Unsteady-State Laminar Flow of Viscoelastic Gel and Air in a Channel: Application to Mucus Transport in a Cough Machine Simulating Trachea

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Abstract—Unsteady laminar flow of two immiscible fluids in a channel due to time-dependent pressure gradient has been considered. One of the fluids is a viscoelastic gel represented by the Maxwell model and the other one is air, which is taken as a Newtonian fluid of low viscosity. An approximate solution valid for small time is obtained by the Laplace transform method. Further, to fully analyze the problem a robust finite-difference scheme, which efficiently handles the sensitive interfacial boundary conditions of the model, has been developed. The analytical and numerical solution qualitative behavior compares favorably with the experimental observations of mucus gel transport in a cough machine. © 2003 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Simultaneous flow of immiscible fluids in a channel/pipe has caught the attention of several researchers [1–4] for its industrial and physiological significance. The analysis of a two-layer flow consisting of viscoelastic gel and air is relevant to the efficient transport of crude oils, lignites, cements, polymer solutions, or clay slurries in an industrial context [5]. Further, it can also throw light on mucus transport in the respiratory tract due to air motion caused by mild forced expiration under pathological conditions. Here the pressure gradient generated during air flow is time dependent.

To understand the mechanism of mucus transport in the lungs, several simulated experimental studies have been conducted with viscoelastic gel in tubes and channels (simulated cough machine) under externally applied constant or time-dependent pressure gradients [6–15]. In partic-
ular, Clarke et al. [8,9] studied experimentally the resistance to airflow, caused by short duration pressure drop, through a tube lined with sputum or with polymer solutions resembling mucus. They found that when the tube was lined with a mucus simulation, airflow resistance was higher than that of the case when the same tube was dry. They further noted that the magnitude of the excess resistance increased with increase in the depth of the fluid. Kim et al. [13] have studied two-phase gas liquid flow in a tube under continuous air flow and pointed out that elasticity of mucus has no effect on its transport speed. King et al. [10–12] have shown that gel clearance in a cough machine decreases with an increase in either the viscosity or the elastic modulus of the mucus gel. They have also found that the mucus transport increases as its thickness increases or as the applied pressure drop increases. Agarwal et al. [14] studied the mucus gel transport by airflow interaction in a miniaturized simulated cough machine with varying gap and found that mucus gel transport increases as the minimum viscosity of mucus gel decreases or as the minimum gap at the air entrance decreases. King et al. [12] have also studied the role of rheological properties of mucus on its transport and have shown that cough clearance decreases with increasing viscosity and yield stress but there is no apparent correlation between cough clearance and elastic modulus.

It may be noted here that few researchers have attempted to explain these experimental observations qualitatively or quantitatively using mathematical models. In view of this, in this paper, we study a planar two-layer laminar flow of viscoelastic gel and air between two infinite plates, relevant to mucus gel transport in a cough machine simulating model trachea, under the following assumptions.

1. The motion in a two-layer fluid system is considered unsteady as it is caused by an instantaneous time-dependent pressure gradient. This flow is assumed to be laminar relevant to the case of mild forced expiration in trachea.
2. The viscoelastic gel is represented by a Maxwell element and air is considered to be a low viscosity Newtonian incompressible fluid.

The current mathematical model describing the flow situation consists of a pair of coupled unsteady partial differential equations with well-defined interfacial, boundary, and initial conditions. For small values of time, where the governing partial differential equations are amenable to the standard Laplace transform method, an approximate solution is derived. Further, to fully analyze the model a numerical scheme in the implicit finite-difference framework has been developed. It is well known that numerical simulation of multilayer flows is highly sensitive to the way the interfacial boundary conditions are handled. In the finite-difference scheme developed here, the interfacial boundary conditions are efficiently handled in an implicit framework and a numerical solution is obtained. The analytical and numerical results are in reasonable agreement with each other and also with the qualitative behavior of the experimental results.

2. MATHEMATICAL MODEL AND ITS SOLUTION

We consider the unsteady-state simultaneous flow of a viscoelastic gel and air between two infinite plates, relevant to mucus gel transport in a cough machine simulating a model trachea. The flow is assumed to be caused by instantaneous pressure gradient generated by air motion simulating mild forced expiration in trachea. The physical situation is shown in Figure 1 where viscoelastic gel \((0 \leq y \leq h_m)\) and air \((h_m \leq y \leq h_a)\) regions are indicated. The equations governing the laminar motion of viscoelastic gel and air under unsteady-state condition can be written as follows.

**Region I** \((0 \leq y \leq h_m)\):

\[
\rho_m \frac{\partial u_m}{\partial t} = \frac{\partial p}{\partial x} + \frac{\partial \tau_m}{\partial y}. \tag{1}
\]
Figure 1. Physical situation of the flow of viscoelastic gel and air in a channel. Viscoelastic gel occupies the region $0 \leq y \leq h_m$ and air occupies the region $h_m \leq y \leq h_a$.

