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Modeling and analysis of the algal bloom in a lake caused by discharge of nutrients

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Abstract

In this paper, a nonlinear model for the algal bloom in a lake caused by excessive flow of nutrients from domestic drainage and water runoff from agricultural fields, is proposed and analyzed. This model considers interactions of cumulative concentration of nutrients, density of algal population, density of detritus and concentration of dissolved oxygen in the lake. It is assumed that detritus, which is formed due to death of algae, supplements the cumulative concentration of the nutrients in the water body, using dissolved oxygen in the process. It is shown that the equilibrium level of algal population is highly dependent on the cumulative rate of discharge of inputs of nutrients in the lake and as this cumulative discharge increases the equilibrium level of algal population increases leading to eutrophication. It is also shown that the equilibrium level of dissolved oxygen decreases but that of detritus increases, as the rate of input of cumulative discharge increases. The same results are also found by numerical solution of the model.

Keywords: Mathematical model; Nutrients; Algal bloom; Detritus; Dissolved oxygen

1. Introduction

Algal bloom in a lake occurs when it becomes rich of nutrients (nitrogen, phosphorus, etc.) by domestic drainage, water run off from agricultural fields, etc. The nutrients may also be formed by detritus of algae and other plant species. Algal bloom and growth of macrophytes cause eutrophication and decreased level of dissolved oxygen because these species covering the water surface allow very little transfer of oxygen from air to enter water by diffusion. Also the resultant oxygen produced by these species on water surface does not get a chance to dissolve in water as most of it goes into atmosphere. Further when algae die and sink to the bottom of the water body, their decay by bacteria further reduces the concentration of dissolved oxygen to levels which are too low to support fish production [3,5–7,12,18,21,23,24,26].

Several investigators have studied the effect of discharge of nutrients in water bodies, such as a lake, causing eutrophication [1–3,7,8,12–15,17–19,24,25]. Jorgenson [16] presented an eutrophication model for a lake using

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ecological concepts. Voinov and Tonkikh [26] have presented a model for eutrophication in macrophyte lakes assuming that the nutrient is supplied only by detritus of algae and macrophytes. They did not consider the additional input of nutrients either from domestic drainage or from water run off from agricultural fields. Dachs et al. [5] investigated the influence of eutrophication on air–water exchange, vertical oxygen flux, etc. in the lake Ontorio. The nitrification in the water column and sediment of a lake and adjoining river system has also been studied [20]. Jayaweera and Asaeda [14] studied biomanipulation in shallow eutrophic lakes by using a mathematical model involving phytoplankton, zooplankton, detritus, bacteria and fish population but they did not consider the supply of nutrients from outside. Some other ecological modeling studies involving phytoplankton, zooplankton and nutrients, relevant to our work, have also been conducted by Busenberg et al. [4] and Hallam [11]. However, they have not considered the density of dissolved oxygen in the modeling process.

Keeping in view of the above, in this paper, a nonlinear model to study algal bloom in a lake is proposed and analyzed by considering that nutrients are supplied to the water body from outside by domestic drainage, water run off from agricultural fields, etc. causing algal bloom. Our main aim is to study effects of different levels of inputs of nutrients on the equilibrium levels of algal bloom and dissolved oxygen.

2. Mathematical model

We model here the algal bloom in a lake caused by nutrients supplied to it from domestic drainage, water run off from agricultural fields, etc. simultaneously as well as due to nutrients formed from detritus. The variables such as concentration of nutrients, density of algae (phytoplankton), density of detritus and concentration of dissolved oxygen are considered. The following assumptions are made in the modeling process:

(i) Nutrients are supplied to the water body from domestic drainage as well as from water run off from agricultural fields with a cumulative rate, which is assumed to be a constant.
(ii) The algal population density is wholly dependent on cumulative concentration of nutrients.
(iii) The detritus in the water body is formed by the death of algae (phytoplankton), which is transformed into nutrients by remineralization process using dissolved oxygen.
(iv) The level of dissolved oxygen in water body increases by air, water interaction at the water surface with a constant rate as well as by photosynthesis/respiration by algae. Its concentration is depleted by its utilization by detritus in formation of nutrients, the amount of which is assumed to be proportional to the concentration of dissolved oxygen.

