PERGAMON Nonlinear Analysis: Real World Applications 4 (2003) 625–637

Nonlinear Analysis
Real World Applications

www.elsevier.com/locate/na

A model for fishery resource with reserve area

B. Dubey[a,*], Peeyush Chandra[b], Prawal Sinha[b]

[a] *Mathematics Group, Birla Institute of Technology & Science, Pilani 333031, India*
[b] *Department of Mathematics, Indian Institute of Technology, Kanpur 208016, India*

Received 18 June 2001; accepted 11 September 2002

Abstract

In this paper, we propose and analyse a mathematical model to study the dynamics of a fishery resource system in an aquatic environment that consists of two zones: a free fishing zone and a reserve zone where fishing is strictly prohibited. Biological and bionomic equilibria of the system are obtained, and criteria for local stability, instability and global stability of the system are derived. It is shown that even if fishery is exploited continuously in the unreserved zone, fish populations can be maintained at an appropriate equilibrium level in the habitat. An optimal harvesting policy is also discussed using the Pantryagin's Maximum Principle.

Keywords: Fishery resource; Stability; Harvesting; Reserved zone

1. Introduction

In a renewable resource (e.g. fishery and forestry) management, maximizing the present value of benefits derived from the resources and its conservation are important problems to be studied. During the last few decades several investigations regarding fishery resource have been conducted [1,5–11,17,21–24]. In particular, Leung and Wang [21] presented a mathematical model for commercial fishing to study the phenomena of non-explosive fishing capital and non-extinctive fishery resources. Clark et al. [10] studied the effects of irreversibility of capital investment upon optimal exploitation policies for renewable resource stocks. Kitabatake [19] developed a dynamic model for fishery resources with predator–prey relationship based on observational data for Lake Kasumigaura in Japan. Ragozin and Brown [30] studied the optimal harvesting policy for predator–prey system in which the prey has no commercial value and the

* Corresponding author. Fax: +91-159-76-44-183.
E-mail addresses: bdubey@bits-pilani.ac.in (B. Dubey), peeyush@iitk.ac.in (P. Chandra), prawal@iitk.ac.in (P. Sinha).

1468-1218/03/$ - see front matter
doi:10.1016/S1468-1218(02)00082-2

predator is selectively harvested. Based on the work of Clark [6, p. 303–314], Chaudhuri [12] proposed a model to study the combined harvesting of two competing fish species and showed that the open-access fishery may have a bionomic equilibrium and one species may be driven to extinction. Mesterton-Gibbons [22] extended the work of Clark [6, p. 303–314] and found criteria for the survival of less productive species as a function of the system parameters and initial stocks. Mesterton-Gibbons [23] also investigated an optimal policy to maximize the present value from the combined harvest of predator and prey. He found an estimate to the true loss of resource value due to catastrophic fall in stock level. Chaudhuri [13] developed a mathematical model to study the dynamic optimization of combined harvest of two competing fish species. Ganguli and Chaudhuri [16] studied the regulation of single-species fishery with taxation as a control variable. Mesterton-Gibbons [24] described a different technique to find the solution of Chaudhuri's model [12] and showed that this technique may be widely applicable in ecological modelling. Chattopadhyay et al. [4] obtained conditions for persistence and global stability in three-species fishery. Mukhopadhyay et al. [25] also studied the selective harvesting of two-species fishery by incorporating discrete time delay in harvesting both the species. Fan and Wang [15] proposed and analysed a model to study the exploitation of a single-fish population modelled by logistic equation with periodic coefficients. Hanson and Ryan [18] investigated effects of large inflationary price fluctuations on the computed harvest strategy for a stochastic Schaefer model. They found that inflationary effects have a pronounced influence on the optimal return. However, optimal harvesting effort levels are much less sensitive to inflationary effects. Pradhan and Chaudhuri [26] discussed the optimal harvesting policy of a single-species fishery with the Gompertz law of population growth. Pradhan and Chaudhuri [27] also studied the growth and exploitation of a schooling fish species. Pradhan and Chaudhuri [28] further developed a dynamic model of two competing fish species with taxation as a control instrument. Pradhan and Chaudhuri [29] also proposed a model to study the selective harvesting in an inshore–offshore fishery. Zhang et al. [32] proposed and analysed a model to study the optimal harvesting policy of a stage structured problem and derived necessary and sufficient condition for the coexistence and extinction of species. Recently, Song and Chen [31] discussed the optimal harvesting policy and stability for a two-species competitive system and derived conditions for the existence of a globally asymptotically stable positive equilibrium and a threshold of harvesting for the mature population. Dubey et al. [14] proposed a dynamic model for a single-species fishery which depends partially on a logistically growing resource. They showed that both the equilibrium density of fish population as well as the *maximum sustainable yield* increase as the resource biomass density increases.

