpi_F)}{q^m}\sum_{x\in\mathbb{F}_{q^{2m}}}\sum_{y\in\mathbb{F}_{q^{2m}}}\mathbb{1}_I(sn_{y+x^q}s)st^{-1}f\\ &= \frac{\theta(-1)^{m+1}\theta(\varpi_F)}{q^m}\sum_{x\in\mathbb{F}_{q^{2m}}} st^{-1}f = \theta(-1)^{m+1}\theta(\varpi_F)q^m st^{-1}f.\end{aligned}$$

Thus, we have $\theta(-1)^{m+1}\theta(\varpi_F)q^m t^{-1}f = \epsilon\varpi_D t^{-1}f$. Evaluating both sides on 1, we obtain that

$$\theta(-1)^{m+1}\theta(\varpi_F)q^m st^{-1}f(1) = \epsilon\varpi_D t^{-1}f(1) \text{ as } \det(\bar{s}) = 1.$$

This gives

$$\theta(-1)^{m+1}\theta(\varpi_F)q^m f\left(\begin{pmatrix}1 & 0\\ 0 & \varpi_D^{-1}\end{pmatrix}\right) = \epsilon f\left(\begin{pmatrix}1 & 0\\ 0 & \varpi_D\end{pmatrix}\right).$$

Using that $f\in\tau\nu^{-1/2}\times\tau\nu^{1/2}$, we get

$$\theta(-1)^{m+1}\theta(\varpi_F)q^m(\mathrm{id}\otimes\tau(\varpi_D^{-1})|\varpi_D|^{1/4})f(1) = \epsilon(\mathrm{id}\otimes\tau(\varpi_D)|\varpi_D|^{-1/4})f(1),$$

and using that $|\varpi_D| = q^{-2m}$ and $\tau(\varpi_D^{-1}) = \tau(\varpi_D)\theta(-1)^{m+1}\theta(\varpi_F)^{-1}$, we conclude

$$q^{m/2}(\mathrm{id}\otimes\tau(\varpi_D))f(1) = \epsilon q^{m/2}(\mathrm{id}\otimes\tau(\varpi_D))f(1).$$

Hence, $\epsilon = 1$ and thus

$$\Phi(\varpi_D\mathbb{1}_I) = \Phi(T(\mathbb{1}_I)) = \varpi_D t^{-1}f = \varpi_D\Phi(\mathbb{1}_I).$$

It follows that the $KD^\times$-representation $\langle K\cdot t^{-1}f\rangle$ is isomorphic to $\mathrm{Ind}_I^K(\tilde{\theta}^q\otimes\tilde{\theta})$ with the ϖ_D-action on the latter given by T.

By [Gar, Proposition 2.0.10], the $D^\times$-representation on the space $\langle K\cdot t^{-1}f\rangle^{I(1),\psi_0}$ is isomorphic to $\tilde{\theta}^{q+1}\mu_c$, where

$$c = \frac{\theta(-1)^{m+1}\theta(\varpi_F)}{q^m}G(\tilde{\theta}^{q-1},\psi_0),$$

and $G(\tilde{\theta}^{q-1},\psi_0) = \sum_{x\in\mathbb{F}_{q^{2m}}}\tilde{\theta}^{q-1}(x)\psi_0(x)$ is the Gauss sum. On the space of the character $\tilde{\theta}^{q+1}\mu_c$, $\mathfrak{o}_D^\times$ acts via the character $\tilde{\theta}^{q+1}$ and ϖ_D acts as the scalar-multiplication by c. To compute the constant c, we need to compute the Gauss sum. Note that, in the Gauss sum,

ψ_0 is viewed as a non-trivial additive character on $\mathbb{F}_{q^{2m}}$ factoring as $\psi_{\mathbb{F}_q} \circ \mathrm{Tr}_{\mathbb{F}_{q^{2m}}/\mathbb{F}_q}$ where $\psi_{\mathbb{F}_q} = \psi_F|_{\mathfrak{o}_F}$. By Hasse-Davenport lifting relation,

$$G(\tilde{\theta}^{q-1}, \psi_0) = (-1)^{m+1} G(\theta^{q-1}, \psi_{\mathbb{F}_q} \circ \mathrm{Tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q})^m.$$