Let n be the cumulative concentration of various nutrients, a be the cumulative density of various algal populations, S be the density of detritus and C be the concentration of dissolved oxygen (DO).We assume that the cumulative rate of discharge of nutrients into the aquatic system from outside in the water body is q (a constant) which is depleted with rate ($\alpha_0 n$) due to natural factors. It is further assumed that the growth rate of nutrients by detritus is ($\pi\delta S$). The depletion of cumulative concentration of nutrients by algae is proportional to the monod interaction of the density of algae and to the concentration of nutrient (i.e. $\beta_1 na/(\beta_{12}+\beta_{11}n)$). Thus, the growth rate of algae, which is assumed to be wholly dependent on the nutrients is proportional to this fraction. The depletion rate of algae caused by mortality is assumed to be proportional to its density a and its depletion rate due to crowding is proportional to a^2. Since parts of the depletion of algae is converted into detritus, the growth rate of detritus is assumed to be proportional to a as well as to a^2 and its natural depletion rate is assumed to be proportional to S. It is considered that the rate of growth of dissolved oxygen through air–water interaction is q_c assumed to be a constant and its natural depletion rate is proportional to its concentration C. It is further assumed that the rate of growth of dissolved oxygen by algae is proportional to a and the depletion of DO caused by transformation of detritus into nutrients is proportional to S.

In view of the above considerations, the system is governed by the following differential equations:

$$\frac{\mathrm{d}n}{\mathrm{d}t} = q - \alpha_0 n - \frac{\beta_1 na}{\beta_{12}+\beta_{11}n} + \pi\delta S,$$
$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{\theta_1\beta_1 na}{\beta_{12}+\beta_{11}n} - \alpha_1 a - \beta_{10}a^2,$$

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \pi_1\alpha_1 a + \pi_2\beta_{10}a^2 - \delta S,$$
$$\frac{\mathrm{d}C}{\mathrm{d}t} = q_c - \alpha_2 C + \lambda_{11}a - \delta_1 S, \tag{2.1}$$

where $n(0) > 0,\ a(0) > 0,\ S(0) > 0,\ C(0) > 0$.

Here α_i's are depletion rate coefficients, β_1, θ_1, δ and δ_1 are constants of proportionality and are positive. The positive constant β_{10} is coefficient corresponding to crowding (flaking off coefficients [22]) of algal population. The proportionality constants π, π_1, π_2 are such that $0 < \pi,\ \pi_1,\ \pi_2 < 1$. It may be pointed out that for feasibility of the model (2.1), the growth rate of algal population should be positive. Hence, from the second equation of the model (2.1), it follows that:

$$\theta_1\beta_1 - \beta_{11}\alpha_1 > 0. \tag{2.2}$$

We also note from (2.1) that the case $q = 0$, $\pi = 0$ is not physically relevant.

3. Analysis of equilibria

The model (2.1) has two non-negative equilibria: namely, a trivial equilibrium $E_1(q/\alpha_0, 0, 0, q_c/\alpha_2)$ and the interior equilibrium $E_2(n^*, a^*, S^*, C^*)$, where n^*, a^*, S^* and C^* are determined from the algebraic equations obtained by putting the right hand side of the model (2.1) to zero.

From the system of algebraic equations, so obtained, it is easy to see that a^* is given by the solution of the following equation:

$$F(a) = \left[\left(\pi\pi_2 - \frac{1}{\theta_1}\right)\beta_{10}a^2 + \left(\pi\pi_1 - \frac{1}{\theta_1}\right)\alpha_1 a + q\right][(\theta_1\beta_1 - \beta_{11}\alpha_1) - \beta_{10}\beta_{11}a] - \beta_{12}\alpha_0(\alpha_1 + \beta_{10}a) = 0. \tag{3.1}$$

From (3.1), we note the following:

(i) $F(0) > 0$ provided $(\theta_1\beta_1 - \beta_{11}\alpha_1)q - \beta_{12}\alpha_0\alpha_1 > 0$.
(ii) $F(\tilde{a}) < 0$ for, $\tilde{a} = (\theta_1\beta_1 - \beta_{11}\alpha_1)/\beta_{10}\beta_{11}$.
(iii) $F'(a) < 0$ in $(0, \tilde{a})$.

Hence $F(a) = 0$ has one and only one positive root (say a^*) in $0 < a < \tilde{a}$.