From the above-mentioned brief literature survey it may be noted here that the effect of a reserve zone in an aquatic habitat on the exploitation of fishery resource has not been modelled and analysed. We model this phenomenon in an aquatic habitat that consists of two zones: one free fishing zone and the other a reserve zone where fishing is not permitted. We derive conditions for the existence of biological and bionomical equilibria and study their stability behaviour. Finally, we discuss the optimal harvesting policy using the Pantryagin's Maximum Principle [3].

2. The model

Consider a fishery habitat, in an aquatic ecosystem, consisting of reserved and unreserved areas. In modelling the system, it is considered that no fishing is permitted in the reserved zone while the unreserved area is an open-access fishery zone. Let $x(t)$ and $y(t)$ be the respective biomass densities of the same fish population inside the unreserved and reserved areas, respectively, at a time t. Let the fish subpopulation of the unreserved area migrate into reserved area at a rate σ_1 and the fish subpopulation of the reserved area migrate into unreserved area at a rate σ_2. Let E be the total effort applied for harvesting the fish population in the unreserved area. We assume that in each zone growth of fish population follows logistic model. Keeping these in view, the dynamics of fish subpopulations in unreserved and reserved areas may be governed by the following autonomous system of differential equations:

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right) - \sigma_1 x + \sigma_2 y - qEx,$$

$$\frac{dy}{dt} = sy\left(1 - \frac{y}{L}\right) + \sigma_1 x - \sigma_2 y, \quad x(0) > 0, y(0) > 0. \tag{2.1}$$

In the above model, r and s are the intrinsic growth rates of fish subpopulation inside the unreserved and reserved areas, respectively, K and L are the carrying capacities of fish species in the unreserved and reserved areas, respectively, q is the catchability coefficient of fish species in the unreserved area. The parameters r, s, q, σ_1, σ_2, K and L are assumed to be positive constants.

We note that if there is no migration of fish population from reserved area to unreserved area (i.e. $\sigma_2 = 0$) and $r - \sigma_1 - qE < 0$, then $\dot{x} < 0$. Similarly, if there is no migration of fish population from unreserved area to reserved area (i.e. $\sigma_1 = 0$) and $s - \sigma_2 < 0$, then $\dot{y} < 0$. Hence throughout our analysis we assume that

$$r - \sigma_1 - qE > 0, \quad s - \sigma_2 > 0. \tag{2.2}$$

3. Existence of equilibria

Equilibria of model (2.1) is obtained by solving $\dot{x} = \dot{y} = 0$. It can be checked that model (2.1) has only two nonnegative equilibria, namely $P_0(0, 0)$ and $P^*(x^*, y^*)$. Here x^*, y^* are the positive solutions of the following algebraic equations:

$$\sigma_2 y = \frac{rx^2}{K} - (r - \sigma_1 - qE)x, \tag{3.1a}$$

$$\sigma_1 x = (\sigma_2 - s)y + \frac{sy^2}{L}. \tag{3.1b}$$

Substituting the value of y from Eq. (3.1a) into Eq. (3.1b), we get a cubic equation in x as

$$ax^3 + bx^2 + cx + d = 0, \tag{3.2}$$

where

$$a = \frac{sr^2}{L\sigma_2^2 K^2},$$

$$b = -\frac{2sr(r-\sigma_1-qE)}{L\sigma_2^2 K},$$

$$c = \frac{s(r-\sigma_1-qE)^2}{L\sigma_2^2} - \frac{(s-\sigma_2)r}{K\sigma_2},$$

$$d = \frac{(s-\sigma_2)}{\sigma_2}(r-\sigma_1-qE) - \sigma_1. \tag{3.3}$$

The above equation has a unique positive solution $x = x^*$ if the following inequalities hold:

$$\frac{s(r-\sigma_1-qE)^2}{L\sigma_2} < \frac{(s-\sigma_2)r}{K}, \tag{3.4a}$$

$$(s-\sigma_2)(r-\sigma_1-qE) < \sigma_1\sigma_2. \tag{3.4b}$$

Knowing the value of x^*, the value of y^* can then be computed from (3.1a). It may be noted here that for y^* to be positive, we must have

$$\frac{rx^*}{K} > r-\sigma_1-qE. \tag{3.4c}$$

4. Dynamical behaviour of equilibria

The dynamical behaviour of equilibria can be studied by computing variational matrices corresponding to each equilibrium. Keeping in view Eq. (2.2), we note that the trivial equilibrium P_0 is unstable. Using the Routh–Hurwitz criteria, it is easy to check that all eigenvalues of the variational matrix corresponding to P^* have negative real parts, and hence P^* is locally asymptotically stable in the x–y plane. This implies that we can find a small circle with centre P^* such that any solution $(x(t), y(t))$ of system (1), which is inside the circle at some time $t = t_1$, will remain inside the circle for all $t \geqslant t_1$ and will tend to (x^*, y^*) as $t \to \infty$.

In the following lemma we show that all solutions of model (2.1) are positive and uniformly bounded. The proof of this lemma is deferred to Appendix A.

Lemma 4.1. *The set*

$$\Omega = \left\{(x,y) \in R_2^+ : w = x + y \leqslant \frac{\mu}{\eta}\right\}$$

is a region of attraction for all solutions initiating in the interior of the positive quadrant, where η is a positive constant and

$$\mu = \frac{K}{4r}(r+\eta-qE)^2 + \frac{L}{4s}(s+\eta)^2.$$

In the following theorem we show that the positive equilibrium P^* is globally asymptotically stable.

Theorem 4.1. *The nontrivial equilibrium P^* is globally asymptotically stable with respect to all solutions initiating in the interior of the positive quadrant.*

Proof. Consider the following positive definite function about P^*:

$$V = \left(x - x^* - x^* \ln \frac{x}{x^*}\right) + \frac{y^*\sigma_2}{x^*\sigma_1}\left(y - y^* - y^* \ln \frac{y}{y^*}\right). \tag{4.1}$$

Differentiating V with respect to time t along the solutions of model (2.1), a little algebraic manipulation yields

$$\frac{dV}{dt} = -\frac{r}{K}(x - x^*)^2 - \frac{y^*\sigma_2 s}{x^*\sigma_1 L}(y - y^*)^2 - \frac{\sigma_2}{x^* xy}(xy^* - x^* y)^2 < 0. \tag{4.2}$$

This shows that dV/dt is negative definite, and hence by Liapunov's theorem on stability [20], it follows that the positive equilibrium P^* is globally asymptotically stable with respect to all solutions initiating in the interior of the positive quadrant. □

The above theorem implies that in an open-access fishery region, if a subregion is reserved where fishing is not allowed and fish populations are harvested only outside the reserved subregion, then in both the reserved and unreserved zones fish species settle down to their respective equilibrium levels, whose magnitudes depend upon the intrinsic growth rates of fish species, their migration coefficients and carrying capacities. This implies that fish populations may be sustained at an appropriate equilibrium level even after continuous harvesting of fish populations in unreserved area.

In the following theorem, we show that system (2.1) does not have any periodic solution.

Theorem 4.2. *System (2.1) cannot have any limit cycle in the interior of the positive quadrant.*

Proof. Let

$$H(x, y) = \frac{1}{xy},$$

$$h_1(x, y) = rx\left(1 - \frac{x}{K}\right) - \sigma_1 x + \sigma_2 y - qEx,$$

$$h_2(x, y) = sy\left(1 - \frac{y}{L}\right) + \sigma_1 x - \sigma_2 y.$$

Clearly $H(x, y) > 0$ in the interior of the positive quadrant of xy-plane. Then we have,

$$\Delta(x, y) = \frac{\partial}{\partial x}(Hh_1) + \frac{\partial}{\partial y}(Hh_2)$$

$$= -\frac{1}{y}\left[\frac{r}{K} + \frac{\sigma_2 y}{x^2}\right] - \frac{1}{x}\left[\frac{s}{L} + \frac{\sigma_1 x}{y^2}\right] < 0.$$