Since viscoelastic gel is considered as a Maxwell fluid, its constitutive equation can be written as follows [16]:

$$\tau_m + \lambda \frac{\partial \tau_m}{\partial t} = \mu_m \frac{\partial u_m}{\partial y}. \quad (2)$$

REGION II ($h_m \leq y \leq h_a$):

$$\rho_a \frac{\partial u_a}{\partial t} = -\frac{\partial p}{\partial x} + \mu_a \frac{\partial^2 u_a}{\partial y^2}. \quad (3)$$

In equations (1)-(3), $p$ is the pressure, $u_a$ and $u_m$ are the velocity components of air and viscoelastic gel in the $x$ direction, respectively, $\mu_a$ and $\rho_a$ are respectively the viscosity and the density of air, $\rho_m$ is the density of viscoelastic gel, $\tau_m$ is the shear stress in the viscoelastic gel layer, $\lambda = \mu_m/G$ is the relaxation time, where $\mu_m$ and $G$ are the viscosity and elastic modulus of viscoelastic gel.

The initial conditions for system (1)-(3) are

$$u_a = 0, \quad u_m = 0, \quad \tau_m = 0, \quad \text{at} \quad t = 0, \quad (4)$$

and the boundary conditions are

$$u_m = 0, \quad \text{at} \quad y = 0, \quad (5)$$

$$u_a = 0, \quad \text{at} \quad y = h_a. \quad (6)$$

Since the velocities and stresses are continuous at the interface $y = h_m$, the matching conditions are

$$u_m = u_a = U_1, \quad \tau_m = \mu_a \frac{\partial u_a}{\partial y}, \quad \text{at} \quad y = h_m, \quad (7)$$

where $U_1$ is the velocity at the air-viscoelastic gel interface which can be determined by using the second condition of (7).

In the following analysis, we assume that the pressure gradient generated due to airflow is time dependent and is represented in the following form:

$$-\frac{\partial p}{\partial x} = \phi_0 f(t) = \phi_p(t), \quad (8)$$

where $\phi_0$ is constant and $f(t)$ is prescribed by the following expression:

$$f(t) = \begin{cases} \frac{27}{4T^3} t(T - t)^2, & 0 \leq t \leq T, \\ 0, & t > T. \end{cases} \quad (9)$$

Here $T$ is the duration of airflow. The function $f(t)$ is shown in Figure 2 for $T = 0.03$ sec.
3. APPROXIMATE SOLUTION

To solve this model, Laplace transform method is applied to system (1)–(7). Solving the resulting equations along with conditions (5)–(7), the transformed velocity components are given by

\[
\bar{u}_m(s) = \bar{U}_1 \frac{\sinh K_m y}{\sinh K_m h_m} + \frac{\bar{\phi}_p}{\psi_m K_m^2} \left[ 1 - \frac{\sinh K_m y + \sinh K_m (h_m - y)}{\sinh K_m h_m} \right],
\]

\[
\bar{u}_a(s) = \bar{U}_1 \frac{\sinh K_a (h_a - y)}{\sinh K_a (h_a - h_m)} + \frac{\bar{\phi}_p}{\mu_a K_a^2} \left[ 1 - \frac{\sinh K_a (h_a - y) + \sinh K_a (y - h_m)}{\sinh K_a (h_a - h_m)} \right],
\]

where \(\bar{u}_m, \bar{u}_a, \bar{U}_1,\) and \(\bar{\phi}_p\) denote the Laplace transform of \(u_m, u_a, U_1,\) and \(\phi_p\), respectively. They are defined as

\[
\bar{u}_m(s) = \int_0^\infty u_m e^{-st} \, dt, \quad \text{for } u_m, \text{ etc.,}
\]

\[
K_m^2 = \frac{sp_m}{\mu_m}, \quad K_a^2 = \frac{sp_a}{\mu_a}, \quad \frac{1}{\psi_m} = \left[ \frac{1}{\mu_m} + \frac{s}{G} \right].
\]

The second condition of (7) gives the expression for \(\bar{U}_1\) as

\[
\bar{U}_1 \left[ \mu_a K_a \coth K_a (h_a - h_m) + \frac{sp_m}{K_m} \coth K_m h_m \right] = \frac{\bar{\phi}_p}{K_a} \tanh \frac{K_a (h_a - h_m)}{2} + \frac{\bar{\phi}_p}{K_m} \tanh \frac{K_m h_m}{2}.
\]

The volumetric flow rate per unit thickness in each layer (\(Q_m\) for viscoelastic gel and \(Q_a\) for air) can be defined as

\[
Q_m = \int_0^{h_m} u_m \, dy, \quad Q_a = \int_{h_m}^{h_a} u_a \, dy.
\]

Therefore, using equations (9) and (10) the flow rate in the transformed form can be found as

\[
\bar{Q}_m = \frac{\bar{U}_1}{K_m} \tanh \frac{K_m h_m}{2} + \frac{\bar{\phi}_p}{sp_m} \left[ h_m - \frac{2 \tanh (K_m h_m/2)}{K_m} \right],
\]

\[
\bar{Q}_a = \frac{\bar{U}_1}{K_a} \tanh \frac{K_a (h_a - h_m)}{2} + \frac{\bar{\phi}_p}{sp_a} \left[ (h_a - h_m) - \frac{2 \tanh (K_a (h_a - h_m)/2)}{K_a} \right],
\]

where the expression for \(\bar{U}_1\) is given by equation (11).