For this value of a^* we get n^*, S^* and C^* by using the equations governing the existence of $E_2(n^*, a^*, S^*, C^*)$, as follows:

$$n^* = \beta_{12}(\alpha_1 + \beta_{10}a^*)/((\theta_1\beta_1 - \beta_{11}\alpha_1) - \beta_{10}\beta_{11}a^*), \tag{3.2}$$
$$S^* = \frac{\pi_1\alpha_1 a^* + \pi_2\beta_{10}a^{*2}}{\delta}, \tag{3.3}$$

and

$$C^* = \frac{1}{\alpha_2}[q_c + \lambda_{11}a^* - \delta_1 S^*], \tag{3.4}$$

which are positive under the following conditions:

$$(\theta_1\beta_1 - \beta_{11}\alpha_1)q - \beta_{12}\alpha_0\alpha_1 > 0 \quad \text{and} \quad q_c + \lambda_{11}a^* - \delta_1 S^* > 0. \tag{3.5}$$

Remarks. To see the effect of cumulative discharge of nutrients on algal bloom as well as on the equilibrium levels of cumulative algal population density and dissolved oxygen, we determine the rates of change of n^*, a^*, S^* and C^* with respect to q. After simple calculation we get that $\frac{\mathrm{d}n^*}{\mathrm{d}q}$, $\frac{\mathrm{d}a^*}{\mathrm{d}q}$ and $\frac{\mathrm{d}S^*}{\mathrm{d}q}$, are all positive showing that as q increases the equilibrium levels of nutrients, algal population and detritus increase.

Also,

$$\frac{dC^*}{dq} = \frac{1}{\alpha_2}\left[\lambda_{11} - \frac{\pi_1\alpha_1\delta_1}{\delta}\right]\frac{da^*}{dq}, \quad \text{which is negative if } (\delta\lambda_{11} - \pi_1\alpha_1\delta_1) < 0. \tag{3.6}$$

Since most algae float on the surface of water, the oxygen formed by algae during photosynthesis may go to the atmosphere and has a little chance to dissolve into the water, thus λ_{11} is very small. Hence, the condition (3.6) would be satisfied easily. Thus, it is concluded that as the cumulative rate of discharge of nutrients in the water body increases, the cumulative density of algal populations increases but the concentration of dissolved oxygen decreases. It is also concluded that the density of detritus also increases due to increase in cumulative discharge of nutrients.

4. Stability analysis

The local stability results of the equilibriums are stated in the following theorem.

Theorem 4.1. *The equilibrium E_1 is locally unstable whenever E_2 exists and the equilibrium E_2 is locally stable.*

This theorem can be proved by calculating eigenvalues of the Jacobian matrix of model (2.1) corresponding to each equilibrium.

In the following, we prove that E_2 is nonlinearly stable under some conditions. For this we need the following lemma, which is stated without proof (see [9,10]).

Lemma 4.1. *The region of attraction for all solutions initiating in the positive octant is given by the set Ω:*

$$\Omega = \left\{(n, a, S, C)/0 \leqslant n + a + S \leqslant \frac{q}{\delta_m}, 0 \leqslant C \leqslant \frac{q_c\delta_m + \lambda_{11}q}{\delta_m\alpha_2}\right\}, \tag{4.1}$$

where $\delta_m = \text{Min}\{\alpha_0, (1-\pi)\delta, (1-\pi_1)\alpha_1\}$.

Theorem 4.2. *The equilibria E_2 is nonlinearly stable in Ω, if the following conditions are satisfied*:

$$\left[\frac{\beta_1\beta_{11}q}{(\beta_{11}q + \beta_{12}\delta_m)(\beta_{12} + \beta_{11}n^*)}\right]^2 n^* < \frac{2}{3}\frac{\beta_{10}\alpha_0}{\theta_1}, \tag{4.2}$$

$$\frac{\pi^2}{\alpha_0} < \frac{8}{27}\frac{\beta_{10}n^*}{\theta_1(\pi_1\alpha_1 + \pi_2\beta_{10}(q/\delta_m + a^*))}. \tag{4.3}$$

It is noted here that the conditions (4.2) and (4.3) are feasible for large α_0, β_{10} and small β_{11} and π. It can also be seen if $\beta_{10} = 0$, these conditions are not satisfied showing the importance of crowding term in the model (2.1).

