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Mathematical modeling and analysis of the depletion of dissolved oxygen in water bodies

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Abstract

In this paper, a nonlinear mathematical model is proposed to study the depletion of dissolved oxygen in a water body caused by industrial and household discharges of organic matters (pollutants). The problem is formulated as a food chain model by considering various interaction processes (biodegradation and biochemical) involving organic pollutants, bacteria, protozoa, an aquatic population and dissolved oxygen. Using stability theory, it is shown that as the rate of introduction of organic pollutants in a water body increases, the concentration of dissolved oxygen decreases due to various interaction processes. It is found that if the organic pollutants are continuously discharged into water body, the concentration of dissolved oxygen may become negligibly small, thus threatening the survival of aquatic populations. However, by using some effort to control the cumulative discharge of these pollutants into the water body, the concentration of dissolved oxygen can be maintained at a desired level.

Keywords: Mathematical model; Water pollution; Dissolved oxygen; Monod interaction

1. Introduction

It is well known that pollutants and toxicants discharged in water bodies are harmful to biological species such as a fish population [10,15,19,22,23]. When household and industrial wastes are discharged into water, the organic matter present in them are uptaken by micro-organisms which convert them into inorganic matter using dissolved oxygen in the process. The density of these micro-organisms (bacteria) increases as the cumulative concentration of organic pollutants increases. As these are part of a food chain involving protozoa and other biological populations in a water body, the level of dissolved oxygen decreases further due to various interactive biochemical and biodegradation processes. It may be noted here that the input of the dissolved oxygen in a water body is mainly due to atmospheric diffusion through the water surface and to a certain extent due to its production by photosynthesis [13,10,18–20]. Therefore, while modeling the depletion of dissolved oxygen in an aquatic ecosystem, all these processes should be taken into account [9].

The study of effect of the organic pollutants on dissolved oxygen in a water body started with the well-known model of Streeter and Phelps [21]. Several workers including Dobbins [3], O' Connor [14], Beck and Young [1] have generalized this model. It may be pointed out here that these models are only linear. However, there exist various nonlinear

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processes in water bodies involving interactions such as organic pollutants with bacteria [6,11,15], phytoplankton with zooplankton in presence of nutrients [2,8], organic chemical distribution in aquatic food chains [22], dissolved oxygen with aquatic populations [15], etc. For a case related to organic pollutants, Rinaldi et al. [15] proposed ecological type nonlinear models involving fast and slow biodegradable pollutants, which implicitly consider the depletion of dissolved oxygen by bacteria and protozoa in a water body. In this study, where simulation analysis has been carried out, several questions regarding the behavior of the nonzero equilibrium including nonlinear stability, etc. remained unanswered. Further some other aspects, such as natural depletion rates of pollutants, crowding of species, growth of oxygen from algae and macrophytes, survival of aquatic population dependent on dissolved oxygen, etc. have not been taken into account in various nonlinear modeling studies. It may be also pointed out here that, the level of dissolved oxygen can be increased by pumping air into the water body, being used for production of fish, and therefore, this aspect needs also to be considered in the modeling process.

In this paper, therefore, a nonlinear ecological type of mathematical model for the depletion of dissolved oxygen in a water body is proposed and analyzed by considering some of the aspects mentioned above involving degradable organic pollutants, bacteria, protozoa and an aquatic population. This model takes into account monod interactions between organic pollutants and bacteria, bacteria and protozoa and a bilinear interaction between protozoa and the aquatic population. It is assumed that a constant rate of input of oxygen is applied into the water body, to increase the level of dissolved oxygen [13,18,19]. A model to control the discharge of organic pollutants and its effect on dissolved oxygen is also proposed and analyzed.

2. Mathematical model

We consider a water body where organic pollutants are discharged in the form of wastes. It is assumed that these pollutants are part of a food chain consisting of bacteria, protozoa and an aquatic population, (for example a fish population) using dissolved oxygen in this water body for various biochemical and biodegradation processes.

Let, T be the cumulative concentration of organic matter (degradable pollutants), B be the density of the bacteria, P be the density of the protozoa, N be the density of an aquatic population, which depends wholly on protozoa and C be the concentration of the dissolved oxygen (DO). It is assumed that the cumulative discharge of organic pollutants into the water body is at a constant rate Q and that the rate of decrease of concentration T due to biochemical and other factors is proportional to T. It is further assumed that the cumulative rate of depletion of T due to bacteria is given by a monod type of interaction involving the density of bacteria B as well as the concentration T (i.e. $TB/(K_{12}+K_{11}T)$). As bacteria wholly depend upon organic pollutants, the growth rate of density of its population is proportional to this term. It is considered that the natural depletion rate of density of bacteria is proportional to B and the decrease of its growth rate due to crowding is proportional to B^2. Further, the depletion rate of density of bacteria by its predator protozoa is assumed to be given by the monod interaction between bacteria and protozoa (i.e. $BP/(K_{21}+K_{22}B)$). Thus, the growth rate of protozoa density is also proportional to this interaction term. The natural depletion rate of protozoa density is assumed proportional to P while its decrease due to crowding is proportional to P^2. The depletion rate of protozoa density by its predator such as a fish population is assumed proportional to (PN) and therefore the corresponding growth rate of density of this predator population is proportional to this product. The natural depletion rate of density of predator population is considered proportional to N and its decrease due to crowding is proportional to N^2. It is also assumed that the rate of growth of concentration of dissolved oxygen by various sources including diffusion, pumping of air into the water body, etc. is q (a constant) and its natural depletion rate is proportional to C. It is assumed, further, that the rate of depletion of dissolved oxygen is proportional to various terms ($K_1TB/(K_{12}+K_{11}T)$, $K_2BP/(K_{21}+K_{22}B)$, K_3PN, $\alpha_1 B$, $\alpha_2 P$, $\alpha_3 N$) representing growth and depletion rates [15].

Keeping in mind the above considerations, the system is governed by the following differential equations:

$$\frac{dT}{dt} = Q - \alpha_0 T - \frac{K_1 TB}{K_{12}+K_{11}T},$$

$$\frac{dB}{dt} = \lambda_1 \frac{K_1 TB}{K_{12}+K_{11}T} - \alpha_1 B - \lambda_{10} B^2 - \frac{K_2 BP}{K_{21}+K_{22}B},$$

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \lambda_2 \frac{K_2 BP}{K_{21} + K_{22} B} - \alpha_2 P - \lambda_{20} P^2 - K_3 PN, \tag{2.1}$$

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \lambda_3 K_3 PN - \alpha_3 N - \lambda_{30} N^2,$$

$$\frac{\mathrm{d}C}{\mathrm{d}t} = q - \alpha_4 C - \lambda_{34} K_3 PN - \lambda_{11}\alpha_1 B - \lambda_{22}\alpha_2 P - \lambda_{33}\alpha_3 N - \lambda_{12}\frac{K_1 TB}{K_{12} + K_{11} T} - \lambda_{23}\frac{K_2 BP}{K_{21} + K_{22} B},$$

where $T(0) = T_0 \geqslant 0$, $B(0) = B_0 \geqslant 0$, $P(0) = P_0 = P_0 \geqslant 0$, $N(0) = N_0 \geqslant 0$, $C(0) = C_0 \geqslant 0$.

In model (2.1) the constants $\lambda_{10}, \lambda_{20}, \lambda_{30}$ are crowding coefficients (flaking off coefficients [15]) of bacteria, protozoa and the aquatic population, respectively. The parameters $K_1, K_2, K_3, K_{11}, K_{12}, K_{21}, K_{22}$ involved in the monod type interactions are all constants. The coefficients $\lambda_1, \lambda_2, \lambda_3, \lambda_{11}, \lambda_{22}, \lambda_{33}, \lambda_{12}, \lambda_{23}, \lambda_{34}$ are positive constants and their magnitudes are less than equal to unity.