The expressions for \(\bar{Q}_m\) and \(\bar{Q}_a\) are complicated and are not amenable to inverse transform. Therefore, we attempt here to find an approximate solution which is valid for small time (i.e., corresponds to large \(s\)). Accordingly, the expressions for \(\bar{Q}_m\) and \(\bar{Q}_a\) are approximated for large \(s\).
By taking $K_m h_m$ and $K_a (h_a - h_m)$ large such that $\tanh K_m h_m \approx 1$, $\tanh K_a (h_a - h_m) \approx 1$, the expressions for $\bar{Q}_m$ and $\bar{Q}_a$ from equations (13) and (14) can be approximated as follows:

$$
\bar{Q}_m = \frac{\bar{\rho}_p}{s \rho_m} \left[ h_m + \frac{1}{K_a} \left( 1 - \frac{\rho_a}{\rho_m} \right) - \frac{1}{K_m} \right],
$$

$$
\bar{Q}_a = \frac{\bar{\rho}_p}{s \rho_a} \left[ (h_a - h_m) - \frac{1}{K_a} \left( 1 - \frac{\rho_a}{\rho_m} \right) - \frac{1}{K_a} \left( 1 + \frac{\rho_a}{\rho_m} \right) \right].
$$

From the expressions for $K_m$ and $K_a$, we have

$$
\frac{\rho_a K_m}{\rho_m K_a} = \left[ \frac{\rho_a \mu_a}{\rho_m \mu_m} \left( 1 + \frac{s \mu_m}{G} \right) \right]^{1/2}.
$$

Since $\rho_a$ and $\mu_a$ are very small, we have $\rho_a K_m / \rho_m K_a \ll 1$ for moderate or large $G$, which gives

$$
\left( 1 + \frac{\rho_a K_m}{\rho_m K_a} \right)^{-1} \approx 1 - \frac{\rho_a K_m}{\rho_m K_a}.
$$

Thus, from the above equations, the expressions for $\bar{Q}_m$ and $\bar{Q}_a$ can be simplified as

$$
\bar{Q}_m = \frac{\bar{\rho}_p}{s \rho_m} \left[ h_m + \frac{1}{K_a} \left( 1 - \frac{\rho_a}{\rho_m} \right) - \frac{1}{K_m} \right],
$$

(15)

$$
\bar{Q}_a = \frac{\bar{\rho}_p}{s \rho_a} \left[ (h_a - h_m) \left( 1 - \frac{\rho_a}{\rho_m} \right) - \frac{1}{K_a} \right].
$$

(16)

From equations (15) and (16), we observe the following:

1. $\bar{Q}_m$ increases as $\bar{\rho}_p$ increases and the air gap $(h_a - h_m)$ increases.

2. $\bar{Q}_m$ increases as $\bar{\rho}_p$ increases. It also increases as the thickness of viscoelastic gel $h_m$ increases.

3. Since $\rho_a K_m \ll \rho_m K_a$, so for moderate or large $G$ from equation (15), we have

$$
\bar{Q}_m \approx \frac{\bar{\rho}_p}{s \rho_m} \left[ h_m + \frac{1}{K_a} - \frac{1}{K_m} \right].
$$

It can be noted from this equation that $\bar{Q}_m$ increases as $K_m$ increases, i.e., as $\mu_m$ decreases or $G$ decreases.

Since the expression for $K_m$ is complicated, we consider the following two cases to find the inverse Laplace transform for $\bar{Q}_m$ and $\bar{Q}_a$.

1. $\lambda$ be such that $s \lambda \gg 1$. In this case, we have

$$
K_m \approx \left( \frac{\rho_m}{G} \right)^{1/2} \left( s + \frac{1}{2 \lambda} \right).
$$

2. $\lambda$ be such that $s \lambda \ll 1$. In this case, we have

$$
K_m \approx \left( \frac{s \rho_m}{\mu_m} \right)^{1/2} \left( 1 + \frac{s \lambda}{2} \right).
$$