This shows that $\Delta(x, y)$ does not change sign and is not identically zero in the positive quadrant of xy-plane. By Bendixson–Dulac criteria, it follows that system (2.1) has no closed trajectory, and hence no periodic solution in the interior of the positive quadrant of xy-plane. Thus, the theorem follows. □

5. Bionomic equilibrium

In the fishery literature, the bionomic equilibrium is said to be achieved when the total revenue obtained by selling the harvested biomass equals the total cost utilized in harvesting it. In such a case, the economic rent is completely dissipated. Let c be the fishing cost per unit effort and p the price per unit biomass of the landed fish. Then net economic revenue at any time t is given by

$$\pi(x, E, t) = (pqx - c)E. \tag{5.1}$$

The bionomic equilibrium is $P_\infty(x_\infty, y_\infty, E_\infty)$, where $x_\infty, y_\infty, E_\infty$ are the positive solutions of

$$\dot{x} = \dot{y} = \pi = 0. \tag{5.2}$$

It may be noted here that if $c > pqx$, i.e. if the fishing cost exceeds the revenue obtained from it, then the economic rent obtained from the fishery becomes negative. Hence the fishery will be closed and no bionomic equilibrium exists. Therefore, for the existence of bionomic equilibrium, it is natural to assume $pqx > c$. We then have

$$x_\infty = \frac{c}{pq}, \tag{5.3a}$$

$$y_\infty = \frac{L}{2s}\left[(s - \sigma_2) + \left\{(s - \sigma_2)^2 + \frac{4s\sigma_1 c}{Lpq}\right\}^{1/2}\right], \tag{5.3b}$$

$$E_\infty = \frac{r}{q}\left(1 - \frac{c}{pqK}\right) - \frac{\sigma_1}{q} + \frac{\sigma_2 p y_\infty}{c}. \tag{5.3c}$$

It is clear that $E_\infty > 0$ if

$$\frac{r}{q}\left(1 - \frac{c}{pqK}\right) > \frac{\sigma_1}{q} - \frac{\sigma_2 p y_\infty}{c}. \tag{5.4}$$

Thus, the bionomic equilibrium $P_\infty(x_\infty, y_\infty, E_\infty)$ exists if, in addition to the second inequality of Eq. (2.2), inequality (5.4) holds. We note that E_∞ increases as σ_2 increases and it decreases as σ_1 increases. Also, y_∞ increases are σ_1 increases and it decreases that σ_2 increases. From Eq. (5.3c) it may be noted that bionomic equilibrium effort does not depend upon the growth rate and carrying capacity of the reserve area.

If $E > E_\infty$, then the total cost utilized in harvesting the fish population would exceed the total revenues obtained from the fishery industry. Hence some of the fishermen would be in loss and naturally they would withdraw their participation from the fishery industry. Hence $E > E_\infty$ cannot be maintained indefinitely. If $E < E_\infty$, then the fishery

is more profitable and hence in an open-access fishery it would attract more and more fishermen. This will have an increasing effect on the harvesting effort. Hence $E < E_\infty$ also cannot be maintained indefinitely.

6. Optimal harvesting policy

In this section we employ the Pontryagins's Maximum Principle to obtain a path of optimal harvesting policy so that if the fish populations inside and outside the reserve zones are kept along this path, then the regulatory agency is assured to achieve its objective. The present value J of a continuous time-stream of revenues is given by

$$J = \int_0^\infty e^{-\delta t}(pqx(t) - c)E(t)\,dt, \tag{6.1}$$

where δ is the instantaneous rate of annual discount. Thus, our objective is to maximize J subject to state equations (2.1) and to the control constraints