It may be pointed out that for feasibility of model (2.1), the growth rates of bacteria and protozoa should be positive. Hence, from the second and third equation of the model, it follows that

$$\lambda_1 K_1 - K_{11}\alpha_1 > 0 \tag{2.2}$$

and

$$\lambda_2 K_2 - K_{22}\alpha_2 > 0. \tag{2.3}$$

Following, Freedman and So [5], the region of attraction Ω for all solutions initiating in the positive octant is given by

$$\Omega = \left\{ 0 \leqslant T \leqslant \frac{Q}{\alpha_0},\ 0 \leqslant B \leqslant R_B,\ 0 \leqslant P \leqslant R_P,\ 0 \leqslant N \leqslant R_N,\ 0 \leqslant C \leqslant \frac{q}{\alpha_4} \right\}, \tag{2.4}$$

where

$$R_B = \frac{\lambda_1 K_1 Q}{\lambda_{10}(K_{11} Q + K_{12}\alpha_0)}, \tag{2.5}$$

$$R_P = \frac{\lambda_1 \lambda_2 K_1 K_2 Q}{\lambda_{20}[K_{22}\lambda_1 K_1 Q + \lambda_{10} K_{21}(K_{11} Q + K_{12}\alpha_0)]}, \tag{2.6}$$

$$R_N = \frac{\lambda_1 \lambda_2 \lambda_3 K_1 K_2 K_3 Q}{\lambda_{20}\lambda_{30}[K_{22}\lambda_1 K_1 Q + \lambda_{10} K_{21}(K_{11} Q + K_{12}\alpha_0)]}. \tag{2.7}$$

It is assumed here that all the initial values of the variables belong to set Ω. In the following, we analyze this model (2.1) by using the qualitative theory of differential [4,7,12,16,17].

3. Equilibrium analysis

System (2.1) has following four nonnegative equilibria in Ω. They are listed below:

(i) $E_1(Q/\alpha_0, 0, 0, 0, q/\alpha_4)$ which always exists.
(ii) $E_2(T_2^*, B_2^*, 0, 0, C_2^*)$ exists, provided the following conditions are satisfied:

$$(\lambda_1 K_1 - K_{11}\alpha_1)Q - K_{12}\alpha_0\alpha_1 > 0, \tag{3.1}$$

$$q - \lambda_{12}\frac{K_1 T_2^* B_2^*}{K_{12} + K_{11} T_2^*} - \lambda_{11}\alpha_1 B_2^* > 0. \tag{3.2}$$

(iii) $E_3(T_3^*, B_3^*, P_3^*, 0, C_3^*)$ exists, provided the following conditions are valid:

$$\lambda_{20}[(\lambda_1 K_1 - K_{11}\alpha_1)Q - K_{12}\alpha_0\alpha_1] + \frac{K_2\alpha_2}{K_{21}}(K_{11}Q + K_{12}\alpha_0) > 0, \tag{3.3}$$

$$\lambda_{10} + \frac{1}{\lambda_{20}}\frac{K_2}{(K_{21}+K_{22}R_B)^2}\left[\frac{\lambda_2 K_2 K_{21}}{K_{21}+K_{22}R_B} - K_{22}\left(\frac{\lambda_2 K_2 R_B}{K_{21}+K_{22}R_B} - \alpha_2\right)\right] > 0, \tag{3.4}$$

$$\frac{\lambda_2 K_2 B_3^*}{K_{21}+K_{22}B_3^*} - \alpha_2 > 0, \tag{3.5}$$

$$q - \lambda_{12}\frac{K_1 T_3^* B_3^*}{K_{12}+K_{11}T_3^*} - \lambda_{23}\frac{K_2 T_3^* B_3^*}{K_{21}+K_{22}B_3^*} - \lambda_{11}\alpha_1 B_3^* - \lambda_{22}\alpha_2 P_3^* > 0. \tag{3.6}$$

(iv) $E_4(T^*, B^*, P^*, N^*, C^*)$ exists whenever the following conditions are satisfied:

$$(\lambda_1 K_1 - K_{11}\alpha_1)Q - K_{12}\alpha_0\alpha_1 + \frac{K_2 m}{K_{21}l}(K_{11}Q + K_{12}\alpha_0) > 0, \tag{3.7}$$

$$\lambda_{10} + \frac{K_2}{(K_{21}+K_{22}R_B)^2 l}\left[\frac{\lambda_{30}\lambda_2 K_2 K_{21}}{(K_{21}+K_{22}R_B)} - K_{22}\left(\frac{\lambda_{30}\lambda_2 K_2 R_B}{(K_{21}+K_{22}R_B)} - m\right)\right] > 0, \tag{3.8}$$

$$\frac{\lambda_{30}\lambda_2 K_2 B^*}{(K_{21}+K_{22}B^*)} - m > 0, \tag{3.9}$$

$$\lambda_3 K_3 P^* - \alpha_3 > 0, \tag{3.10}$$

$$q - \lambda_{12}\frac{K_1 T^* B^*}{K_{12}+K_{11}T^*} - \lambda_{23}\frac{K_2 B^* P^*}{K_{21}+K_{22}B^*} - \lambda_{34}K_3 P^* N^* - \lambda_{11}\alpha_1 B^* - \lambda_{22}\alpha_2 P^* - \lambda_{33}\alpha_3 N^* > 0, \tag{3.11}$$

where $l = (\lambda_{20}\lambda_{30} + \lambda_3 K_3^2)$, $m = \lambda_{30}\alpha_2 - K_3\alpha_3$.

Proof. The equilibria E_i $(i = 1, 2, 3, 4)$ of system (2.1) are obtained by solving the following set of simultaneous algebraic equations:

$$Q - \alpha_0 T - \frac{K_1 TB}{K_{12}+K_{11}T} = 0, \tag{3.12}$$

$$\frac{\lambda_1 K_1 TB}{K_{12}+K_{11}T} - \alpha_1 B - \lambda_{10}B^2 - \frac{K_2 BP}{K_{21}+K_{22}B} = 0, \tag{3.13}$$

$$\frac{\lambda_2 K_2 BP}{K_{21}+K_{22}B} - \alpha_2 P - \lambda_{20}P^2 = 0, \tag{3.14}$$

$$\lambda_3 K_3 PN - \alpha_3 N - \lambda_{30}N^2 = 0, \tag{3.15}$$

$$q - \alpha_4 C - \lambda_{12}\frac{K_1 TB}{K_{12}+K_{11}T} - \lambda_{23}\frac{K_2 BP}{K_{21}+K_{22}B} - \lambda_{34}K_3 PN - \lambda_{11}\alpha_1 B - \lambda_{22}\alpha_2 P - \lambda_{33}\alpha_3 N = 0. \tag{3.16}$$

The equilibrium $E_1(Q/\alpha_0, 0, 0, 0, q/\alpha_4)$ exists obviously. We show the existence of other equilibria as follows.

Existence of E_2: for the equilibrium $E_2(T_2^*, B_2^*, 0, 0, C_2^*)$, the values of T_2^*, B_2^* and C_2^* are obtained by solving the following algebraic equations:

$$B = \frac{(Q - \alpha_0 T)}{K_1}\left(K_{11} + \frac{K_{12}}{T}\right), \tag{3.17}$$

$$B = \frac{1}{\lambda_{10}}\left[\frac{\lambda_1 K_1 T}{K_{12}+K_{11}T} - \alpha_1\right]. \tag{3.18}$$

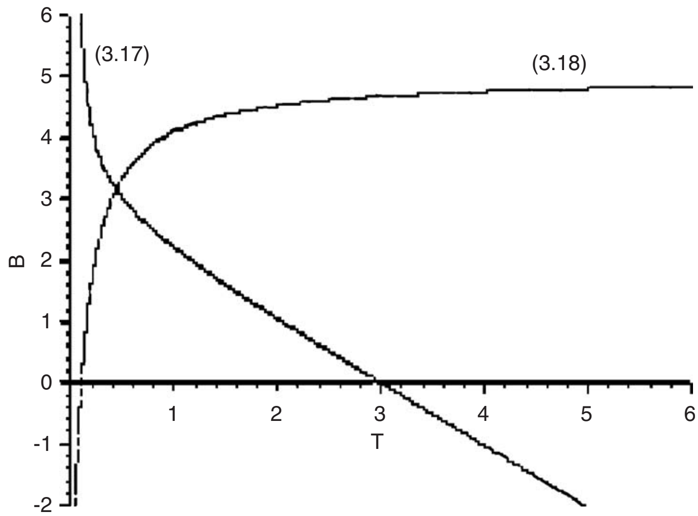

Fig. 1. Existence of (T_2^*, B_2^*).