3.1. The Case $s \lambda \gg 1$

Under this approximation, the expressions for $\bar{Q}_m$ and $\bar{Q}_a$ (from equations (15) and (16)), can be approximated as follows:

$$
\bar{Q}_m = \frac{\bar{\rho}_p}{s \rho_m} \left[ h_m + \left( 1 - \frac{\rho_a}{\rho_m} \right) \left\{ \left( \frac{\mu_a}{s \rho_a} \right)^{1/2} - \left( \frac{\mu_a}{G \rho_m} \right)^{1/2} \left( 1 + \frac{1}{2 \lambda} \right) \right\} - \left( \frac{G}{\rho_m} \right)^{1/2} \frac{1}{s + 1/2 \lambda} \right],
$$

$$
\bar{Q}_a = \frac{\bar{\rho}_p}{s \rho_a} \left[ (h_a - h_m) - \left( 1 - \frac{\rho_a}{\rho_m} \right) \left\{ \left( \frac{\mu_a}{s \rho_a} \right)^{1/2} - \left( \frac{\mu_a}{G \rho_m} \right)^{1/2} \left( 1 + \frac{1}{2 \lambda} \right) \right\} - \left( \frac{\mu_a}{s \rho_a} \right)^{1/2} \right].
$$
By taking the inverse Laplace transform and using equation (5), we have the following:

\[
Q_m = \frac{27\phi_0}{4\rho_m T^3} \left[ T^2 f_1(t) - 2T \{ 2f_2(t) + f_3(t-T)u(t-T) \} + 6 \{ f_5(t) - f_3(t-T)u(t-T) \} \right],
\]

\[
Q_a = \frac{27\phi_0}{4\rho_a T^3} \left[ T^2 f_4(t) - 2T \{ 2f_5(t) + f_6(t-T)u(t-T) \} + 6 \{ f_6(t) - f_6(t-T)u(t-T) \} \right],
\]

where

\[
f_1(t) = \frac{h_m t^2}{2} + \frac{8t^2}{15} \left( 1 - \frac{\rho_a}{\rho_m} \right) \left( \frac{\mu_a}{\rho_a} \right)^{1/2} \left\{ t \left( \frac{t}{\pi} \right)^{1/2} - \frac{15}{16} \left( \frac{\rho_a \mu_a}{G \rho_m} \right)^{1/2} \left( 1 + \frac{t}{6\lambda} \right) \right\} - 8\lambda^3 \left( \frac{G}{\rho_m} \right)^{1/2} \left\{ 1 - \frac{t}{2\lambda} + \frac{t^2}{8\lambda^2} - \exp \left( -\frac{t}{2\lambda} \right) \right\},
\]

\[
f_2(t) = \frac{h_m t^3}{6} + \frac{16t^3}{105} \left( 1 - \frac{\rho_a}{\rho_m} \right) \left( \frac{\mu_a}{\rho_a} \right)^{1/2} \left\{ t \left( \frac{t}{\pi} \right)^{1/2} - \frac{35}{32} \left( \frac{\rho_a \mu_a}{G \rho_m} \right)^{1/2} \left( 1 + \frac{t}{8\lambda} \right) \right\} - 16\lambda^4 \left( \frac{G}{\rho_m} \right)^{1/2} \left\{ \exp \left( -\frac{t}{2\lambda} \right) + \frac{t^3}{48\lambda^3} - \frac{t^2}{8\lambda^2} - \frac{t}{2\lambda} - 1 \right\},
\]

\[
f_3(t) = \frac{h_m t^4}{24} + \frac{32t^4}{945} \left( 1 - \frac{\rho_a}{\rho_m} \right) \left( \frac{\mu_a}{\rho_a} \right)^{1/2} \left\{ t \left( \frac{t}{\pi} \right)^{1/2} - \frac{315}{256} \left( \frac{\rho_a \mu_a}{G \rho_m} \right)^{1/2} \left( 1 + \frac{t}{10\lambda} \right) \right\} - 32\lambda^5 \left( \frac{G}{\rho_m} \right)^{1/2} \left\{ 1 - \frac{t}{2\lambda} + \frac{t^2}{8\lambda^2} - \frac{t^3}{32\lambda^3} + \frac{t^4}{384\lambda^4} - \exp \left( -\frac{t}{2\lambda} \right) \right\},
\]

\[
f_4(t) = \frac{(h_a - h_m) t^2}{2} - \frac{8t^2}{15} \left( 1 - \frac{\rho_a}{\rho_m} \right) \left( \frac{\mu_a}{\rho_a} \right)^{1/2} \left\{ t \left( \frac{t}{\pi} \right)^{1/2} - \frac{15}{16} \left( \frac{\rho_a \mu_a}{G \rho_m} \right)^{1/2} \left( 1 + \frac{t}{6\lambda} \right) \right\} - \frac{8\lambda^2}{15} \left( \frac{\mu_a t}{\pi \rho_a} \right)^{1/2},
\]

\[
f_5(t) = \frac{(h_a - h_m) t^3}{6} - \frac{16t^3}{105} \left( 1 - \frac{\rho_a}{\rho_m} \right) \left( \frac{\mu_a}{\rho_a} \right)^{1/2} \left\{ t \left( \frac{t}{\pi} \right)^{1/2} - \frac{35}{32} \left( \frac{\rho_a \mu_a}{G \rho_m} \right)^{1/2} \left( 1 + \frac{t}{8\lambda} \right) \right\} - \frac{16t^2}{105} \left( \frac{\mu_a t}{\pi \rho_a} \right)^{1/2},
\]