$$0 \leqslant E \leqslant E_{\max}. \tag{6.2}$$

The associated Hamiltonian is given by

$$\begin{aligned} H &= e^{-\delta t}(pqx - c)E + \lambda_1(t)\left[rx\left(1 - \frac{x}{K}\right) - \sigma_1 x + \sigma_2 y - qEx\right] \\ &\quad + \lambda_2(t)\left[sy\left(1 - \frac{y}{L}\right) + \sigma_1 x - \sigma_2 y\right] \\ &= \sigma(t)E + \lambda_1(t)\left[rx\left(1 - \frac{x}{K}\right) - \sigma_1 x + \sigma_2 y\right] \\ &\quad + \lambda_2(t)\left[sy\left(1 - \frac{y}{L}\right) + \sigma_1 x - \sigma_2 y\right], \end{aligned} \tag{6.3}$$

where λ_1, λ_2 are the adjoint variables and $\sigma(t) = e^{-\delta t}(pqx - c) - \lambda_1 qx$ is called the switching function [6].

Since H is linear in the control variable E, the optimal control will be a combination of bang–bang control and singular control. The optimal control $E(t)$ which maximizes H must satisfy the following conditions:

$$E = E_{\max}, \quad \text{when } \sigma(t) > 0, \text{ i.e. when } \lambda_1(t)e^{\delta t} < p - \frac{c}{qx}. \tag{6.4a}$$

$$E = 0, \quad \text{when } \sigma(t) < 0, \text{ i.e. when } \lambda_1(t)e^{\delta t} > p - \frac{c}{qx}. \tag{6.4b}$$

$\lambda_1(t)e^{\delta t}$ is the usual shadow price [6] and $p - (c/qx)$ is the net economic revenue on a unit harvest. This shows that $E = E_{\max}$ or zero according to the shadow price is less than or greater than the net economic revenue on a unit harvest. Economically, condition (6.4a) implies that if the profit after paying all the expenses is positive, then it is beneficial to harvest up to the limit of available effort. Condition (6.4b) implies that when the shadow price exceeds the fisherman's net economic revenue on a unit harvest, then the fisherman will not exert any effort.

When $\sigma(t) = 0$, i.e. when the shadow price equals the net economic revenue on a unit harvest, then the Hamiltonian H becomes independent of the control variable $E(t)$, i.e. $\partial H/\partial E = 0$. This is the necessary condition for the singular control $E^*(t)$ to be optimal over the control set $0 < E^* < E_{\max}$.

Thus, the optimal harvesting policy is

$$E(t) = \begin{cases} E_{\max}, & \sigma(t) > 0, \\ 0, & \sigma(t) < 0, \\ E^*, & \sigma(t) = 0. \end{cases} \tag{6.5}$$

When $\sigma(t) = 0$, it follows that

$$\lambda_1 qx = e^{-\delta t}(pqx - c) = e^{-\delta t}\frac{\partial \pi}{\partial E}. \tag{6.6}$$

This implies that the user's cost of harvest per unit of effort equals the discounted value of the future marginal profit of the effort at the steady-state level.

Now in order to find the path of singular control we utilize the Pontryagin's Maximum Principle. Using this principle, we obtain (see Appendix B) an equation to the singular path as

$$\frac{B_2}{B_1 + \delta} = p - \frac{c}{qx^*}, \tag{6.7}$$

where

$$B_1 = \frac{rx^*}{K} + \frac{\sigma_2 y^*}{x^*}, \quad B_2 = pqE + \frac{A_2\sigma_1}{A_1 + \delta},$$
$$A_1 = \frac{sy^*}{L} + \frac{\sigma_1 x^*}{y^*}, \quad A_2 = \sigma_2\left(p - \frac{c}{qx^*}\right). \tag{6.8}$$

Using (3.1a) and (3.1b), A_1, A_2, B_1, B_2 can be written as

$$A_1 = \sigma_2 - s + \frac{2sx^*}{L\sigma_2}\left[\frac{rx^*}{K} - (r - \sigma_1 - qE)\right],$$
$$A_2 = \frac{\sigma_2}{qx^*}(pqx^* - c),$$
$$B_1 = \frac{2rx^*}{K} - (r - \sigma_1 - qE),$$
$$B_2 = pqE + \frac{C_1}{qx^*}, \tag{6.9}$$

where

$$C_1 = \frac{\sigma_1\sigma_2(pqx^* - c)}{A_1 + \delta}.$$

Thus, Eq. (6.7) can be written as

$$pqx^* - c = h(x^*), \tag{6.10}$$

where

$$h(x^*) = \frac{pq^2Ex^* + C_1}{B_1 + \delta}.$$

Let

$$F(x^*) = h(x^*) - (pqx^* - c). \tag{6.11}$$

Then positive root of $F(x^*) = 0$ gives the optimal level of fish population $x^* = x_\delta$. It may be noted that there exists a unique $x^* = x_\delta$ in the interval $0 < x_\delta < K$ if the following inequalities hold:

$$F(0) < 0, \quad F(K) > 0, \quad F'(x^*) > 0 \quad \text{for } x^* > 0. \tag{6.12}$$