From the isocline given by (3.17), it is easy to note the following:

(i) $B=0$ at $T=Q/\alpha_0$
(ii) $B\to\infty$ as $T\to 0$,
(iii) $\frac{\mathrm{d}B}{\mathrm{d}T}=-\frac{K_{12}}{T^2}\frac{(Q-\alpha_0 T)}{K_1}-\frac{\alpha_0}{K_1}\left(K_{11}+\frac{K_{12}}{T}\right)$, which is negative for $0\leqslant T\leqslant Q/\alpha_0$. Thus B decreases as T increases.

From the isocline given by (3.18), it is noted that $B<0$ at $T=0$ and $B=0$ at $T=K_{12}\alpha_1/(\lambda_1 K_1-K_{11}\alpha_1)$, which is positive. Further $\mathrm{d}B/\mathrm{d}T>0$ and hence B increases for all $T>0$.

Thus, the two isoclines (3.17) and (3.18) intersect at T_2^*, B_2^* (see Fig. 1) provided $(\lambda_1 K_1-K_{11}\alpha_1)Q-K_{12}\alpha_0\alpha_1>0$.

Using these values of T_2^* and B_2^*, we get C_2^* from (3.16) as follows:

$$C_2^*=\frac{1}{\alpha_4}\left[q-\lambda_{12}\frac{K_1T_2^*B_2^*}{K_{12}+K_{11}T_2^*}-\lambda_{11}\alpha_1B_2^*\right], \tag{3.19}$$

which is positive provided the right-hand side of (3.19) is positive. This gives condition (3.2).

Thus, $E_2(T_2^*, B_2^*, 0, 0, C_2^*)$ exists, provided conditions (3.1) and (3.2) are satisfied.

Existence of E_3: for the equilibrium $E_3(T_3^*, B_3^*, P_3^*, 0, C_3^*)$, the values of T_3^* and B_3^* are given by the following algebraic equations:

$$B=\frac{(Q-\alpha_0T)}{K_1}\left(K_{11}+\frac{K_{12}}{T}\right) \tag{3.20}$$

and

$$\frac{\lambda_1K_1T}{K_{12}+K_{11}T}-\alpha_1=\lambda_{10}B+\frac{K_2}{(K_{21}+K_{22}B)}\frac{1}{\lambda_{20}}\left(\frac{\lambda_2K_2B}{K_{21}+K_{22}B}-\alpha_2\right). \tag{3.21}$$

Isocline (3.20) is the same as (3.17), hence its behavior is also the same.

From the isocline given by (3.21), it is noted that B increases with T in $0\leqslant B\leqslant R_B$ (R_B is given by (2.5)), provided inequality (3.4) is satisfied.

Also $B=0$ at

$$T=K_{12}\left(\alpha_1-\frac{K_2\alpha_2}{\lambda_{20}K_{21}}\right)\bigg/\left((\lambda_1K_1-K_{11}\alpha_1)+K_{11}\frac{K_2\alpha_2}{\lambda_{20}K_{21}}\right),$$

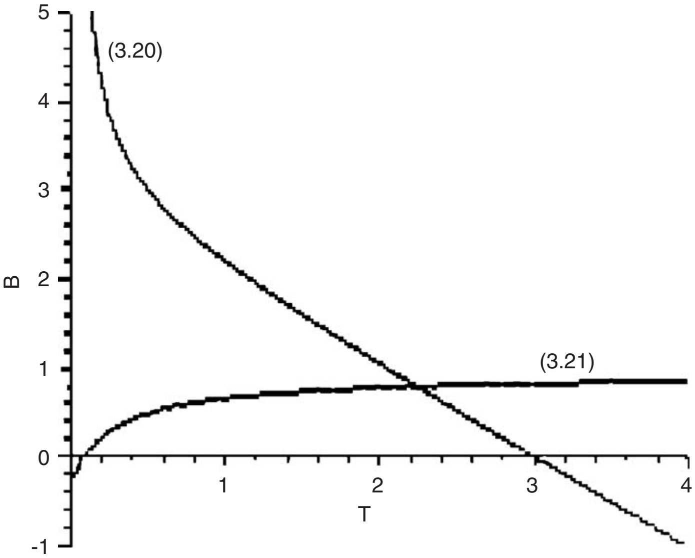

Fig. 2. Existence of (T_3^*, B_3^*).

which is positive or negative depending on the sign of $\bar{\alpha}_1 = (\alpha_1 - K_2\alpha_2/\lambda_{20}K_{21})$. It is noted that the value of B at $T = 0$ is given by the following cubic equation:

$$\begin{aligned}&\lambda_{10}\lambda_{20}K_{22}^2B^3 + (2\lambda_{10}\lambda_{20}K_{21}K_{22} + \lambda_{20}K_{22}^2\alpha_1)B^2\\&\quad + (\lambda_{10}\lambda_{20}K_{21}^2 + K_2(\lambda_2K_2 - K_{22}\alpha_2) + \lambda_{20}\alpha_1K_{21}K_{22})B + \lambda_{20}K_{21}^2\bar{\alpha}_1 = 0.\end{aligned} \tag{3.22}$$

It is clear that one root of Eq. (3.22) is positive or negative depending on whether $\bar{\alpha}_1$ is negative or positive, respectively.

Thus, from the above analysis we conclude that the two isoclines (3.20) and (3.21) intersect at a unique point (T_3^*, B_3^*) in the interior of first quadrant (see Fig. 2, which is drawn for $\bar{\alpha}_1 > 0$) provided conditions (3.3) and (3.4) are satisfied.

Using these values of T_3^*, B_3^* in Eqs. (3.14) and (3.16), we get

$$P_3^* = \frac{1}{\lambda_{20}}\left(\frac{\lambda_2K_2B_3^*}{K_{21} + K_{22}B_3^*} - \alpha_2\right), \tag{3.23}$$

$$C_3^* = \frac{1}{\alpha_4}\left[q - \lambda_{12}\frac{K_1T_3^*B_3^*}{K_{12} + K_{11}T_3^*} - \lambda_{23}\frac{K_2B_3^*P_3^*}{K_{21} + K_{22}B_3^*} - \lambda_{11}\alpha_1B_3^* - \lambda_{22}\alpha_2P_3^*\right] \tag{3.24}$$

which are positive under conditions (3.5) and (3.6), respectively.

Thus E_3 exists provided conditions (3.3)–(3.6) are satisfied.

Keeping in mind model (2.1), it is noted on physical consideration that the growth rate of bacteria in the absence of protozoa is more than when protozoa is present (as protozoa predate bacteria). Thus $B_2^* > B_3^*$. (This result can also be proved from model (2.1) by using a comparison theorem [7].)

Existence of E_4: for the equilibrium $E_4(T^*, B^*, P^*, N^*, C^*)$, T^* and B^* are obtained by solving the following algebraic equations:

$$B = \frac{(Q - \alpha_0T)}{K_1}\left(K_{11} + \frac{K_{12}}{T}\right) \tag{3.25}$$

and

$$\frac{\lambda_1K_1T}{K_{12} + K_{11}T} - \alpha_1 - \lambda_{10}B - \frac{K_2}{(K_{21} + K_{22}B)}\frac{1}{l}\left[\frac{\lambda_{30}\lambda_2K_2B}{K_{21} + K_{22}B} - m\right] = 0, \tag{3.26}$$

where $l = (\lambda_{20}\lambda_{30} + \lambda_3K_3^2)$ and $m = (\lambda_{30}\alpha_2 - K_3\alpha_3)$.

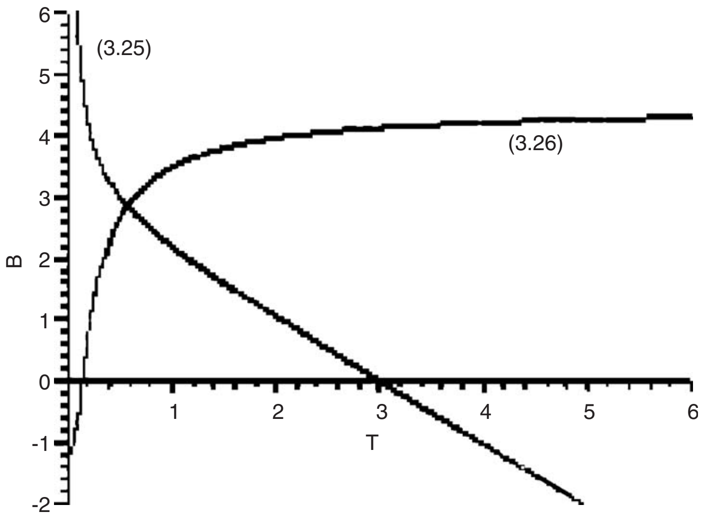

Fig. 3. Existence of (T^*, B^*).

