Mathematical Modeling of the Survival of a Biological Species in Polluted Water Bodies

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Submitted by V. Sree Hari Rao

Abstract

In this paper, a nonlinear mathematical model is proposed to study the depletion of dissolved oxygen in a water body caused by industrial and household discharges of organic matters (pollutants). The problem is modelled by considering various nonlinear processes involving organic pollutants, bacteria, protozoa, dissolved oxygen and a biological species wholly dependent on it. The effect of depleted level of dissolved oxygen on the survival of biological species in such an aquatic system is studied. Using stability theory, it is shown that not only the concentration of dissolved oxygen decreases due to various biodegradation and biochemical processes but also the survival of the biological species is threatened. It is found that if the organic pollutants continue to be discharged into the water body, the concentration of dissolved oxygen may become negligibly small and the biological species wholly dependent on it is doomed to extinction.

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1. Introduction

It is well known that pollutants, discharged in a water body such as a lake, are harmful to biological species living in such a habitat. Inorganic pollutants affect the growth of biological species directly when they are uptaken by biological species [13],[15], but organic pollutants threaten their survival indirectly by lowering down the concentration of dissolved oxygen (DO) in a water body [6], [8],[12], [18].

Specifically, when an organic matter is discharged into a water body, it becomes part of the food chain involving bacteria, protozoa and other biological populations using DO in various biochemical and biodegradation processes and decreasing its concentration [20]. If the cumulative rate of discharge of organic wastes is very high, the dissolved oxygen decreases further and it may not be enough for survival of various biological species such as fishes of different varieties [7]. In addition, when the level of dissolved oxygen near the water surface falls, anaerobes rise towards it and begin to attack the waste matter, thus, producing foul smelling hydrogen sulphide gas and making the water turbid. In such a case, sunlight cannot penetrate water and without it, the algae cannot produce their food as well as oxygen, useful for other species [10], [16], [17]. Thus, the role of dissolved oxygen in a water body is very important from the point of survival of biological species in an aquatic ecosystem.

The study of effect of organic pollutants on dissolved oxygen in a water body started with the well-known model of [19]. Since then many investigators including [1],[2], [11] have studied this problem. It may be pointed out here that these models are only linear and do not take into account the non linear processes that exist in a water body due to interactions of dissolved oxygen with organic matter, bacteria, protozoa and other biological populations in the food chain [7]. In this regard, some ecological non-linear models involving fast and slow biodegradable pollutants have been proposed which implicitly take into account the depletion of dissolved oxygen by bacteria and protozoa in a water body [4],[8],[12]. In these studies, however, only simulation analysis have been conducted and several questions regarding the nonlinear behavior of the nonzero equilibrium have remained unanswered. Also in these studies, the survival of other aquatic populations such as a fish dependent on DO has not been considered in the modeling process. Several other aspects such as natural depletion rates of pollutants, crowding of populations,
growth of oxygen from algae and macrophytes, etc., have also not been considered in these models [12].

In this paper, therefore, a food chain type non-linear mathematical model is proposed to study the survival of a biological population in a dissolved oxygen depleted water body such as a lake. In this model, the nonlinear biodegradation processes involving interactions of organic pollutants, bacteria and protozoa are assumed to be in the form of monod kinetics. Further, the biological species is assumed to be wholly dependent on dissolved oxygen to visualize its maximum impact on species survival.

2. Mathematical Model:

We consider a food chain type system in a water body consisting of organic matters (pollutants), bacteria, protozoa, and a biological population whose growth rate is wholly dependent on the concentration of dissolved oxygen. Let $T$ be the cumulative concentration of organic matters (pollutants), $B$ be the density of bacteria, $P$ be the density of protozoa, $C$ be the concentration of dissolved oxygen (DO), $F$ be the density of a biological population, the growth rate of which is wholly dependent on dissolved oxygen. We assume that the cumulative rate of discharge of organic pollutants into the water body is $Q$. The rate of depletion of cumulative pollutants concentration $T$ caused by natural factors is assumed to be proportional to its concentration. It is assumed further that the cumulative rate of depletion of organic pollutants by bacteria is in the form of monod type interaction involving the cumulative concentration of the organic pollutants, $T$, and the density of bacteria, $B$, (i.e. $\frac{TB}{K_{12}+K_{11}T}$). As bacteria wholly depend upon organic pollutants, the growth rate of bacteria is proportional to this amount (i.e. $\frac{TB}{K_{12}+K_{11}T}$).

The natural depletion rate of bacteria is assumed to be proportional to its density $B$ and the decrease of growth rate due to crowding is proportional to $B^2$. Similarly the depletion rate of bacteria by its predator protozoa is also assumed to be given by the monod type interaction of bacteria involving protozoa and so is the growth rate of protozoa. The natural depletion rate of protozoa is assumed to be proportional to $P$ and its decrease of growth rate due to crowding is proportional to $P^2$. We consider that the rate of growth of dissolved oxygen by various sources
is \( q \) (assumed a constant) and its natural depletion rate is proportional to its concentration \( C \). It is assumed that the depletion rate of dissolved oxygen caused by various interactions processes, as mentioned above, is proportional to terms, \( \left( \frac{K_1TB}{K_{12}+K_{11}T} \right) \), \( \left( \frac{K_2BP}{K_{21}+K_{22}B} \right) \), \( K_3PN \), \( \alpha_1B \), \( \alpha_2P \). It is assumed further that the depletion of dissolved oxygen by the biological population with density \( F \) is proportional to \( CF \) as it is wholly dependent on it. Hence, the growth rate of biological population is proportional to \( CF \). The depletion of biological population with density \( F \) due to crowding is assumed to be proportional to \( F^2 \) and its natural depletion rate is proportional to \( F \).