\[
f_6(t) = \frac{(h_a - h_m) t^4}{24} - \frac{32t^4}{945} \left( 1 - \frac{\rho_a}{\rho_m} \right) \left( \frac{\mu_a}{\rho_a} \right)^{1/2} \left\{ t \left( \frac{t}{\pi} \right)^{1/2} - \frac{315}{256} \left( \frac{\rho_a \mu_a}{G \rho_m} \right)^{1/2} \left( 1 + \frac{t}{10\lambda} \right) \right\} - \frac{32t^4}{945} \left( \frac{\mu_a t}{\pi \rho_a} \right)^{1/2}.
\]

3.2. The Case \( s\lambda \ll 1 \)

From equations (15) and (16), the expressions for \( \bar{Q}_m \) and \( \bar{Q}_a \) are approximated as follows:

\[
\bar{Q}_m = \frac{\phi_p}{s \rho_m} \left[ h_m + \left( 1 - \frac{\rho_a}{\rho_m} \right) \left( \frac{\mu_a}{s \rho_a} \right)^{1/2} \left\{ 1 - \left( \frac{\rho_a \mu_a}{s \rho_m \rho_m} \right)^{1/2} \left( 1 + \frac{s\lambda}{2} \right) \right\} - \left( \frac{\mu_m}{s \rho_m} \right)^{1/2} \left( 1 - \frac{s\lambda}{2} \right) \right],
\]

\[
\bar{Q}_a = \frac{\phi_p}{s \rho_a} \left[ (h_a, h_m) \left( 1 - \frac{\rho_a}{\rho_m} \right) \left( \frac{\mu_a}{s \rho_a} \right)^{1/2} \left\{ 1 - \left( \frac{\rho_a \mu_a}{s \rho_m \mu_m} \right)^{1/2} \left( 1 + \frac{s\lambda}{2} \right) \right\} \left( \frac{\mu_a}{s \rho_a} \right)^{1/2} \right].
\]
By inverse Laplace transform and using boundary condition (5), we have the following:

\[ Q_m = \frac{27\phi_0}{4\rho_mT^3} \left[ T^2 f_7(t) - 2T \left\{ 2f_8(t) + f_8(t-T)u(t-T) \right\} + 6 \left\{ f_9(t) - f_9(t-T)u(t-T) \right\} \right], \quad (19) \]

\[ Q_a = \frac{27\phi_0}{4\rho_aT^3} \left[ T^2 f_{10}(t) - 2T \left\{ 2f_{11}(t) + f_{11}(t-T)u(t-T) \right\} + 6 \left\{ f_{12}(t) - f_{12}(t-T)u(t-T) \right\} \right] \quad (20) \]

where

\[
\begin{align*}
    f_7(t) &= \frac{h_m t^2}{2} + \frac{8t}{15} \left( 1 - \frac{\rho_a}{\rho_m} \right) \left( \frac{\mu_a t}{\rho_a} \right)^{1/2} \left\{ t - \left( \frac{\rho_a \mu_a}{\rho_m \mu_m} \right)^{1/2} \left( t + \frac{5\lambda}{4} \right) \right\} \\
    &\quad - \frac{8t}{15} \left( \frac{\mu_m t}{\rho_m} \right)^{1/2} \left( t - \frac{5\lambda}{4} \right), \\
    f_8(t) &= \frac{h_m t^3}{6} + \frac{16t^2}{105} \left( 1 - \frac{\rho_a}{\rho_m} \right) \left( \frac{\mu_a t}{\rho_a} \right)^{1/2} \left\{ t - \left( \frac{\rho_a \mu_a}{\rho_m \mu_m} \right)^{1/2} \left( t + \frac{7\lambda}{4} \right) \right\} \\
    &\quad - \frac{16t^2}{105} \left( \frac{\mu_m t}{\rho_m} \right)^{1/2} \left( t - \frac{7\lambda}{4} \right), \\
    f_9(t) &= \frac{h_m t^4}{24} + \frac{32t^3}{945} \left( 1 - \frac{\rho_a}{\rho_m} \right) \left( \frac{\mu_a t}{\rho_a} \right)^{1/2} \left\{ t - \left( \frac{\rho_a \mu_a}{\rho_m \mu_m} \right)^{1/2} \left( t + \frac{9\lambda}{4} \right) \right\} \\
    &\quad - \frac{32t^3}{945} \left( \frac{\mu_m t}{\rho_m} \right)^{1/2} \left( t - \frac{9\lambda}{4} \right), \\
    f_{10}(t) &= \frac{(h_a - h_m) t^2}{2} - \frac{8t}{15} \left( 1 - \frac{\rho_a}{\rho_m} \right) \left( \frac{\mu_a t}{\rho_a} \right)^{1/2} \left\{ t - \left( \frac{\rho_a \mu_a}{\rho_m \mu_m} \right)^{1/2} \left( t + \frac{5\lambda}{4} \right) \right\} \\
    &\quad - \frac{8t^2}{15} \left( \frac{\mu_a t}{\rho_a} \right)^{1/2}, \\
    f_{11}(t) &= \frac{(h_a - h_m) t^3}{6} - \frac{16t^2}{105} \left( 1 - \frac{\rho_a}{\rho_m} \right) \left( \frac{\mu_a t}{\rho_a} \right)^{1/2} \left\{ t - \left( \frac{\rho_a \mu_a}{\rho_m \mu_m} \right)^{1/2} \left( t + \frac{7\lambda}{4} \right) \right\} \\
    &\quad - \frac{16t^2}{105} \left( \frac{\mu_a t}{\rho_a} \right)^{1/2}, \\
    f_{12}(t) &= \frac{(h_a - h_m) t^4}{24} - \frac{32t^3}{945} \left( 1 - \frac{\rho_a}{\rho_m} \right) \left( \frac{\mu_a t}{\rho_a} \right)^{1/2} \left\{ t - \left( \frac{\rho_a \mu_a}{\rho_m \mu_m} \right)^{1/2} \left( t + \frac{9\lambda}{4} \right) \right\} \\
    &\quad - \frac{32t^3}{945} \left( \frac{\mu_a t}{\rho_a} \right)^{1/2}.
\end{align*}
\]