Isocline (3.25) is the same as (3.17); hence, its behavior is also the same.

From isocline (3.26), it is noted that under condition (3.8) B increases with T in $0 \leqslant B \leqslant R_B$.

It is also noted that $B = 0$ at

$$T = K_{12}\left(\alpha_1 - \frac{K_2 m}{K_{21} l}\right) \Big/ \left\{(\lambda_1 K_1 - K_{11}\alpha_1) + \frac{K_{11}K_2 m}{K_{21} l}\right\}$$

which is positive or negative depending upon the sign of $\tilde{\alpha}_1 = (\alpha_1 - (K_2/K_{21})(m/l))$.

Further, for this isocline, the value of B at $T = 0$ is given by the following cubic equation:

$$\lambda_{10}K_{22}^2 B^3 + (2\lambda_{10}K_{21}K_{22} + K_{22}^2\alpha_1)B^2 + \left(\lambda_{10}K_{21}^2 + \frac{K_2}{l}(\lambda_{30}\lambda_2 K_2 - mK_{22}) + 2K_{21}K_{22}\alpha_1\right)B + K_{21}^2\tilde{\alpha}_1 = 0. \tag{3.27}$$

One root of Eq. (3.27) is positive or negative depending on the negative or positive value of $\tilde{\alpha}_1$, respectively.

Thus, from the above discussion, we conclude that the two isoclines (3.25) and (3.26) intersect at a unique point (T^*, B^*) in the interior of first quadrant (see Fig. 3, drawn for $\tilde{\alpha}_1 > 0$) provided (3.7) and (3.8) are satisfied.

Using these values of $T^*$and B^*we get,

$$P^* = \frac{1}{l}\left[\frac{\lambda_{30}\lambda_2 K_2 B^*}{K_{21} + K_{22}B^*} - m\right], \tag{3.28}$$

$$N^* = \frac{1}{\lambda_{30}}(\lambda_3 K_3 P^* - \alpha_3), \tag{3.29}$$

$$C^* = \frac{1}{\alpha_4}\left[q - \lambda_{12}\frac{K_1 T^* B^*}{K_{12} + K_{11}T^*} - \lambda_{23}\frac{K_2 B^* P^*}{K_{21} + K_{22}B^*} - \lambda_{34}K_3 P^* N^* - \lambda_{11}\alpha_1 B^* - \lambda_{22}\alpha_2 P^* - \lambda_{33}\alpha_3 N^*\right] \tag{3.30}$$

which are positive under conditions (3.9)–(3.11).

Thus E_4 exists, provided conditions (3.7)–(3.11) are satisfied. □

On physical consideration from model (2.1) it is noted here that growth rate of protozoa in the absence of lower order aquatic population is greater than that in its presence (as protozoa is predated by this aquatic population). Hence $P_3^* > P^*$. (This result can also be proved from model (2.1) by using a comparison theorem [7].)

Remark 1.1. Using Eqs. (3.25) and (3.26) we can check that under condition (3.8), $\mathrm{d}T^*/\mathrm{d}Q > 0$ and $\mathrm{d}B^*/\mathrm{d}T^* > 0$. This gives $\mathrm{d}B^*/\mathrm{d}Q = (\mathrm{d}B^*/\mathrm{d}T^*)(\mathrm{d}T^*/\mathrm{d}Q) > 0$.

Also, from Eq. (3.28), we note that

$$\frac{\mathrm{d}P^*}{\mathrm{d}Q} = \frac{1}{l}\,\frac{\lambda_{30}\lambda_2 K_2 K_{21}}{(K_{21} + K_{22}B^*)^2}\,\frac{\mathrm{d}B^*}{\mathrm{d}Q} > 0.$$

Similarly from Eq. (3.29), we get $\mathrm{d}N^*/\mathrm{d}Q = (\lambda_3 K_3/\lambda_{30})(\mathrm{d}P^*/\mathrm{d}Q) > 0$.

Hence T^*, B^*, P^* and N^* increase as Q increases.

Also from Eq. (3.30) it can be seen that $\mathrm{d}C^*/\mathrm{d}Q < 0$, $\mathrm{d}C^*/\mathrm{d}\lambda_{ij} < 0$ and $\mathrm{d}C^*/\mathrm{d}q > 0$.

The first inequality implies that as the rate of discharge of organic pollutants 'Q' increases, the equilibrium level of dissolved oxygen C^* decreases. The second inequality suggests that C^* further decreases due to presence of various degradation processes. Also, it can be noted from the third inequality that C^* increases as the rate of input of dissolved oxygen 'q' increases.

4. Stability analysis

In our analysis, we assume that all the above equilibria exist. The local stability behavior of these equilibria has been studied and the result is stated in the following theorem:

Theorem 1. *The equilibrium E_i $(i = 1, 2$ or $3)$ is unstable whenever E_{i+1} exists. The equilibrium E_4 is locally asymptotically stable provided the following condition is satisfied*:

$$\lambda_{10} - \frac{K_2 K_{22}}{K_{21}^2}\,P^* > 0. \tag{4.1}$$

The proof is given in Appendix A.

Theorem 2. *The equilibrium E_4 is nonlinearly stable in Ω provided the following two conditions are satisfied*:

$$\lambda_{10} - \frac{K_2 K_{22}}{K_{21}^2}\,P^* > 0, \tag{4.2}$$

$$\frac{\alpha_0 T^*}{\lambda_1}\left[\lambda_{10} - \frac{K_2 K_{22}}{K_{21}^2}\,P^*\right] - \left[\frac{K_1 K_{12} T^*}{(K_{12} + K_{11}T^*)}\,\frac{Q}{(K_{12}\alpha_0 + K_{11}Q)}\right]^2 > 0. \tag{4.3}$$

The proof is given in Appendix B.

We note from the above that if $\lambda_{10} = 0$, the above conditions are never satisfied. This shows that crowding coefficient of bacteria population density (i.e. flaking off term) stabilizes the system.

The above theorems imply that the concentration of dissolved oxygen decreases as the discharge rate of organic pollutants increases. They also show that if the cumulative rate of discharge of pollutants is very high, the concentration of dissolved oxygen may become negligible caused by various degradation processes.

5. A model for control of organic pollutants

Here, we propose and analyze a nonlinear mathematical model to study the effect of control of discharge rate of organic pollutants on dissolved oxygen by using some effort [15]. Let e be the effort applied to control the discharge of organic pollutants. It is assumed that the growth rate of effort applied to control the organic pollutants is proportional to the cumulative concentration of organic pollutants (i.e. $r_1 T$), and that the rate of natural depletion of effort is proportional to e (i.e. $r_{10}e$). It is assumed further that the rate of decrease of organic pollutants in the water body is proportional to effort e (i.e. γe). In the following model the other notations and assumptions for growth rates and depletion rates of organic pollutants, bacteria, protozoa, lower order aquatic population and dissolved oxygen are the same as in model (2.1).

Keeping the above considerations in mind, the control dynamics of the system is governed by the following system of differential equations:

$$\frac{\mathrm{d}T}{\mathrm{d}t} = Q - \alpha_0 T - \frac{K_1 TB}{K_{12} + K_{11} T} - \gamma e,$$

$$\frac{\mathrm{d}B}{\mathrm{d}t} = \lambda_1 \frac{K_1 TB}{K_{12} + K_{11} T} - \alpha_1 B - \lambda_{10} B^2 - \frac{K_2 BP}{K_{21} + K_{22} B},$$

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \lambda_2 \frac{K_2 BP}{K_{21} + K_{22} B} - \alpha_2 P - \lambda_{20} P^2 - K_3 PN,$$

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \lambda_3 K_3 PN - \alpha_3 N - \lambda_{30} N^2,$$

$$\frac{\mathrm{d}C}{\mathrm{d}t} = q - \alpha_4 C - \lambda_{34} K_3 PN - \lambda_{11}\alpha_1 B - \lambda_{22}\alpha_2 P - \lambda_{33}\alpha_3 N - \lambda_{12}\frac{K_1 TB}{K_{12} + K_{11} T} - \lambda_{23}\frac{K_2 BP}{K_{21} + K_{22} B},$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = r_1 T - r_{10} e, \tag{5.1}$$

where $T(0) > 0,\ B(0) > 0,\ P(0) > 0,\ N(0) > 0,\ C(0) > 0,\ e(0) > 0$.