Keeping in view of these considerations, this system is governed by the following differential equations.

\[
\begin{align*}
\frac{dT}{dt} &= Q - \alpha_0 T - \frac{K_1TB}{K_{12} + K_{11}T} \\
\frac{dB}{dt} &= \frac{\lambda_1 K_1 TB}{K_{12} + K_{11}T} - \alpha_1 B - \frac{\lambda_{10}B^2 - \frac{K_2BP}{K_{21} + K_{22}B}}{K_{21} + K_{22}B} \\
\frac{dP}{dt} &= \frac{\lambda_2 K_2 BP}{K_{21} + K_{22}B} - \alpha_2 P - \frac{\lambda_{20}P^2}{K_{21} + K_{22}B} \\
\frac{dC}{dt} &= q - \alpha_3 C - \frac{\lambda_{12}K_1TB}{K_{12} + K_{11}T} - \frac{\lambda_{23}K_2BP}{K_{21} + K_{22}B} - \frac{\lambda_{11}\alpha_1 B - \lambda_{20}\alpha_2 P - K_3CF}{K_{21} + K_{22}B} \\
\frac{dF}{dt} &= \frac{\lambda_3 K_3 CF - \alpha_4 F - \lambda_{30}F^2}{K_{21} + K_{22}B}
\end{align*}
\]

where \( T(0) > 0, B(0) > 0, P(0) > 0, C(0) > 0, F(0) > 0 \).

Here, the coefficients \( \alpha_i \)'s are depletion rate coefficients, \( \lambda_i \)'s , and \( K_i \)'s (i=1,2,3) are proportionality constants, which are positive. The constants \( \lambda_{10} \), \( \lambda_{20} \), \( \lambda_{30} \) are coefficients corresponding to crowding terms(flaking off coefficients [12] of bacteria, protozoa and fish population respectively with respect to the aquatic habitat. It may be pointed out that for feasibility of the model (1), the growth rate of bacteria and protozoa should be positive. Hence, from the second and third equation of the model (1), it follows that

\[
\lambda_1 K_1 - K_{11} \alpha_1 > 0 \\
\lambda_3 K_2 - K_{22} \alpha_2 > 0
\]

Following [3], [13], [14] the region of attraction \( \Omega \) for all solutions initi-
surviving in the positive octant is given by

\[ \Omega := \left\{ \begin{array}{l}
0 \leq T \leq \frac{Q}{\alpha_0}, 0 \leq B \leq R_B, 0 \leq P \leq R_P, 0 \leq C \leq \frac{q}{\alpha_3}, \\
0 \leq F \leq \frac{\lambda_3 K_3 q}{\lambda_3 \alpha_3} \end{array} \right\} \quad (4) \]

where

\[ R_B = \frac{\lambda_1 K_1 Q}{\lambda_{10} (K_{11} Q + K_{12} \alpha_0)}, \]

and

\[ R_P = \frac{\lambda_1 \lambda_2 K_1 K_2 Q}{\lambda_{20} [\lambda_1 K_1 K_{22} Q + \lambda_{10} K_{21} (K_{11} Q + K_{12} \alpha_0)]}. \]

3. Equilibrium Analysis

System (1) has following six non-zero equilibria. They are listed below.

1. \( E_1 = \left( \frac{Q}{\alpha_0}, 0, 0, \frac{q}{\alpha_3}, 0 \right) \) always exists.

2. \( E_2 = \left( \frac{Q}{\alpha_0}, 0, 0, C_2^*, F_2^* \right) \) exists, provided the following condition is satisfied:

\[ \lambda_3 K_3 q - \alpha_3 \alpha_4 > 0 \quad (5) \]

3. \( E_3 = \left( T_3^*, B_3^*, 0, C_3^*, 0 \right) \) exists, provided the following conditions are satisfied:

\[ (\lambda_1 K_1 - K_{11} \alpha_1) Q - K_{12} \alpha_0 \alpha_1 > 0 \quad (6) \]

\[ q - \frac{\lambda_1 K_1 T_3^* B_3^*}{K_{12} + K_{11} T_3^*} - \lambda_1 \alpha_1 B_3^* > 0 \quad (7) \]

4. \( E_4 = \left( T_4^*, B_4^*, 0, C_4^*, F_4^* \right) \) exists, provided the following conditions are satisfied:

\[ (\lambda_1 K_1 - K_{11} \alpha_1) Q - K_{12} \alpha_0 \alpha_1 > 0 \quad (8) \]

\[ \lambda_3 K_3 q_1^* - \alpha_3 \alpha_4 > 0 \quad (9) \]