It may be noted here that our analytical solution is obtained under the assumptions that \( K_m h_m \) and \( K_a (h_a - h_m) \) are large. This implies that there are certain restrictions on \( h_m \) and \( (h_a - h_m) \) as well as on \( \mu_m, \mu_a, \) and \( t \). To overcome this difficulty, the governing equations have also been solved numerically. The method is described in the following section.

4. NUMERICAL SOLUTION

For the computational simulation of transient nonsymmetrical two layer fluid flow model of viscoelastic gel and air, a new implicit finite-difference scheme has been evolved. Equations (1)–(3) governing the flow of viscoelastic gel and air together with the matching conditions (7), boundary
conditions (5) and (6) and initial conditions (7) have been solved using a new implicit scheme. For the simulation purposes, a vertical cross-section \( ABC \) in the planar two layer flow domain has been considered. Segment \( AB \) corresponds to the gel layer and \( BC \) to the air layer. A finite-difference grid consisting of \( n_1 \) points in the gel layer and \( n_2 \) points in the air layer has been introduced. If \( t_g \) is the target time for the numerical simulation and \( \delta t \) is the chosen time step size, then the target time will be reached in \( M(= t_g/\delta t) \) steps. The finite-difference grid employed for the flow simulations is given in Figure 3. At a typical grid point \( P_1 \) in the region \( AB \), the field variables are \( u_{m_i}^j \), and at a typical grid point \( P_2 \) in the region \( BC \), the field variables are \( u_{n_i}^j \), where \( 0 \leq i \leq n_1 + n_2 - 1, 0 \leq j \leq M \).

**Figure 3. Finite-difference grid.**

### 4.1. Development of the Finite-Difference Scheme

The implicit finite-difference analog of equations (1)–(3) along with the initial and boundary conditions (4)–(6) and the matching condition (7) at the interface is solved simultaneously for \( u_{m_i}^j \) and \( u_{n_i}^j \). The implicit finite-difference analog is obtained by using forward differences for temporal derivatives and Crank Nicholson based central differences for spatial derivatives as follows.

**For Viscoelastic Gel Layer.**

\[
\begin{align*}
    c_1 u_{m_{i+1}}^{j+1} + c_2 u_{m_i}^{j+1} + c_1 u_{m_i}^{j-1} &= \phi_0 \left( f(t) + \frac{d(f(t))}{dt} \right)^{j-1} \delta t^2 + R_1 u_{m_i}^{j-1} - \rho_m u_{m_i}^{j-2}, \\
    &\quad \text{for } 0 < i < n_1, \quad 0 \leq j \leq M,
\end{align*}
\]

where, \( f(t) \) is given by (8) and

\[
\begin{align*}
    c_1 &= -\frac{\mu_m \delta t^2}{2y_1^2}, \quad c_2 = \rho_m (\lambda + \delta t) + \frac{\mu_m \delta t^2}{y_1^2}, \quad R_1 = \rho_m (2\lambda + \delta t) - \frac{\mu_m \delta t^2}{y_1^2}.
\end{align*}
\]

The finite-difference analog of the boundary condition (5) at the grid point \( i = 0 \) is the following:

\[
u_{m_1}^{j+1} = 0.
\]
The finite-difference representation of the interface conditions (7) at \( i = n_1 \) is given by

\[
I_1 u_{m_{n_1}}^{i-1} + I_2 u_{m_{n_1}}^i + I_3 u_{a_{n_1+1}}^i = I_4 \left( u_{a_{n_1}}^{i-1} - u_{a_{n_1+1}}^{i-1} \right),
\]

(22)

where \( I_1 = -\left( \mu_m \delta y_1 \right) \), \( I_2 = -(I_1 + I_3) \), \( I_3 = -\left( \mu_a \delta y_2 \right) - I_4 \), \( I_4 = \lambda \mu_a \delta t \delta y_2 \), \( \delta y_1 \) and \( \delta y_2 \) are the spatial step sizes in Regions I and II, respectively.