The proof of this theorem is given in the Appendix.

The above discussions imply that under certain conditions, the system variables would attain their equilibrium values and the cumulative densities of algae, detritus increase as the cumulative rate of discharge of nutrients increases but the concentration of dissolved oxygen decreases.

5. Numerical solution and discussion

To check the feasibility of our analysis regarding the existence of E_2 and the corresponding stability conditions, we conduct some numerical computation of model (2.1) by choosing the following values of the parameters:

$$\begin{gathered} q = 4.0, \quad \alpha_0 = 1.0, \quad \pi = 0.1, \quad \delta = 0.5, \quad \beta_1 = 1.0, \quad \beta_{12} = 0.1, \quad \beta_{11} = 1.0, \\ \theta_1 = 1, \quad \alpha_1 = 0.5, \quad \beta_{10} = 0.1, \quad \pi_1 = 0.9, \quad \pi_2 = 0.1, \quad q_c = 10.0, \quad \alpha_2 = 1.0, \quad \lambda_{11} = 0.25, \quad \delta_1 = 2.0. \end{gathered} \tag{5.1}$$

It is found that under the above set of parameters, conditions for the existence of interior equilibrium $E_2(n^*, a^*, S^*, C^*)$ are satisfied and E_2 is given by

$$n^* = 0.773891, \quad a^* = 3.855692, \quad S^* = 3.767450, \quad C^* = 3.429023.$$

The eigenvalues of the Jacobian matrix M corresponding to this equilibrium E_2 are obtained as, -1.0, -0.463627, $-0.963411 + 0.388688i$, $-0.963411 - 0.388688i$.

Thus, it has eigenvalues, which are either negative or have negative real parts. Hence E_2 is locally stable.

It is pointed out here that for the above set of parameters, the conditions for nonlinear stability (4.2) and (4.3) are also satisfied.

Further for the above set of parameters, a computer generated graphs of n versus C and a versus C obtained by solving the model (2.1) using MAPPLE 7.0 are shown in Figs. 1 and 2, respectively, which also shows the nonlinear stability of (n^*, C^*) and (a^*, C^*).

Similarly the graphs of variables a, S and C are drawn with respect to time t to see the effects of various parameters.

From Fig. 3, we note that as the cumulative rate of input of nutrients i.e. q increases the equilibrium level of the cumulative density of algal populations increases. It is also noted from this figure that the percentage increase in the equilibrium level of cumulative density when the input of nutrients q is doubled is about 23%. From Figs. 4 and 5, we note that the density of detritus increases but the concentration of dissolved

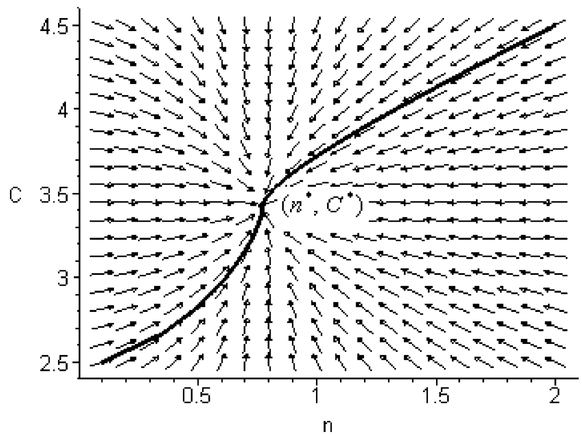

Fig. 1. Nonlinear stability in n–C plane.

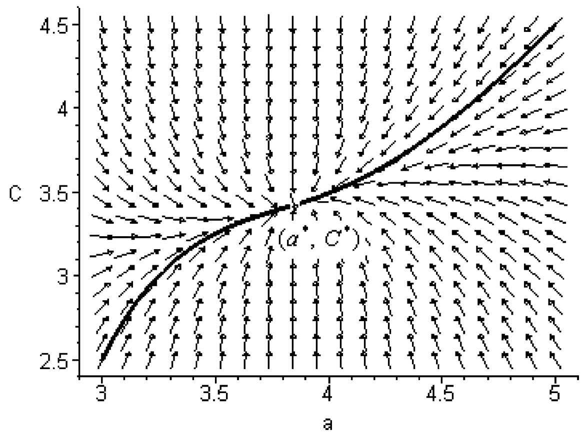

Fig. 2. Nonlinear stability in a–C plane.