5. \( E_5 = \left( T_5^*, B_5^*, P_3^*, C_5^*, 0 \right) \) exists, provided the following conditions
are satisfied:

\[
\lambda_{20} \left[ (\lambda_1 K_1 - K_{11} \alpha_1) Q - K_{12} \alpha_0 \alpha_1 \right] + \frac{K_2 \alpha_2}{K_{21}} (K_{11} Q + K_{12} \alpha_0) > 0 \tag{10}
\]

\[
\lambda_{10} + \frac{K_2}{\lambda_{20} (K_{21} + K_{22} R_B)^2} \left[ \frac{\lambda_2 K_2 K_{21}}{(K_{21} + K_{22} R_B)} \right] - K_{22} \left( \frac{\lambda_2 K_2 R_B}{(K_{21} + K_{22} R_B)} - \alpha_2 \right) > 0 \tag{11}
\]

\[
\frac{\lambda_2 K_2 B^*_5}{(K_{21} + K_{22} B^*_5)} - \alpha_2 > 0 \tag{12}
\]

\[
q = \lambda_{12} \frac{K_1 T^*_5 B^*_3}{K_{12} + K_{11} T^*_5} - \lambda_{23} \frac{K_2 B^*_5 P^*_5}{K_{12} + K_{11} B^*_5}
- \lambda_{11} \alpha_1 B^*_5 - \lambda_{22} \alpha_2 P^*_5 > 0 \tag{13}
\]

6. \( E_6 = (T^*, B^*, P^*, C^*, F^*) \) exists, provided the following conditions are satisfied:

\[
\lambda_{20} \left[ (\lambda_1 K_1 - K_{11} \alpha_1) Q - K_{12} \alpha_0 \alpha_1 \right] + \frac{K_2 \alpha_2}{K_{21}} (K_{11} Q + K_{12} \alpha_0) > 0 \tag{14}
\]

\[
\lambda_{10} + \frac{K_2}{\lambda_{20} (K_{21} + K_{22} R_B)^2} \left[ \frac{\lambda_2 K_2 K_{21}}{(K_{21} + K_{22} R_B)} \right] - K_{22} \left( \frac{\lambda_2 K_2 R_B}{(K_{21} + K_{22} R_B)} - \alpha_2 \right) > 0 \tag{15}
\]

\[
\frac{\lambda_2 K_2 B^*}{K_{21} + K_{22} B^*} - \alpha_2 > 0 \tag{16}
\]

\[
\lambda_3 K_3 q^*_2 - \alpha_3 > 0 \tag{17}
\]

Here we note that

\[
T^*_3 = T^*_4, B^*_3 = B^*_4, T^*_5 = T^*, B^*_5 = B^*, P^*_5 = P^* \tag{18}
\]

\[
q_1^* = q - \frac{\lambda_{12} K_1 T^*_3 B^*_3}{K_{12} + K_{11} T^*_3} - \lambda_{11} \alpha_1 B^*_3 \tag{19}
\]

and

\[
q_2^* = q - \frac{K_1 T^*_5 B^*_5}{K_{12} + K_{11} T^*_5} - \lambda_{23} \frac{K_2 B^*_5 P^*_5}{K_{21} + K_{22} B^*_5}
- \lambda_{11} \alpha_1 B^*_5 - \lambda_{22} \alpha_2 P^*_5 \tag{20}
\]
The existence of these equilibria is shown in Appendix A.

Remark 1. Using equations (42) and (43) we can check that under the condition (15), $\frac{dT^*}{dQ} > 0$ and $\frac{dB^*}{dQ} > 0$. This gives $\frac{dB^*}{dQ} > 0$.

Also, from equation (45), we note that $\frac{dT^*}{dQ} = \frac{\lambda_2 K_2 K_{21}}{\lambda_2 (K_{21} + K_{22} B^*)^2} \frac{dB^*}{dQ}$

Hence $T^*$, $B^*$, $P^*$ increase as $Q$ increases.

And from equation (46) we note that,

$$\frac{dC^*}{dQ} = -\frac{1}{\alpha_3} \left[ \frac{\lambda_1 K_1 K_{12} B^*}{(K_{12} + K_{11} T^*)^2} \frac{dT^*}{dQ} + \left( \frac{\lambda_1 K_1 T^*}{(K_{12} + K_{11} T^*)} + \frac{\lambda_23 K_2 K_{21} P^*}{(K_{21} + K_{22} B^*)^2} \right) \frac{dB^*}{dQ} + \left( \frac{\lambda_23 K_2 B^*}{(K_{21} + K_{22} B^*)} + \lambda_22 \alpha_2 \right) \frac{dP^*}{dQ} \right]$$

Since $\frac{dT^*}{dQ}$, $\frac{dB^*}{dQ}$ and $\frac{dP^*}{dQ}$ are positive. Hence, from above equation (21) we observe that $\frac{dC^*}{dQ}$ is negative. This implies that as rate of discharge of organic pollutants $'Q'$ increases, then the equilibrium level of dissolved oxygen $C^*$ decreases.

From equation (48), we further note that $\frac{dF^*}{dQ} = \frac{\lambda_4 K_4}{\lambda_{10}} \frac{dC^*}{dQ}$, which is negative since $\frac{dC^*}{dQ}$ is negative. Thus the density of biological population decreases as the rate of discharge of organic pollutants $'Q'$ into the water body increases.