**FOR AIR LAYER.** For \( n_1 < i < n_1 + n_2 - 1 \) the grid points correspond to the air layer and the scheme is as follows:

\[
d_1 u_{a_{i+1}}^i + d_2 u_{a_{i}}^i + d_1 u_{a_{i-1}}^i = (\phi_p)^{i-1} \delta t + R_2 u_{a_{i-1}}^{i-1} - d_1 \left( u_{a_{i+1}}^{i-1} + u_{a_{i-1}}^{i-1} \right),
\]

(23)

where \( d_1 = -\left( \mu_a \delta t / 2 \delta y_2^2 \right) \), \( d_2 = \rho_a \), \( \phi_p = \mu_a \delta t / \delta y_2^2 \), \( R_2 = \rho_a - \mu_a \delta t / \delta y_2^2 \).

The interface corresponding to \( i = n_1 \) has already been taken care of.

At \( i = n_1 + n_2 - 1 \), corresponding to the no-slip condition at the boundary can be written as follows:

\[
u_{a_{n_1+n_2-1}}^{i+1} = 0.
\]

(24)

### 4.2. Algorithm for the Simulation of the Flow Dynamics

**Notations**

\( j \) time level \( t_c \) current time \( t_g \) target time

**Begin Step 0.** \( j \leftarrow 0 \), \( t_c \rightarrow 0 \)

**Step 1.** If \( j = 0 \) then

Based on the finite-difference analog (21)-(24) of the governing equations (1),(2), matching conditions (7) and the boundary conditions (5),(6) solve for \( u_{a_{i}}^j \) and \( u_{d_{i}}^j \), using the initial conditions (4)

Else

Solve simultaneously for \( u_{d_{i}}^j \) and \( u_{m_{i}}^j \) based on (21)-(24) using \( u_{d_{i}}^{j-1} \), \( u_{m_{i}}^{j-1} \), and \( u_{m_{i}}^{j-2} \).

**End**

**Step 2.** Check for termination

If \( t_c < t_g \) then

- Store \( u_{d_{i}}^j \), \( u_{m_{i}}^j \)
- Update \( t_c \), \( j \)

goto step 1

endif

**Step 3.** Evaluate \( Q_a \) and \( Q_m \) at \( t_c = t_g \) using (12)

Stop

The scheme is validated by

(i) testing for the steady-state solutions for large values of \( t \) when constant pressure gradient is applied

(ii) checking for the convergence of the solution with increasing grid points and time steps.

It is worth reiterating that the Achilles Heel in the numerical simulation of the multilayer fluid flow problem is in handling the interface conditions and the current scheme is sound in this respect. The scheme is fully implicit and has been found to be stable for all the practical time steps.

### 5. RESULTS AND DISCUSSION

To see the effect of various parameters on gel flow rate, the values of these parameters are chosen keeping in mind the application of the model to cough machine simulating trachea. Hence, following King et al. [11] and Agarwal et al. [14], the numerical computations are made by taking

\[ h_a = 1.2 \text{ cm}, \quad \rho_a = 1.0 \times 10^{-3} \text{ gm/cm}^3, \quad \rho_m = 1.0 \text{ gm/cm}^3, \quad \mu_a = 2.0 \times 10^{-4} \text{ poise} \]
along with the following range of the values for the other parameters:

\[ h_m : 0.10 - 0.20 \text{ cm}; \quad \phi_0 : 0.0 - 4.0 \times 10^2 \text{ gm/cm}^2 \text{ sec}^2; \]
\[ \mu_m : 0.5 - 10.0 \text{ poise}; \quad G : 10.0 - \infty \text{ dyne/cm}^2. \]

It may be pointed out here that for defining the function \( f(t) \) in \( \phi_p(t) \), we have taken \( T = 0.03 \text{ sec} \). Further, to see the effect of cough on transportation of the mucus gel, the results have been presented at the time instant \( t = t_g = 0.02 \text{ sec} \), which corresponds to the time after the peak of the pressure pulse has been achieved.