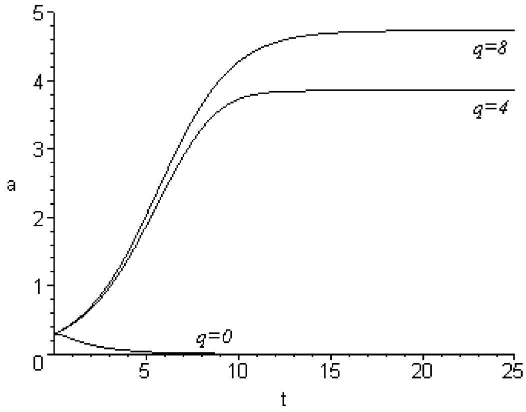

Fig. 3. Variation of a with respect to time t.

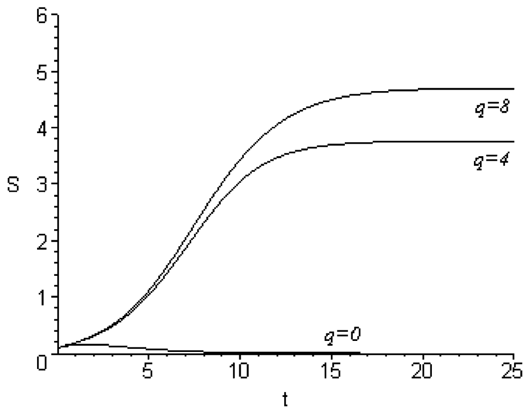

Fig. 4. Variation of S with respect to time t.

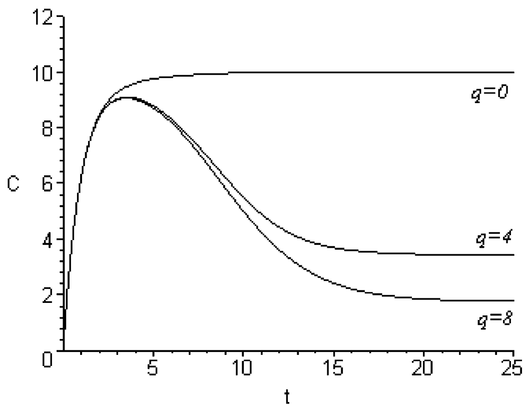

Fig. 5. Variation of C with respect to time t.

oxygen decreases as q increases. In Fig. 5, the plot of C with respect to t is also shown from which it is clear that as q increases, the concentration of dissolved oxygen C decreases. In Fig. 6, the plot of S with t for different values of π_2 is shown keeping other parameters are fixed as given above. From this figure it is clear that

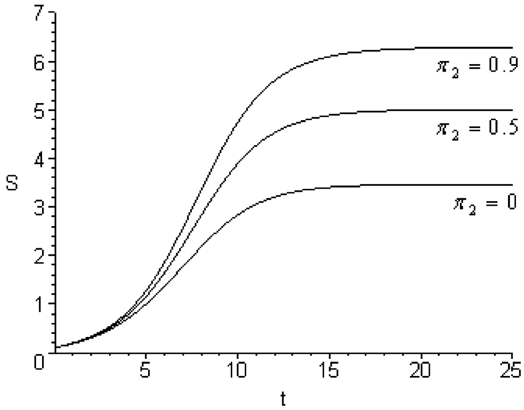

Fig. 6. Variation of S with respect to time t for different π_2.

Table 1
Variation of equilibrium values E_2 with q

q	n^*	a^*	S^*	C^*
0.1	0.1	0.0	0.0	10.0
1.0	0.176938	1.389088	1.288770	7.769732
2.0	0.291995	2.448945	2.323997	5.964243
3.0	0.472377	3.252829	3.139236	4.534752
4.0	0.773891	3.855692	3.767450	3.429023

the density of detritus increases as the death rate due to crowding (i.e. $\beta_{10}a^2$) increases showing the importance of this term in the model (2.1).

In the following Table 1, effects of cumulative rate of input of nutrients (i.e. q) on equilibrium levels of different variables are also shown. It is noted here that as q increases n^*, a^* and S^* increases but C^* decreases as expected.