4. Stability Analysis

In our analysis we assume that all the above equilibria exist. The local stability behavior of these equilibria is then studied by computing the eigen values of the Jacobian matrix corresponding to each equilibria. Here it is found that $E_{i+1}$ exists only when $E_i$ $(i=1,2,3,4,5)$ is unstable.

The local stability behavior of $E_6$ is studied by using Liapunov’s method [9], which is given in the form of following theorem.

Theorem 2. The equilibrium $E_i$ $(i=1,2,3,4,5)$ is unstable whenever $E_{i+1}$ exists. The equilibria $E_6$ is locally asymptotically stable if the following condition is satisfied:

$$\lambda_{10} > \frac{K_2 K_{22}}{K_{21}^2} P^*$$

(22)
Proof. The proof of this theorem is given in Appendix B.

We further note that $E_6$ is nonlinearly asymptotically stable under certain conditions as stated in the following theorem:

**Theorem 3.** The equilibrium $E_6$ is non-linearly stable in $\Omega$ if the following conditions are satisfied:

$$\lambda_{10} > \frac{K_2K_{22}}{K_{21}} P^*$$  \hspace{1cm} (23)

$$\left[\frac{K_1T^*Q}{K_0(K_{11} + K_{21}T^*)}\right]^2 < \frac{\alpha_0T^*}{\alpha_1} \left[\lambda_{10} - \frac{K_2K_{22}}{K_{21}} P^*\right]$$  \hspace{1cm} (24)

Here we note that the condition (23) is same as the condition (22).

Proof. The proof of this theorem is given in Appendix C.

Keeping in view the remarks in §3 theorems 2. and 3. imply that if the rate of discharge of pollutants in a water body increases, the concentration of dissolved oxygen decreases, leading to decrease of the density of the biological population which is wholly dependent on it. It is also noted that if the rate of discharge of pollutants is very high, the biological population may be doomed to extinction.

5. Numerical Example

To check the feasibility of our analysis regarding the existence of $E_6$ and corresponding stability conditions, we conduct some numerical computation by choosing the following values of the parameters in model (1).

- $Q = 3.0$, $\alpha_0 = 1.0$, $K_1 = 1.0$, $K_{11} = 1.0$, $K_{12} = 0.1$
- $\lambda_1 = 1.0$, $\alpha_1 = 0.5$, $\lambda_{10} = 0.1$, $K_2 = 1.0$, $K_{21} = 2.0$, $K_{22} = 1.0$
- $\lambda_2 = 1.0$, $\alpha_2 = 0.5$, $\lambda_{20} = 1.0$, $q = 10.0$, $\alpha_3 = 2.0$
- $\lambda_{12} = \lambda_{23} = 1.5$, $\lambda_{11} = \lambda_{22} = 1.0$, $\lambda_3 = 1.0$, $K_3 = 1.0$, $\alpha_3 = 0.1$, $\lambda_{30} = 1.0$

It is found that under the above set of parameters, conditions for the existence of interior equilibrium $E_6 = (T^*, B^*, P^*, C^*, F^*)$ are satisfied and $E_6$ is given by

- $T^* = 0.475414$, $B^* = 3.055616$, $P^* = 0.1044$, $C^* = 1.382587$, $F^* = 1.282587$
The eigenvalues of the Jacobian matrix $M$ corresponding to this equilibrium $E_6$ are obtained as, 

$$
-0.11261, -1.10387 + 0.30852i, -1.10387 - 0.30852i, -2.28259 + 0.83937i, -2.28259 - 0.83937i
$$

which are either negative or have negative real parts. Hence $E_6$ is locally stable.

It is pointed out here that for the above set of parameters, the conditions for local stability and nonlinear stability are also satisfied.

Further, for the above set of parameters, a computer generated graph of $T$ versus $B$ and $T$ versus $F$ are shown in Fig.1 and Fig.2, which indicates the global stability of $(T^*, B^*)$ in the $T-B$ plane and $(T^*, F^*)$ in $T-F$ plane. Similarly the graphs of variables $B$, $C$ and $F$ are drawn with respect to time $t$ to see the effect of rate of input of organic pollutants i.e. $Q$ on above variables.

From fig.3, we note that as the cumulative rate of input of organic pollutants i.e. $Q$ increases the equilibrium level of the cumulative density of bacterial population increases. From fig. 4 it is clear that as $Q$ increases the equilibrium level of dissolved oxygen decreases and it may become zero for some large value of $Q$. In fig.5., the plot of $F$ with respect to $t$ is also shown from which it is clear that as $Q$ increases, the density of aquatic population $F$ decreases and it may become zero for large value
Figure 2: Global Stability in $T - F$ Plane

Figure 3: Variation of $B$ with respect to time $t$. 
Figure 4: Variation of $C$ with respect to time $t$.

Figure 5: Variation of $F$ with respect to time $t$. 
of $Q$.

In the following Table no. 1, effects of cumulative rate of input of organic pollutants (i.e. $Q$) on equilibrium levels of different variables are shown. It is noted here that as $Q$ increases $T^*$, $B^*$ and $P^*$ increases but $C^*$ and $F^*$ decreases as expected.