To compute $G(\theta^{q-1}, \psi_{\mathbb{F}_q} \circ \mathrm{Tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}) = \sum_{x \in \mathbb{F}_{q^2}} \theta^{q-1}(x)\psi_{\mathbb{F}_q}(\mathrm{Tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(x))$, let us fix a set $\{x_i\}$ of coset representatives for $\mathbb{F}_{q^2}^\times/\mathbb{F}_q^\times$. We abbreviate $\mathrm{Tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}$ as Tr. Then

$$\begin{aligned}
G(\theta^{q-1}, \psi_{\mathbb{F}_q} \circ \mathrm{Tr}) &= \sum_{x_i} \sum_{y \in \mathbb{F}_q^\times} \theta^{q-1}(x_i y)\psi_{\mathbb{F}_q}(\mathrm{Tr}(x_i y)) \\
&= \sum_{x_i} \sum_{y \in \mathbb{F}_q^\times} \theta^{q-1}(x_i)\psi_{\mathbb{F}_q}(\mathrm{Tr}(x_i) y) \\
&= \sum_{x_i, \mathrm{Tr}(x_i)=0} (q-1)\theta^{q-1}(x_i) + \sum_{x_i, \mathrm{Tr}(x_i)\neq 0} -\theta^{q-1}(x_i) \\
&= \sum_{x_i, \mathrm{Tr}(x_i)=0} q\theta^{q-1}(x_i) - \sum_{x_i} \theta^{q-1}(x_i) \\
&= \sum_{x_i, \mathrm{Tr}(x_i)=0} q\theta^{q-1}(x_i).
\end{aligned}$$

Note that if $\mathrm{Tr}(x) = \mathrm{Tr}(y) = 0$ for some $x, y \in \mathbb{F}_{q^2}^\times$ then x and y belong to the same coset in $\mathbb{F}_{q^2}^\times/\mathbb{F}_q^\times$, i.e., $\frac{x}{y} \in \mathbb{F}_q^\times$. This is clear because if $\mathrm{Tr}(x) = \mathrm{Tr}(y) = 0$ then $(\frac{x}{y})^{q-1} = \frac{x^q y}{x y^q} = \frac{-xy}{-xy} = 1$. There always exists an element $x_0 \in \mathbb{F}_{q^2}^\times$ with $\mathrm{Tr}(x_0) = 0$: for $\mathbb{F}_{q^2}^\times = \langle \alpha \rangle$, $x_0 = \alpha^{\frac{q+1}{2}}$. Hence, $\mathbb{F}_{q^2}^\times/\mathbb{F}_q^\times$ has a unique coset of trace 0 elements. Therefore

$$G(\theta^{q-1}, \psi_{\mathbb{F}_q} \circ \mathrm{Tr}) = q\theta(x_0^q x_0^{-1}) = q\theta(-x_0 x_0^{-1}) = q\theta(-1).$$

Thus,

$$G(\tilde{\theta}^{q-1}, \psi_0) = (-1)^{m+1} q^m \theta(-1)^m \text{ and } c = (-1)^{m+1}\theta(-\varpi_F).$$

An easy computation then gives that the character $\tilde{\theta}^{q+1}\mu_c$ is the character $\theta \circ \mathrm{Nr}_{D/F}$ multiplied with the unramified character $\mu_{(-1)^{m+1}}$. Hence, the $D^\times$-representation $\mathrm{Sp}(\tau)_{N,\psi}$ is isomorphic to $(\theta \circ \mathrm{Nr}_{D/F})\mu_{(-1)^{m+1}}$. □

Remark 4.10. *We remark that in contrast with odd d, for $d = 2$, the above theorem implies that the $D^\times$-representation $\mathrm{Sp}(\tau)_{N,\psi}$ is isomorphic to $\bigwedge^2(\tau)$ if and only if $\theta(-1)^m = \omega_\tau(-1) = (-1)^m = (-1)^{\frac{n}{2}}$. In particular, for the quaternionic division algebra D (i.e. $n = 2$), the twisted Jacquet module of $\mathrm{Sp}(\tau)$ is the exterior square representation $\bigwedge^2(\tau)$ if and only if the central character of τ is odd.*

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