In (5.1) γ is the control rate of cumulative density of organic pollutants, r_1 is the growth rate of effort and r_{10} is its depletion rate.

As before we can find that the set Ω_c, defined below, is a region of attraction for model (5.1).

$$\Omega_c = \left\{ 0 \leqslant T \leqslant \frac{Q}{\alpha_0}, 0 \leqslant B \leqslant R_B, 0 \leqslant P \leqslant R_P, 0 \leqslant N \leqslant R_N, 0 \leqslant C \leqslant \frac{q}{\alpha_4}, 0 \leqslant e \leqslant \frac{r_1 Q}{r_{10}\alpha_0} \right\}, \tag{5.2}$$

where all the initial values of the variables belong to set Ω_c.

5.1. Equilibrium analysis

Model (5.1) has four equilibria in Ω_c as described below (the proof of existence of each of these equilibria is similar as discussed for model (2.1)).

(i) $\tilde{E}_1\left(\frac{r_{10}Q}{r_{10}\alpha_0 + \gamma r_1}, 0, 0, 0, \frac{q}{\alpha_4}, \frac{r_1 Q}{r_{10}\alpha_0 + \gamma r_1}\right)$.

(ii) $\tilde{E}_2(\tilde{T}_2, \tilde{B}_2, 0, 0, \tilde{C}_2, \tilde{e}_2)$ exists provided the following are satisfied:

$$(\lambda_1 K_1 - K_{11}\alpha_1) Q - K_{12}\alpha_1(\alpha_0 + \gamma r_1/r_{10}) > 0, \tag{5.3}$$

and

$$q - \lambda_{12}\frac{K_1 \tilde{T}_2 \tilde{B}_2}{K_{12} + K_{11}\tilde{T}_2} - \lambda_{11}\alpha_1 \tilde{B}_2 > 0, \tag{5.4}$$

where $\tilde{T}_2$, $\tilde{B}_2$, $\tilde{C}_2$ and $\tilde{e}_2$ are given by the following equations:

$$Q - \left(\alpha_0 + \frac{\gamma r_1}{r_{10}}\right)\tilde{T}_2 - \frac{K_1 \tilde{T}_2 \tilde{B}_2}{K_{12} + K_{11}\tilde{T}_2} = 0,$$

$$\frac{\lambda_1 K_1 \tilde{T}_2 \tilde{B}_2}{K_{12} + K_{11}\tilde{T}_2} - \alpha_1 - \lambda_{10}\tilde{B}_2 = 0,$$

$$\tilde{C}_2 = \frac{1}{\alpha_4}\left[q - \lambda_{12}\frac{K_1\tilde{T}_2\tilde{B}_2}{K_{12}+K_{11}\tilde{T}_2} - \lambda_{11}\alpha_1\tilde{B}_2\right],$$

$$\tilde{e} = \frac{r_1}{r_{10}}\tilde{T}_2.$$

(iii) $\tilde{E}_3(\tilde{T}_3, \tilde{B}_3, \tilde{P}_3, 0, \tilde{C}_3, \tilde{e}_3)$ exists provided the following conditions are satisfied:

$$\lambda_{20}\left[(\lambda_1 K_1 - K_{11}\alpha_1)Q - K_{12}\alpha_1\left(\alpha_0 + \frac{\gamma r_1}{r_{10}}\right)\right] + \frac{K_2\alpha_2}{K_{21}}\left[K_{11}Q + K_{12}\left(\alpha_0 + \frac{\gamma r_1}{r_{10}}\right)\right] > 0, \tag{5.5}$$

$$\lambda_{10} + \frac{K_2}{\lambda_{20}(K_{21}+K_{22}R_B)^2}\left[\frac{\lambda_2 K_2 K_{21}}{K_{21}+K_{22}R_B} - K_{22}\left(\frac{\lambda_2 K_2 R_B}{K_{21}+K_{22}R_B} - \alpha_2\right)\right] > 0, \tag{5.6}$$

$$\frac{\lambda_2 K_2\tilde{B}_3}{K_{21}+K_{22}\tilde{B}_3} - \alpha_2 > 0, \tag{5.7}$$

$$q - \lambda_{12}\frac{K_1\tilde{T}_3\tilde{B}_3}{K_{12}+K_{11}\tilde{T}_3} - \lambda_{23}\frac{K_2\tilde{B}_3\tilde{P}_3}{K_{21}+K_{22}\tilde{B}_3} - \lambda_{11}\alpha_1\tilde{B}_3 - \lambda_{22}\alpha_2\tilde{P}_3 > 0, \tag{5.8}$$

where $\tilde{T}_3$, $\tilde{B}_3$, $\tilde{P}_3$, $\tilde{C}_3$ and $\tilde{e}_3$ are given by the following equations:

$$Q - \left(\alpha_0 + \frac{\gamma r_1}{r_{10}}\right)\tilde{T}_3 - \frac{K_1\tilde{T}_3\tilde{B}_3}{K_{12}+K_{11}\tilde{T}_3} = 0,$$

$$\frac{\lambda_1 K_1\tilde{T}_3}{K_{12}+K_{11}\tilde{T}_3} - \alpha_1 = \lambda_{10}\tilde{B} + \frac{K_2}{\lambda_{20}(K_{21}+K_{22}\tilde{B}_3)}\left[\frac{\lambda_2 K_2\tilde{B}_3}{K_{21}+K_{22}\tilde{B}_3} - \alpha_2\right],$$

$$\tilde{P}_3 = \frac{1}{\lambda_{20}}\left[\frac{\lambda_2 K_2\tilde{B}_3}{K_{21}+K_{22}\tilde{B}_3} - \alpha_2\right],$$

$$\tilde{C}_3 = \frac{1}{\alpha_4}\left[q - \lambda_{12}\frac{K_1\tilde{T}_3\tilde{B}_3}{K_{12}+K_{11}\tilde{T}_3} - \lambda_{23}\frac{K_2\tilde{B}_3\tilde{P}_3}{K_{21}+K_{22}\tilde{B}_3} - \lambda_{11}\alpha_1\tilde{B}_3 - \lambda_{22}\alpha_2\tilde{P}_3\right],$$

$$\tilde{e}_3 = \frac{r_1}{r_{10}}\tilde{T}_3.$$

(iv) $\tilde{E}_4$ $(\tilde{T}, \tilde{B}, \tilde{P}, \tilde{N}, \tilde{C}, \tilde{e})$ exists, provided the following conditions are satisfied:

$$(\lambda_1 K_1 - K_{11}\alpha_1)Q - K_{12}\alpha_1\left(\alpha_0 + \frac{\gamma r_1}{r_{10}}\right) + \frac{K_2 m}{K_{21}l}\left(K_{11}Q + K_{12}\left(\alpha_0 + \frac{\gamma r_1}{r_{10}}\right)\right) > 0, \tag{5.9}$$

$$\lambda_{10} + \frac{K_2}{(K_{21}+K_{22}R_B)^2 l}\left[\frac{\lambda_{30}\lambda_2 K_2 K_{21}}{(K_{21}+K_{22}R_B)} - K_{22}\left(\frac{\lambda_{30}\lambda_2 K_2 R_B}{(K_{21}+K_{22}R_B)} - m\right)\right] > 0, \tag{5.10}$$

$$\frac{\lambda_{30}\lambda_2 K_2\tilde{B}}{(K_{21}+K_{22}\tilde{B})} - m > 0, \tag{5.11}$$

$$\lambda_3 K_3 \tilde{P} - \alpha_3 > 0, \tag{5.12}$$

$$q - \frac{\lambda_{12} K_1 \tilde{T}\, \tilde{B}}{K_{12} + K_{11}\tilde{T}} - \frac{\lambda_{23} K_2 \tilde{B}\, \tilde{P}}{K_{21} + K_{22}\tilde{B}} - \lambda_{34} K_3 \tilde{P}\, \tilde{N} - \lambda_{11}\alpha_1 \tilde{B} - \lambda_{22}\alpha_2 \tilde{P} - \lambda_{33}\alpha_3 \tilde{N} > 0, \tag{5.13}$$