<table>
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<th>$T^*$</th>
<th>$B^*$</th>
<th>$P^*$</th>
<th>$C^*$</th>
<th>$F^*$</th>
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</table>

Table no. 1

6. Conclusion

In this paper, we have proposed and analyzed a nonlinear model to study the survival of a biological species in a dissolved oxygen-depleted water body such as a lake, caused by a discharge of organic pollutants. The problem is modelled by considering various non-linear processes involving organic pollutants, bacteria, protozoa, dissolved oxygen and a biological species, such as a fish population, wholly dependent on it. It has been assumed that oxygen is depleted by various non-linear biochemical and biodegradation processes occurring in the water body. It has been shown that as the rate of discharge of organic pollutants in the water body increases, the equilibrium concentration of dissolved oxygen decreases due to various interactive processes under certain conditions. Due to this the density of the biological population which is wholly dependent on it also decreases. If the rate of discharge of organic pollutants is very high, the equilibrium concentration of dissolved oxygen becomes negligible and the survival of the biological population may be threatened.
References


SURVIVAL OF BIOLOGICAL SPECIES

APPENDIX

A Existence of Equilibria:

The equilibria of model (1) are obtained by the following system of algebraic equations.

\[ Q - \alpha_0 T - \frac{K_1 TB}{K_{12} + K_{11} T} = 0 \]  
\[ \frac{\lambda_1 K_1 TB}{K_{12} + K_{11} T} - \alpha_1 B - \lambda_{10} B^2 - \frac{K_2 BP}{K_{21} + K_{22} B} = 0 \]  
\[ \frac{\lambda_2 K_2 BP}{K_{21} + K_{22} B} - \alpha_2 P - \lambda_{20} P^2 = 0 \]  
\[ q - \alpha_3 C - \frac{\lambda_{12} K_1 TB}{K_{12} + K_{11} T} - \frac{\lambda_{23} K_2 BP}{K_{21} + K_{22} B} - \lambda_{11} \alpha_1 B - \lambda_{22} \alpha_2 P - K_3 CF = 0 \]  
\[ \lambda_3 K_3 CF - \alpha_4 F - \lambda_{30} F^2 = 0 \]

From these equations, the existence of \( E_1 = (\frac{Q}{\alpha_0}, 0, 0, \frac{q}{\alpha_3}, 0) \) is obvious.

**Existence of \( E_2 \):** For the equilibrium \( E_2(\frac{Q}{\alpha_0}, 0, 0, C_2^*, F_2^*) \), the values of \( C_2^* \) and \( F_2^* \) are obtained by solving the following algebraic equations:

\[ q - \alpha_3 C - K_3 CF = 0 \]  
\[ \lambda_3 K_3 C - \alpha_4 - \lambda_{30} F = 0 \]

Eliminating \( C \) between equations (30) and (31), we find that \( F \) is given by a quadratic equation, where roots are:

\[ F = \frac{-(\lambda_{30} \alpha_3 + K_3 \alpha_4) \pm \sqrt{(\lambda_{30} \alpha_3 + K_3 \alpha_4)^2 + 4 \lambda_{30} K_3 (\lambda_3 K_3 q - \alpha_3 \alpha_4)}}{2 \lambda_{30} K_3} \]  

Hence, for \( F \) to be positive in equation (32), the following condition must be satisfied:

\[ \lambda_3 K_3 q - \alpha_3 \alpha_4 > 0 \]

Let the condition (33) be satisfied and \( F_2^* \) be the positive value of \( F \) in equation (32).
Then substituting this value of $F = F_2^*$ in equation (31) we get

$$C_2^* = \frac{1}{\lambda_3 K_3} (\alpha_4 + \lambda_{30} F_2^*) > 0$$  \hspace{1cm} (34)

Hence $E_2(Q, 0, 0, C_2^*, F_2^*)$ exists provided the condition (5) is satisfied.

**Existence of $E_3$** : For the equilibrium $E_3(T_3^*, B_3^*, 0, C_3^*, 0)$, the values of $T_3^*$ and $B_3^*$ are obtained by solving the following algebraic equations:

$$B = (Q - \alpha_0 T)(\frac{K_{12}}{K_{1T}} + \frac{K_{11}}{K_{1}})$$  \hspace{1cm} (35)

$$B = \frac{1}{\lambda_{10}} [\frac{\lambda_1 K_{1T}}{K_{12} + K_{11} T} - \alpha_1]$$  \hspace{1cm} (36)

From the isocline given by (35), we note the following:

(i) $B = 0$ at $T = \frac{Q}{\alpha_0}$

(ii) $B \rightarrow \infty$ as $T \rightarrow 0$.

(iii) $\frac{dB}{dT} = -\frac{K_{12} Q + K_{11} \alpha_0 T^2}{K_{1T}^2}$, which is negative for $T > 0$. Thus $B$ decreases as $T$ increases.