6. Conclusions

In this paper, we have proposed and analyzed algal bloom in a lake caused by excessive supply of nutrients from domestic drainage as well as water run off from agricultural fields, etc. The modeling analysis shows that as the supply of nutrients in the water body increases, the cumulative density of algal population increases and algal bloom occurs causing eutrophication. It has also been shown that the density of detritus increases correspondingly leading to a decrease in the concentration of dissolved oxygen (as the net production of oxygen, formed due to photosynthesis by floating algae does not affect the concentration of dissolved oxygen), threatening the survival of fish population. By numerical solution of the model it has been shown again that algal population increases as the cumulative rate of input of nutrients increases. It is also found that algal population also increases as the density of detritus increases. Further modeling research is needed to predict quantitatively the effect of decreased level of dissolved oxygen on the growth and survival of fish population in the lake.

Appendix

Proof of Theorem 4.2. To prove this theorem, we consider the following positive definite function:

$$V = \frac{1}{2}(n - n^*)^2 + m_1\left(a - a^* - a^* \ln\frac{a}{a^*}\right) + \frac{1}{2}m_2(S - S^*)^2 + \frac{1}{2}m_3(C - C^*)^2, \tag{A1}$$

where m_1, m_2 and m_3 are positive constants, to be chosen appropriately.

Differentiating V with respect to t along the solution of (2.1), using (3.1)–(3.4), choosing $m_1 = \frac{n^*}{\theta_1}$ and after some algebraic manipulations $\frac{dV}{dt}$ reduces in the following form:

$$\begin{aligned}\frac{dV}{dt} = &-\frac{\beta_1\beta_{12}a}{(\beta_{12}+\beta_{11}n)(\beta_{12}+\beta_{11}n^*)}(n-n^*)^2 - \frac{1}{2}p_{11}(n-n^*)^2 + p_{12}(n-n^*)(a-a^*) - \frac{1}{2}p_{22}(a-a^*)^2 \\ &- \frac{1}{2}p_{11}(n-n^*)^2 + p_{13}(n-n^*)(S-S^*) - \frac{1}{2}p_{33}(S-S^*)^2 - \frac{1}{2}p_{22}(a-a^*)^2 + p_{23}(a-a^*)(S-S^*) \\ &- \frac{1}{2}p_{33}(S-S^*)^2 - \frac{1}{2}p_{22}(a-a^*)^2 + p_{24}(a-a^*)(C-C^*) - \frac{1}{2}p_{44}(C-C^*)^2 - \frac{1}{2}p_{33}(S-S^*)^2 \\ &+ p_{34}(S-S^*)(C-C^*) - \frac{1}{2}p_{44}(C-C^*)^2,\end{aligned}$$

where $p_{11} = \alpha_0$, $p_{22} = \frac{2}{3}\frac{n^*\beta_{10}}{\theta_1}$, $p_{33} = \frac{2}{3}m_2\delta$, $p_{44} = m_3\alpha_2$, $p_{12} = -\frac{\beta_1\beta_{11}nn^*}{(\beta_{12}+\beta_{11}n)(\beta_{12}+\beta_{11}n^*)}$, $p_{13} = \pi\delta$, $p_{23} = m_2(\pi_1\alpha_1 + \pi_2\beta_{10}(a+a^*))$, $p_{24} = m_3\lambda_{11}$, $p_{34} = -m_3\delta_1$.

Thus, sufficient conditions for $\frac{dV}{dt}$ to be negative definite in Ω are that the following inequalities hold:

$$p_{12}^2 < p_{11}p_{22}, \quad p_{13}^2 < p_{11}p_{33}, \quad p_{23}^2 < p_{22}p_{33}, \quad p_{24}^2 < p_{22}p_{44}, \quad p_{34}^2 < p_{33}p_{44}.$$

The first condition i.e. $p_{12}^2 < p_{11}p_{22}$ and using region of attraction Ω gives condition (4.2).

From the rest of inequalities we can choose $m_2 > 0$ if the condition (4.3) is satisfied, then we can also choose m_3 positive appropriately. Hence V is a Lyapunov's function with respect to E_2 whose domain contains the region of attraction Ω, proving the theorem. □

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