where $l = (\lambda_{20}\lambda_{30} + \lambda_3 K_3^2)$, $m = \lambda_{30}\alpha_2 - K_3\alpha_3$ and $\tilde{T}, \tilde{B}, \tilde{P}, \tilde{N}, \tilde{C}, \tilde{e}$ are given by the following equations:

$$Q - \left(\alpha_0 + \frac{\gamma r_1}{r_{10}}\right)\tilde{T} - \frac{K_1 \tilde{T}\, \tilde{B}}{K_{12} + K_{11}\tilde{T}} = 0, \tag{5.14}$$

$$\frac{\lambda_1 K_1 \tilde{T}}{K_{12} + K_{11}\tilde{T}} - \alpha_1 = \lambda_{10}\tilde{B} + \frac{K_2}{l(K_{21} + K_{22}\tilde{B})}\left[\frac{\lambda_{30}\lambda_2 K_2 \tilde{B}}{(K_{21} + K_{22}\tilde{B})} - m\right], \tag{5.15}$$

$$\tilde{P} = \frac{1}{l}\left[\frac{\lambda_{30}\lambda_2 K_2 \tilde{B}}{K_{21} + K_{22}\tilde{B}} - m\right], \tag{5.16}$$

$$\tilde{N} = \frac{1}{\lambda_{30}}(\lambda_3 K_3 \tilde{P} - \alpha_3), \tag{5.17}$$

$$\tilde{C} = \frac{1}{\alpha_4}\left[q - \frac{\lambda_{12} K_1 \tilde{T}\, \tilde{B}}{K_{12} + K_{11}\tilde{T}} - \frac{\lambda_{23} K_2 \tilde{B}\, \tilde{P}}{K_{21} + K_{22}\tilde{B}} - \lambda_{34} K_3 \tilde{P}\, \tilde{N} - \lambda_{11}\alpha_1 \tilde{B} - \lambda_{22}\alpha_2 \tilde{P} - \lambda_{33}\alpha_3 \tilde{N}\right], \tag{5.18}$$

$$\tilde{e} = \frac{r_1}{r_{10}}\tilde{T}. \tag{5.19}$$

In this case also it can be checked that as the rate of control of organic pollutants γ increases, the equilibrium level of the dissolved oxygen concentration $\tilde{C}$ increases.

As in the previous case, for model (5.1) also we can easily show that $\tilde{E}_i$ $(i = 1, 2$ or $3)$ is unstable whenever $\tilde{E}_{i+1}$ exists. The local stability behavior of the equilibrium $\tilde{E}_4(\tilde{T}, \tilde{B}, \tilde{P}, \tilde{N}, \tilde{C}, \tilde{e})$ is stated in the following theorem.

Theorem 3. *The equilibrium $\tilde{E}_4$ is locally asymptotically stable if*

$$\lambda_{10} - \frac{K_2 K_{22}}{K_{21}^2}\,\tilde{P} > 0. \tag{5.20}$$

The proof is given in Appendix C.

Theorem 4. *The equilibrium $\tilde{E}_4$ is nonlinearly stable in Ω_c if the following conditions are satisfied*:

$$\lambda_{10} - \frac{K_2 K_{22}}{K_{21}^2}\,\tilde{P} > 0, \tag{5.21}$$

$$\frac{\alpha_0 \tilde{T}}{\lambda_1}\left[\lambda_{10} - \frac{K_2 K_{22}}{K_{21}^2}\tilde{P}\right] - \left[\frac{K_1 K_{12}\tilde{T}}{(K_{12} + K_{11}\tilde{T})}\frac{Q}{(K_{12}\alpha_0 + K_{11}Q)}\right]^2 > 0. \tag{5.22}$$

In this case also, we may note the stabilizing effect of λ_{10}. Further, these theorems imply that if an appropriate effort is applied to control organic pollutants, the equilibrium level of dissolved oxygen can be maintained at a desired level. (The proof is given in Appendix D.)

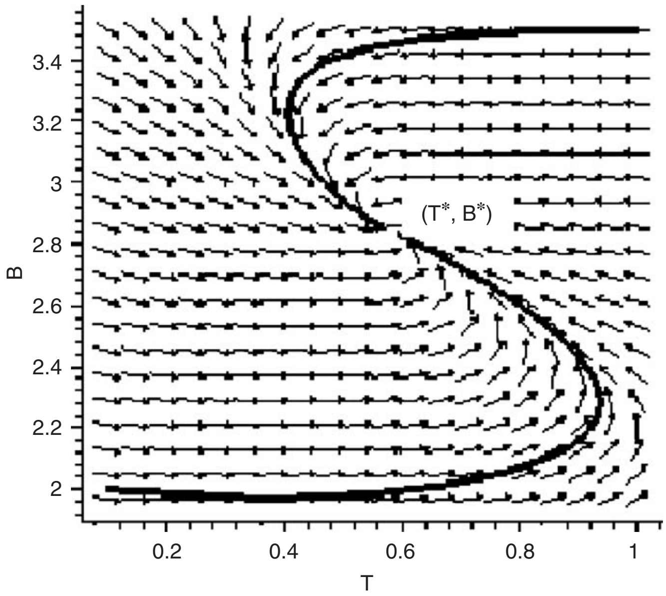

Fig. 4. Global stability in T–B plane.

6. Numerical example

To check the feasibility of our analysis regarding the existence of E_4 and corresponding stability conditions, we conduct some numerical computation by choosing the following values of the parameters in model (2.1):

$$Q = 3.0,\quad \alpha_0 = 1.0,\quad K_1 = 1.0,\quad K_{12} = 0.1,\quad K_{11} = 1.0,$$

$$\lambda_1 = 1.0,\quad \alpha_1 = 0.5,\quad \lambda_{10} = 0.1,\quad K_2 = 1.0,\quad K_{21} = 2.0,\quad K_{22} = 1.0,$$

$$\lambda_2 = 1.0,\quad \alpha_2 = 0.02,\quad \lambda_{20} = 0.25,\quad K_3 = 1.0,$$

$$\lambda_3 = 1.0,\quad \alpha_3 = 0.2,\quad \lambda_{30} = 0.25,\quad q = 10.0,\quad \alpha_4 = 2.0,\quad \lambda_{12} = \lambda_{23} = \lambda_{34} = \lambda_{11} = \lambda_{22} = \lambda_{33} = 0.25.$$

It is found that under the above set of parameters, conditions for the existence of interior equilibrium $E_4(T^*, B^*, P^*, N^*, C^*)$ are satisfied and E_4 is given by

$$T^* = 0.572971,\quad B^* = 2.850616,\quad P^* = 0.321807,\quad N^* = 0.487229,\quad C^* = 3.924467.$$

The eigenvalues of the Jacobian matrix M corresponding to this equilibrium E_4 are

$$-2,\quad -0.926092 + 0.217815\,i,\quad -0.926092 - 0.217815\,i,$$
$$-0.112787 + 0.404036\,i,\quad -0.112787 - 0.404036\,i,$$

which are either negative or have negative real parts. Hence E_4 is locally stable.

It is pointed out here that for the above set of parameters, the conditions for local stability (4.1) and nonlinear stability (4.2) and (4.3) are also satisfied.

Further, for the above set of parameters model (2.1) computer generated graphs of T verses B and T verses C are shown in Figs. 4 and 5, which indicates the global stability of (T^*, B^*) in the TB-plane and (T^*, C^*) in TC-plane.