Similarly from equation (36), we note that $B = 0$ for $T = \frac{K_{12} \alpha_1}{\lambda_1 K_{12} - K_{11} \alpha_1}$, $B = \frac{-\alpha_1}{\lambda_{10}}$ for $T = 0$, and

$$\frac{dB}{dT} = \frac{\lambda_1 K_{12} K_{11}}{\lambda_{10} (K_{12} + K_{11} T)^2} > 0 \quad \text{for} \quad T > 0$$  \hspace{1cm} (37)

From the above analysis we note that the two isoclines (35) and (36) intersect at a unique point $(T_3^*, B_3^*)$ in the interior of first quadrant provided $\frac{K_{12} \alpha_1}{\lambda_1 K_{12} - K_{11} \alpha_1} < \frac{Q}{\alpha_0}$

i.e. $(\lambda_1 K_{1} - K_{11} \alpha_1)Q - K_{12} \alpha_0 \alpha_1 > 0$, [See Fig.6]  \hspace{1cm} (38)

Using these values of $T_3^*$ and $B_3^*$ in equation (28), we get

$$C_3^* = \frac{1}{\alpha_3} [q - \frac{\lambda_{12} K_{13} T_3^* B_3^*}{K_{12} + K_{11} T_3^*} - \lambda_{11} \alpha_1 B_3^*]$$  \hspace{1cm} (39)

which is positive if right hand side of (39) is positive, giving condition (7).
Hence the equilibrium point $E_3$ exists provided the conditions (6) and (7) are satisfied.

**Existence of $E_4$:** For the equilibrium $E_4(T_4^*, B_4^*, 0, C_4^*, F_4^*)$, we note that $T_4^* = T_3^*$ and $B_4^* = B_3^*$. The values of $C_4^*$ and $F_4^*$ are obtained by solving the following algebraic equations:

$$q_1^* - \alpha_3 C - K_3 CF = 0 \quad (40)$$
$$\lambda_3 K_3 C - \alpha_4 - \lambda_{30} F = 0 \quad (41)$$

where $q_1^*$ is given by equation (19).

Again, we can see that equations (40) and (41) are similar as equations (30) and (31).

Hence $C_4^*$ and $F_4^*$ exist, provided the condition (9) is satisfied.

Hence $E_4(T_4^*, B_4^*, 0, C_4^*, F_4^*)$ exists, provided the conditions (8) and (9) are satisfied.

**Existence of $E_5$:** For the equilibrium $E_5(T_5^*, B_5^*, P_5^*, C_5^*, 0)$, the
values of $T_5^*$ and $B_5^*$ are given by following algebraic equations:

$$B = (Q - \alpha_0 T) \left( \frac{K_{12}}{K_1 T} + \frac{K_{11}}{K_1} \right)$$  \hspace{1cm} (42)

$$\lambda_1 K_1 T \left( K_{11} - K_{12} T \right) - \alpha_1 = \lambda_{10} B + \frac{K_2}{\lambda_{20} (K_{21} + K_{22} B)} - \left[ \frac{\lambda_2 K_2 B}{K_{21} + K_{22} B} - \alpha_2 \right]$$  \hspace{1cm} (43)

The isocline (42) is the same as (35), hence its behavior is also the same.

From the isocline given by (43), it is noted that $B$ increases with $T$ in $0 \leq B \leq R_B$ [$R_B$ is given by (4)], provided the inequality (11) is satisfied.

Also $B = 0$ at $T = \frac{K_{12}(\lambda_2 K_{21} \alpha_1 - K_{22} \alpha_2)}{\lambda_{20} (K_{21} + K_{22} B)}$, which is positive or negative depending on the sign of $(\lambda_2 K_{21} \alpha_1 - K_{22} \alpha_2)$. It is noted that the value $B$ at $T = 0$ is given by the following cubic equation

$$\lambda_{10} \lambda_{20} K_{22}^2 B^3 + (2 \lambda_{10} \lambda_{20} K_{21} K_{22} + \lambda_{20} K_{22}^2 \alpha_1) B^2 + (\lambda_{10} \lambda_{20} K_{21}^2 + K_2 (\lambda_2 K_2 - K_{22} \alpha_2) + K_{22} \alpha_2 + 2 \lambda_{20} K_{21} K_{22} \alpha_1) B + K_{21} (\lambda_{20} K_{21} \alpha_1 - K_{22} \alpha_2) = 0$$  \hspace{1cm} (44)

It is clear that one root of equation (44) is positive or negative depending on whether $(\lambda_{20} K_{21} \alpha_1 - K_{22} \alpha_2)$ is negative or positive respectively.

Thus, from the above analysis we note that the graph of two isoclines (42) and (43) intersect at a unique point $(T_5^*, B_5^*)$ in the interior of first quadrant, drawn for $(\lambda_{20} K_{21} \alpha_1 - K_{22} \alpha_2) > 0$ provided conditions (10) and (11) are satisfied, [See Fig. 7].

Using these values of $T_5^*, B_5^*$ in equations (27) and (28), we get

$$P_5^* = \frac{1}{\lambda_{20}} \left( \frac{\lambda_2 K_2 B_5^*}{K_{21} + K_{22} B_5^*} - \alpha_2 \right)$$  \hspace{1cm} (45)

$$C_5^* = \frac{1}{\alpha_3} \left[ q - \frac{\lambda_{12} K_1 T_5^* B_5^*}{K_{12} + K_{11} T_5^*} - \frac{\lambda_{23} K_2 B_5^* P_5^*}{K_{21} + K_{22} B_5^*} - \lambda_{11} \alpha_1 B_5^* - \lambda_{22} \alpha_2 P_5^* \right]$$  \hspace{1cm} (46)

which is positive under the conditions (12) and (13) respectively. Thus, $E_5$ exists provided the conditions (10) - (13) are satisfied.