The influence of pressure gradient and the thickness of viscoelastic gel on air flow and gel flow rates has been analyzed both analytically and numerically by finite-difference method (FDM). The results are presented in Figure 4 for \( h_m = 0.10, 0.15, 0.20 \text{ cm} \). These calculations are carried out with \( \mu_m = 10.0 \text{ poise} \) and \( G = 50.0 \text{ dyn/cm}^2 \). From Figures 4a and 4b, it is clear that for a

![Figure 4](image)

**Figure 4.** Variation of \( Q_a \) and \( Q_m \) with \( \phi_0 \) for different \( h_m \) (solid line → numerical value, \( \square \) → analytical value). The labels 1-3 correspond to \( h_m = 0.10 \text{ cm}, 0.015 \text{ cm}, \) and \( 0.20 \text{ cm} \), respectively.
fixed total thickness of the channel, gel flow rate increases with its thickness whereas the air flow rate decreases with the increase in thickness of gel layer. In Figure 4a, one can find a good agreement between the numerical and analytical solutions for air flow rates with pressure gradient at all the values of gel layer thickness. In Figure 4b, it is observed that the analytical solution for gel flow rate is in good agreement with the corresponding numerical values for large gel thickness. However, there is a mild deviation in the two solutions for small thickness of gel layer. Figure 4a also suggests that air flow resistance is more when gel layer is present in the channel compared to the dry case. Also in these figures one can notice that both gel and air flow rates increase with the magnitude of pressure gradient ($\phi_0$).

Figure 5 illustrates the effects of viscosity and elastic modulus of viscoelastic gel on its flow rate. In this case, $\phi_0 = 2.50 \times 10^2 \text{gm/cm}^2$, $h_m = 0.2 \text{cm}$, $\mu_m = 0.5, 1.0, 2.0, 4.0 \text{poise}$. In Figure 5a, variation of gel flow rate, as obtained from FDM calculations, with elastic modulus for various values of gel viscosity has been presented. Here, we find that the gel flow rate increases as its viscosity decreases. It is clear that gel flow rate increases as the elastic modulus decreases. Further, from these figures it is also clear that for large gel viscosity, $\mu_m$, a small decrease from the moderate values of gel elastic modulus brings in a steep raise in the gel flow rates. It may also be noted that the effect of $\mu_m$ is insignificant at higher values of $G^{-1}$. In Figures 5b and 5c.

![Figure 5](image-url)
results for gel flow rate for small and large values of elastic modulus have been presented in
different ranges of $G^{-1}$ and a comparison of the numerical and analytical solution has been
made. It is observed that the approximate analytical solution is in good agreement with the
numerical results for small values of $G$ and $\mu_m$. However, as the analytical solutions are obtained
under various approximations, they are at variance with the numerical results for higher values
of $\mu_m$. Also, it may be noted that analytical results are not valid for all values of the parameters.
Figure 5c shows the influence of large elastic modulus on gel flow rate ($G > 10^3$ dyne/cm$^2$) and
it is clear from this figure that gel flow rate does not vary significantly with elastic modulus for
large values of $G$.

6. APPLICATION TO MUCUS TRANSPORT
IN A SIMULATED COUGH MACHINE

It is pointed out here that our analysis is based on the basic assumption that the flow is laminar.
In the simulated cough machine generally the flow becomes turbulent. Though the quantitative
comparison of the results found here with the experimental results of a simulated cough machine
is not possible but a qualitative trend can be found. Thus, we mention the following results in
order.

1. Clarke et al. [8,9] in their experiment pointed out that air flow resistance is more in a
viscous lined tube compared to the dry case. Figure 4a shows that air flow resistance is
more if thick viscous liquid is present in the channel.

2. King et al. [10-12] and Agarwal et al. [14,15] in their experiments related to mucus trans-
port simulating trachea observed that mucus transport increases with its thickness. Fig-
ure 4b shows that mucus transport increases as its thickness increases.

3. Scherer and Burtz [6], Scherer [7], and King et al. [12] in their experimental work mentioned
that mucus transport increases as its viscosity decreases. Figure 5 shows similar result.

4. King et al. [11] and King [10] in their experimental work pointed out that elasticity has
negative effect on mucus transport. Figure 5b shows that for small or moderate elastic
modulus, mucus transport increases as its elastic modulus decreases. However, for large
values of the elastic modulus Figure 5c shows that mucus transport is almost independent
of its elastic modulus. This is also in agreement with the observations of Kim et al. [13]
and King et al. [12] who noted that there is no apparent relationship between mucus
transport and the elastic modulus of mucus.
7. CONCLUSION

In this paper, unsteady-state simultaneous flow of incompressible immiscible fluids in a channel due to time-dependent pressure gradient is analyzed by considering one fluid as a viscoelastic gel which is represented by Maxwell model and the other one is air which is a Newtonian fluid. The governing equations are solved by Laplace transform method and an approximate expressions for the flow rates are obtained that are valid for small time. A numerical scheme in the implicit finite-difference framework is also developed to fully analyze the model. The analytical and numerical solutions are compared which are in good agreement with each other for small time and large thickness of fluid layers.

REFERENCES