7. Conclusion

In this paper, we have proposed and analyzed a nonlinear mathematical model for the depletion of dissolved oxygen due to discharge of organic pollutants in a water body by considering biodegradation and biochemical processes in

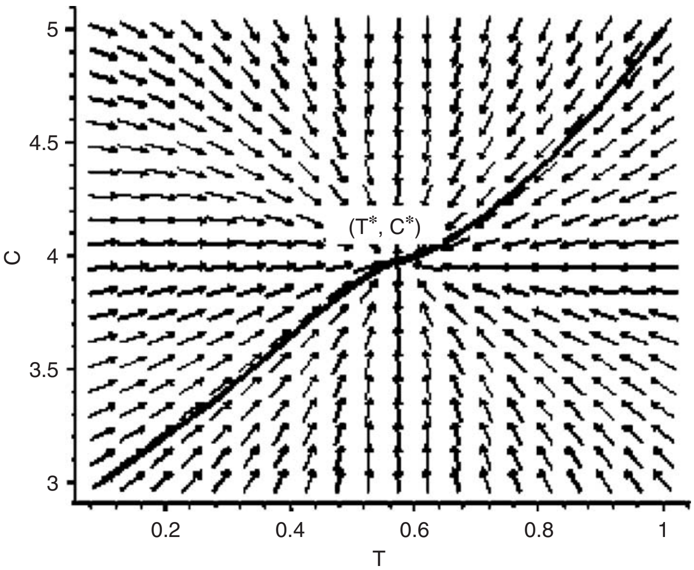

Fig. 5. Global stability in T–C plane.

the food chain involving bacteria, protozoa and an aquatic population (for example a fish population) in a water body. It has been shown that if the cumulative rate of discharge of pollutants increases, the equilibrium level of dissolved oxygen decreases, the amount of which depends on the rate of growth of oxygen as well as on various biodegradation and biochemical processes in the water body. It has also been shown that under certain conditions, if the cumulative rate of introduction of water pollutants is too high, the equilibrium concentration of the dissolved oxygen may become negligibly small, threatening the survival of biological species in a water body.

A model to control the cumulative discharge of organic pollutants in a water body is also proposed and analyzed. It has been shown that the equilibrium concentration of the dissolved oxygen can be maintained at a desired level by using an appropriate effort to control the cumulative discharge of organic pollutants.

Appendix A. Local stability analysis

The general variational matrix M for system (2.1) is given as follows:

$$M = \begin{bmatrix} -f_1(T,B) & -\dfrac{K_1 T}{(K_{12}+K_{11}T)} & 0 & 0 & 0 \\ \dfrac{\lambda_1 K_1 K_{12} B}{(K_{12}+K_{11}T)^2} & f_2(T,B,P) & -\dfrac{K_2 B}{(K_{21}+K_{22}B)} & 0 & 0 \\ 0 & \dfrac{\lambda_2 K_2 K_{21} P}{(K_{21}+K_{22}B)^2} & f_3(B,P,N) & -K_3 P & 0 \\ 0 & 0 & \lambda_3 K_3 N & f_4(P,N) & 0 \\ -g_1(T,B) & -g_2(T,B,P) & -g_3(B,N) & -g_4(P) & -\alpha_4 \end{bmatrix},$$

where

$$f_1(T,B) = \left[\alpha_0 + \frac{K_1 K_{12} B}{(K_{12}+K_{11}T)^2}\right],$$

$$f_2(T,B,P) = \left[\frac{\lambda_1 K_1 T}{(K_{12}+K_{11}T)} - \alpha_1 - 2\lambda_{10} B - \frac{K_2 K_{21} P}{(K_{21}+K_{22}B)^2}\right],$$

$$f_3(B,P,N) = \frac{\lambda_2 K_2 B}{(K_{21}+K_{22}B)} - \alpha_2 - 2\lambda_{20} P - K_3 N, \quad f_4(P,N) = \lambda_3 K_3 P - \alpha_3 - 2\lambda_{30} N,$$

$$g_1(T,B)=\frac{\lambda_{12}K_1K_{12}B}{(K_{12}+K_{11}T)^2},\quad g_2(T,B,P)=\frac{\lambda_{12}K_1T}{K_{12}+K_{11}T}+\frac{\lambda_{23}K_2K_{21}P}{(K_{21}+K_{22}B)^2}+\lambda_{11}\alpha_1,$$

$$g_3(B,N)=\frac{\lambda_{23}K_2B}{K_{21}+K_{22}B}+\lambda_{22}\alpha_2+\lambda_{34}K_3N,\quad g_4(P)=\lambda_{34}K_3P+\lambda_{33}\alpha_3. \tag{A.1}$$

We use eigenvalue method to study the local stability behavior of equilibria E_i $(i=1,2,3)$.

Let M_i be the matrix obtained from M after substituting for E_i.

For the equilibrium E_1, we note that one of the eigenvalues of M_1 is $((\lambda_1K_1-K_{11}\alpha_1)Q-K_{12}\alpha_0\alpha_1)/(K_{12}\alpha_0+K_{11}Q)$, which is positive whenever E_2 exists (condition (3.1)). Hence, E_1 is unstable whenever E_2 exists.

For the equilibrium $E_2(T_2^*,B_2^*,0,0,C_2^*)$, we note that one of the eigenvalues of M_2 is $(\lambda_2K_2B_2^*/(K_{21}+K_{22}B_2^*)-\alpha_2)$, which is positive whenever E_3 exists (as $B_3^*<B_2^*$). Thus, E_2 is unstable whenever E_3 exists.

For the equilibrium $E_3(T_3^*,B_3^*,P_3^*,0,C_3^*)$, we note that one of the eigenvalues of M_3 is $(\lambda_3K_3P_3^*-\alpha_3)$, which is positive whenever E_4 exists (as $P^*<P_3^*$). This shows that E_3 is unstable whenever E_4 exists.

Since in our study E_4 is the most interesting equilibrium from ecological point of view and its behavior cannot be described in a simple manner from M_4, we discuss its behavior by using Lyapunov's method.

Proof of Theorem 1. We linearize system (2.1) by using the transformations

$$T=T^*+\tau,\ B=B^*+b,\ P=P^*+p,\ N=N^*+n,\ C=C^*+c. \tag{A.2}$$

Now we consider the following positive definite function:

$$V=\frac{1}{2}\left(\tau^2+\frac{m_1}{B^*}b^2+\frac{m_2}{P^*}p^2+\frac{m_3}{N^*}n^2+m_4c^2\right), \tag{A.3}$$

and use the linearized system of (2.1) to get

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t}&=-f_1(T^*,B^*)\tau^2-m_1f_5(B^*,P^*)b^2,\\ &-m_2\lambda_{20}p^2-m_3\lambda_{30}n^2-m_4\alpha_4c^2,\\ &-m_4g_1(T^*,B^*)\tau c-m_4g_2(T^*,B^*,P^*)bc-m_4g_3(B^*,N^*)pc-m_4g_4(P^*)nc,\end{aligned} \tag{A.4}$$

where the functions f_1 and g_i $(i=1,2,3,4)$ are given by (A.1), and

$$f_5(B,P)=\lambda_{10}-(K_2K_{22}P/(K_{21}+K_{22}B)^2).$$

Now choosing $m_1=T^*(K_{12}+K_{11}T^*)/(\lambda_1K_{12})$, $m_2=m_1(K_{21}+K_{22}B^*)/(\lambda_2K_{21})$, $m_3=m_2/\lambda_3$ we note that the coefficient of b^2 is negative under condition (4.1). Thus $\mathrm{d}V/\mathrm{d}t$ can be made negative definite by appropriately choosing a positive value for m_4 following usual methods of nonlinear analysis [4,7,16,17].