Keeping in view the model (1) it is noted that growth rate of bacteria in absence of protozoa is greater than in its presence as bacteria
is predated by protozoa. Therefore, using a comparison theorem [5], it can be easily seen that the population of bacteria is greater in absence of protozoa than when it is present. Thus $B_4^* > B_5^*$.

**Existence of $E_6$**: For the equilibrium $E_6(T^*, B^*, P^*, C^*, F^*)$, it may be noted that $T^* = T_5^*$, $B^* = B_5^*$ and $P^* = P_5^*$. The values of and are obtained by solving the following algebraic equations:

\[
q_2^* - \alpha_3 C - K_3 CF = 0 \quad (47)
\]

and

\[
\lambda_3 K_3 C - \alpha_4 - \lambda_{30} F = 0 \quad (48)
\]

where $q_2^*$ is given by equation (20).

Again we can note that equations (47) and (48) are similar as equations (30) and (31). Hence positive $C^*$ and $F^*$ exists, provided the condition (17) is satisfied. Thus, $E_6(T^*, B^*, P^*, C^*, F^*)$ exists, provided the conditions (14 - 17) are satisfied.
B Stability Analysis:

The general variational matrix of the model (1) is given by

\[ M = \begin{pmatrix}
    f_1(T, B) & -\frac{K_1}{T} & 0 & 0 & 0 \\
    -\frac{\lambda_1 K_1 K_{12}}{(K_{12} + K_{11} T)^2} & f_2(T, B, P) & -\frac{K_2}{B} & 0 & 0 \\
    0 & -\frac{\lambda_3 K_3 K_{21}}{K_{21} + K_{22} B^2} & f_3(B, P) & 0 & 0 \\
    g_1(T, B) & g_2(T, B, P) & g_3(B) & -(\alpha_3 + K_3 F) & -K_3 C \\
    0 & 0 & 0 & -\lambda_3 K_3 F & f_4(C, F)
\end{pmatrix} \]

where

\[ f_1(T, B) = -\left(\frac{K_1}{T} + \frac{K_1 K_{12} B}{(K_{12} + K_{11} T)^2}\right), \]
\[ f_2(T, B, P) = -\left(\frac{\lambda_1 K_1 K_{12}}{(K_{12} + K_{11} T)^2} - 2\lambda_2 - 2\lambda_2 P\right), \]
\[ f_3(B, P) = -\left(\frac{\lambda_3 K_3 K_{21}}{K_{21} + K_{22} B^2} + \frac{K_3 K_{21} P}{(K_{21} + K_{22} B)^2} + \lambda_1\alpha_1\right), \]
\[ g_1(T, B) = -\left(\frac{\lambda_1 K_1 K_{11}}{(K_{12} + K_{11} T)^2}\right), \]
\[ g_2(T, B, P) = -\left(\frac{\lambda_3 K_3 K_{11}}{(K_{12} + K_{11} T)^2}\right), \]
\[ g_3(B) = -\left(\frac{\lambda_3 K_3 K_{22}}{(K_{21} + K_{22} B)^2} + \lambda_2\alpha_2\right). \]

Let \( M_i \) be the matrix obtained from \( M \) after substituting for \( E_i (i = 1, 2, 3, 4, 5) \). For local stability of \( E_i \), we check the negativity of eigenvalues of \( M_i \).

For the equilibrium \( E_1 \), we note that one of the eigenvalues of \( M_1 \) is \( \frac{(\lambda_1 K_3 Q - \alpha_3\alpha_1)}{\alpha_3} \), which is positive whenever \( E_2 \) exists (condition (5)). Hence \( E_1 \) is unstable whenever \( E_2 \) exists.

For the equilibrium \( E_2\left(\frac{Q_0}{\alpha_3}, 0, 0, C_2^*, F_2^*\right) \), we note that one of the eigenvalues of \( M_2 \) is \( \frac{(\lambda_1 K_3 Q - \alpha_3\alpha_1)}{\alpha_3} \), which is positive whenever \( E_3 \) exists (condition (6)). Thus \( E_2 \) is unstable whenever \( E_3 \) exists.

For the equilibrium \( E_3\left(T_3^*, B_3^*, 0, C_3^*, 0\right) \), we note that one of the eigenvalues of \( M_3 \) is \( \frac{\lambda_1 K_3 Q - \alpha_3\alpha_1}{\alpha_3} \), which is positive whenever \( E_4 \) exists (condition (9)). Thus \( E_3 \) is unstable whenever \( E_4 \) exists.

For the equilibrium \( E_4\left(T_4^*, B_4^*, 0, C_4^*, F_4^*\right) \), we note that one of the eigenvalues of \( M_4 \) is \( \frac{\lambda_1 K_3 B_4^*}{\alpha_3} \), which is positive whenever \( E_5 \) exists (as \( B_4^* > B_5^* \)). Thus \( E_4 \) is unstable whenever \( E_5 \) exists.

For the equilibrium \( E_5\left(T_5^*, B_5^*, P_5^*, C_5^*, 0\right) \), we note that one of the eigenvalues of \( M_5 \) is \( \frac{\lambda_1 K_3 Q - \alpha_3\alpha_1}{\alpha_3} \), which is positive whenever \( E_6 \) exists (condition (17)). Thus \( E_5 \) is unstable whenever \( E_6 \) exists.