Appendix B. Proof of Theorem 2

To prove this theorem we consider the following positive definite function:

$$\begin{aligned}V=&(T-T^*)^2/2+m_1(B-B^*-B^*\ln(B/B^*))+m_2(P-P^*-P^*\ln(P/P^*))\\ &+m_3(N-N^*-N^*\ln(N/N^*))+m_4(C-C^*)^2/2.\end{aligned} \tag{B.1}$$

Now using system (2.1) we get,

$$
\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} = & -\frac{K_1K_{12}B(T-T^*)^2}{(K_{12}+K_{11}T)(K_{12}+K_{11}T^*)} - m_1K_2K_{22}P^*\left(\frac{1}{K_{21}^2}-\frac{1}{(K_{21}+K_{22}B)(K_{21}+K_{22}B^*)}\right)(B-B^*)^2 \\
& - \alpha_0(T-T^*)^2 - m_1\left[\lambda_{10}-\frac{K_2K_{22}}{(K_{21}+K_{22}B^*)^2}P^*\right](B-B^*)^2 - m_2\lambda_{20}(P-P^*)^2 \\
& - m_3\lambda_{30}(N-N^*)^2 - m_4\alpha_4(C-C^*)^2 \\
& + (T-T^*)(B-B^*)\left[\frac{K_1K_{12}T^*T}{(K_{12}+K_{11}T)(K_{12}+K_{11}T^*)}\right] \\
& - m_4(T-T^*)(C-C^*)\left[\frac{\lambda_{12}K_1K_{12}B}{(K_{12}+K_{11}T)(K_{12}+K_{11}T^*)}\right] \\
& - m_4(B-B^*)(C-C^*)\left[\frac{\lambda_{12}K_1T^*}{K_{12}+K_{11}T^*}+\frac{\lambda_{23}K_2K_{21}P}{(K_{21}+K_{22}B)(K_{21}+K_{22}B^*)}+\lambda_{11}\alpha_1\right] \\
& - m_4(P-P^*)(C-C^*)\left[\frac{\lambda_{23}K_2B^*}{K_{21}+K_{22}B^*}+\lambda_{34}K_3N^*+\lambda_{22}\alpha_2\right] \\
& - m_4(N-N^*)(C-C^*)[\lambda_{34}K_3P+\lambda_{33}\alpha_3].
\end{aligned} \tag{B.2}
$$

Note that

$$
\frac{1}{K_{21}^2}-\frac{1}{(K_{21}+K_{22}B)(K_{21}+K_{22}B^*)}>0
$$

inside the region of attraction Ω.

Now choosing $m_1 = T^*/\lambda_1, m_2 = m_1(K_{21}+K_{22}B^*)/(\lambda_2K_{21}), m_3 = m_2/\lambda_3$, we note that $\mathrm{d}V/\mathrm{d}t$ can be made negative definite inside Ω if

$$
\lambda_{10}-\frac{K_2K_{22}}{K_{21}^2}P^*>0, \tag{B.3}
$$

$$
\frac{\alpha_0T^*}{\lambda_1}\left[\lambda_{10}-\frac{K_2K_{22}}{K_{21}^2}P^*\right]-\left[\frac{K_1K_{12}T^*}{(K_{12}+K_{11}T^*)}\frac{Q}{(K_{12}\alpha_0+K_{11}Q)}\right]^2>0 \tag{B.4}
$$

by making an appropriate choice for m_4 as pointed out earlier.

Appendix C. Proof of Theorem 3

We linearize system (5.1) by using the following transformations:

$$
T=\tilde{T}+\tau,\quad B=\tilde{B}+b,\quad P=\tilde{P}+p,\quad N=\tilde{N}+n,\quad C=\tilde{C}+c,\quad e=\tilde{e}+e_1.
$$

Now using the following positive definite function:

$$
V=\frac{1}{2}\left(\tau^2+\frac{m_1}{\tilde{B}}b^2+\frac{m_2}{\tilde{P}}p^2+\frac{m_3}{\tilde{N}}n^2+m_4c^2+m_5e_1^2\right), \tag{C.1}
$$

and use the linearized system of (5.1) to get

$$
\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} = & -f_1(\tilde{T},\tilde{B})\tau^2 - m_1f_5(\tilde{B},\tilde{P})b^2 - m_2\lambda_{20}p^2 - m_3\lambda_{30}n^2 - m_4\alpha_4c^2 - \frac{\gamma r_{10}}{r_1}e_1^2 \\
& - m_4g_1(\tilde{T},\tilde{B})\tau c - m_4g_2(\tilde{T},\tilde{B},\tilde{P})bc - m_4g_3(\tilde{B},\tilde{N})pc - m_4g_4(\tilde{P})nc
\end{aligned} \tag{C.2}
$$

and m_1's chosen as positive constants in the following manner: $m_1=\tilde{T}(K_{12}+K_{11}\tilde{T})/(\lambda_1K_1), m_2=m_1(K_{21}+K_{22}\tilde{B})/(\lambda_2K_{21}), m_3=m_2/\lambda_3, m_5=\gamma/r_1$.

If the following condition is satisfied:

$$\lambda_{10} - \frac{K_2 K_{22}}{K_{21}^2}\,\tilde{P} > 0.$$

then $\mathrm{d}V/\mathrm{d}t$ can be made negative definite by appropriate choice of $m_4 > 0$. Hence $\tilde{E}_4$ is locally stable under above condition.

Appendix D. Proof of Theorem 4

We prove this theorem by using the following positive definite function:

$$V = \frac{1}{2}(T-\tilde{T})^2 + m_1\left(B - \tilde{B} - \tilde{B}\,\ln\frac{B}{\tilde{B}}\right) + m_2\left(P - \tilde{P} - \tilde{P}\,\ln\frac{P}{\tilde{P}}\right) + m_3\left(N - \tilde{N} - \tilde{N}\,\ln\frac{N}{\tilde{N}}\right) + \frac{1}{2}m_4(C-\tilde{C})^2 + \frac{1}{2}m_5(e-\tilde{e})^2, \tag{D.1}$$

where m_1, m_2, m_3, m_4, m_5 are some positive constants to be chosen appropriately.

Using system (5.1) with the constants m_1, m_2, m_3 and m_5 chosen as follows: $m_1 = \tilde{T}(K_{12}+K_{11}\tilde{T})/(\lambda_1 K_1)$, $m_2 = m_1(K_{21}+K_{22}\tilde{B})/(\lambda_2 K_{21})$, $m_3 = m_2/\lambda_3$, $m_5 = \gamma/r_1$, we get

$$\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} = &- \frac{K_1 K_{12} B (T-\tilde{T})^2}{(K_{12}+K_{11}T)(K_{12}+K_{11}\tilde{T})} \\
&- m_1 K_2 K_{22}\tilde{P}\left(\frac{1}{K_{21}^2} - \frac{1}{(K_{21}+K_{22}B)(K_{21}+K_{22}\tilde{B})}\right)(B-\tilde{B})^2 - m_5 r_{10}(e-\tilde{e})^2 \\
&- \alpha_0 (T-\tilde{T})^2 - m_1\left[\lambda_{10} - \frac{K_2 K_{22}}{K_{21}^2}\,\tilde{P}\right](B-\tilde{B})^2 \\
&- m_2\lambda_{20}(P-\tilde{P})^2 - m_3\lambda_{30}(N-\tilde{N})^2 - m_4\alpha_4(C-\tilde{C})^2 \\
&+ (T-\tilde{T})(B-\tilde{B})\left[-\frac{K_1 K_{12}\tilde{T}T}{(K_{12}+K_{11}T)(K_{12}+K_{11}\tilde{T})}\right] \\
&- m_4(T-\tilde{T})(C-\tilde{C})\left[\frac{\lambda_{12}K_1 K_{12} B}{(K_{12}+K_{11}T)(K_{12}+K_{11}\tilde{T})}\right] \\
&- m_4(B-\tilde{B})(C-\tilde{C})\left[\frac{\lambda_{12}K_1\tilde{T}}{K_{12}+K_{11}\tilde{T}} + \frac{\lambda_{23}K_2 K_{21} P}{(K_{21}+K_{22}B)(K_{21}+K_{22}\tilde{B})} + \lambda_{11}\alpha_1\right] \\
&- m_4(P-\tilde{P})(C-\tilde{C})\left[\frac{\lambda_{23}K_2\tilde{B}}{K_{21}+K_{22}\tilde{B}} + \lambda_{34}K_3\tilde{N} + \lambda_{22}\alpha_2\right] \\
&- m_4(N-\tilde{N})(C-\tilde{C})[\lambda_{34}K_3 P + \lambda_{33}\alpha_3].
\end{aligned} \tag{D.2}$$

Now $\mathrm{d}V/\mathrm{d}t$ can be made negative definite in Ω_c provided the following conditions (as given in the theorem) are satisfied:

$$\lambda_{10} - \frac{K_2 K_{21}}{K_{21}^2}\,\tilde{P} > 0$$

and

$$\frac{\alpha_0\tilde{T}}{\lambda_1}\left[\lambda_{10} - \frac{K_2 K_{22}}{K_{21}^2}\,\tilde{P}\right] - \left[\frac{K_1 K_{12}\tilde{T}}{K_{12}+K_{11}\tilde{T}}\,\frac{Q}{K_{12}\alpha_0 + K_{11}Q}\right]^2 > 0$$

for a suitable choice of $m_4 > 0$.

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