Since in our study \( E_6 \) is the most interesting equilibrium from ecological point of view and its behavior cannot be described in a simple
manner from $M_6$, we discuss its behavior by using Lyapunov’s method.

**Proof of theorem 2.**: Linearizing the system (1) by using the following transformations

$$T = T^* + \tau, B = B^* + b, P = P^* + p, C = C^* + c, F = F^* + f$$

(49)

and using the following positive definite function

$$V = \frac{1}{2}(\tau^2 + \frac{m_1}{B^*}b^2 + \frac{m_2}{P^*}p^2 + \frac{m_3}{C^*}c^2 + \frac{m_4}{F^*}f^2)$$

(50)

(where $m_1, m_2, m_3, m_4$ are some positive constants to be chosen appropriately).

Choosing $m_1 = \frac{T^*(K_{12} + K_{11}T^*)}{\lambda_1 K_{12}}, m_2 = m_1 \frac{(K_{21} + K_{22}B^*)}{\lambda_2 K_{21}},$ and $m_4 = m_3 \frac{C^*}{\lambda_3}$, \(dV\) is simplified as

$$\frac{dV}{dt} = - \left[ \alpha_0 + \frac{K_1 K_{12} B}{(K_{12} + K_{11}T^*)^2} \right] \tau^2 - m_1 \left[ \lambda_{10} \frac{K_2 K_{22} P^*}{(K_{21} + K_{22}B^*)^2} \right] b^2$$

$$- m_2 \lambda_{20} p^2 - m_3 (\alpha_3 + K_3 F^*) c^2 - m_4 \lambda_{30} f^2$$

$$- m_3 g_1 (T^*, B^*) \tau c - m_3 g_2 (T^*, B^*, P^*) b c - m_3 g_3 (B^*) pc$$

(51)

Here we note that if $\lambda_{10} > \frac{K_2 K_{22} P^*}{K_{21}}$, then coefficients of $b^2$ will be negative. Thus $\frac{dV}{dt}$ can be made negative definite by appropriately choosing a positive value for $m_3$ following usual methods of nonlinear analysis.

**C Proof of theorem 3.**

To prove this theorem we consider the following positive definite function,

$$V = \frac{1}{2}(T - T^*)^2 + m_1 (B - B^* - B^* \ln \frac{B}{B^*}) + m_2 (P - P^* - P^* \ln \frac{P}{P^*})$$

$$+ \frac{1}{2} m_3 (C - C^*)^2 + m_4 (F - F^* - F^* \ln \frac{F}{F^*})$$

(52)

Choosing $m_1 = \frac{T^*}{\lambda_1}, m_2 = m_1 \frac{(K_{21} + K_{22}B^*)}{\lambda_2 K_{21}},$ and $m_4 = m_3 \frac{C^*}{\lambda_3}$, \(dV\) reduces in the following form
\[
\frac{dV}{dt} = -\frac{K_1 K_{12} B (T - T^*)^2}{(K_{12} + K_{11} T)(K_{12} + K_{11} T^*)}
- m_1 K_2 K_{22} P^* \left( \frac{1}{K_{21}^2} - \frac{1}{(K_{21} + K_{22} B)(K_{21} + K_{22} B^*)} \right) (B - B^*)^2
- m_4 \lambda_{10} (F - F^*)^2 - \alpha_0 (T - T^*)^2 - m_1 \lambda_{10} - K_{22} P^* \left( \frac{K_{21}}{K_{21}^2} \right) (B - B^*)^2
- m_2 \lambda_{20} (P - P^*)^2 - m_3 \alpha_3 (C - C^*)^2
- (T - T^*)(B - B^*) \left[ \frac{K_1 K_{12} T^* T}{(K_{12} + K_{11} T)(K_{12} + K_{11} T^*)} \right]
- m_4 (T - T^*)(C - C^*) \left[ \frac{\lambda_{12} K_1 K_{12} B}{(K_{12} + K_{11} T)(K_{12} + K_{11} T^*)} \right]
- m_4 (B - B^*)(C - C^*) \left[ \frac{\lambda_{12} K_1 T^* + \lambda_{23} K_2 K_{21} P}{K_{12} + K_{11} T^*} + \frac{\lambda_{23} K_2 K_{21} P}{(K_{21} + K_{22} B)(K_{21} + K_{22} B^*)} + \lambda_{11} \alpha_1 \right]
- m_4 (P - P^*)(C - C^*) \left[ \frac{\lambda_{23} K_2 B^*}{K_{21} + K_{22} B^*} + \lambda_{22} \alpha_2 \right]
\] (53)

Note that \( \frac{1}{K_{21}^2} - \frac{1}{(K_{21} + K_{22} B)(K_{21} + K_{22} B^*)} > 0 \) inside region of attraction and \( \lambda_{10} - \frac{K_2 K_{22} P^*}{K_{21}^2} > 0 \) by condition (22).

Hence \( \frac{dV}{dt} \) can be made negative definite inside \( \Omega \) if

\[
\lambda_{10} > \frac{K_2 K_{22} P^*}{K_{21}^2}
\]

by making an appropriate choice for \( m_3 \) as pointed out earlier.

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