Knowing the value of $x^* = x_\delta$, the optimal level of fish population $y^* = y_\delta$ in the reserved zone and the optimal level of effort are given by, respectively,

$$y^* = y_\delta = \frac{1}{2s}[L(s - \sigma_2) + \{L^2(s - \sigma_2)^2 + 4sL\sigma_1 x_\delta\}^{1/2}], \tag{6.13}$$

$$E = E_\delta = \frac{1}{qx_\delta}\left[rx_\delta\left(1 - \frac{x_\delta}{K}\right) - \sigma_1 x_\delta + \sigma_2 y_\delta\right]. \tag{6.14}$$

It may be noted here that $y_\delta > 0$ if the second inequality of (2.2) is satisfied, and $E_\delta > 0$ if

$$rx_\delta\left(1 - \frac{x_\delta}{K}\right) + \sigma_2 y_\delta > \sigma_1 x_\delta. \tag{6.15}$$

From Eqs. (B.5) and (B.7) of Appendix B, we note that $\lambda_i(t)e^{\delta t}$ $(i=1,2)$ is independent of time in an optimum equilibrium. Hence they satisfy the transversality condition at ∞, i.e. they remain bounded as $t \to \infty$.

From Eq. (6.7) we also note that

$$pqx^* - c = \frac{B_2 qx^*}{B_1 + \delta} \to 0 \quad \text{as } \delta \to \infty.$$

Thus, the net economic revenue $\pi(x_\infty, y, E, t) = 0$.

This implies that in case of infinite discount rate, the net economic revenue to the society becomes zero and hence the fishery would remain closed. This shows that high interest rate will cause high inflation rate. If the inflation rate increases rapidly, the real value of the resource decreases. Hence the owner of the resource stock prefers to exploit it at no-profit–no-loss basis.

7. Conclusions

In this paper, a mathematical model has been proposed and analysed to study the dynamics of fishery resource. In modelling the system it has been assumed that the aquatic ecosystem consists of two zones: one free fishing zone and the other reserved

zone where fishing is strictly prohibited. It has also been assumed that fish populations are growing logistically inside and outside the reserved zone and they migrate from reserved zone to unreserved area and vice verse.

Using stability theory of ordinary differential equations it has been proved that the interior equilibrium exists under certain conditions and it is globally asymptotically stable. It has also been shown that the system under consideration does not have any limit cycle. It has been found that the bionomic equilibrium effort is independent of the growth rate and carrying capacity of the reserved zone. It has been further found that if a reserved zone is created in an open-access fishery region where fishing is not allowed and harvesting of fish populations is permitted only outside the reserved zone, then the fish populations settle down at their respective equilibrium levels inside as well as outside the reserved zone. The magnitudes of these equilibria mainly depend upon intrinsic growth rates and migration coefficients of the fish population and the carrying capacities of the unreserved and reserved zones. It has been noted that even under continuous harvesting of fish species outside the reserved zone, the fish population may be maintained at an appropriate equilibrium level.

Using the Pontryagin's Maximum Principle, the optimal harvesting policy has been discussed. It has been found that the total user's cost of harvest per unit of effort equals to the discounted value of the future marginal profit of the effort at the steady-state level. It has also been noted that if the discount rate increases, then the economic rent decreases and even may tend to zero if the discount rate tend to infinity. Thus, it has been concluded that high interest rate will cause high inflation rate.

Recently, ratio-dependent system interactions have attracted many researchers as ratio-dependent models produce richer dynamics. The dynamics of ratio-dependent fishery models is also an important area of research which is left for future investigations.

Acknowledgements

The authors are grateful to the reviewers for their critical evaluations and suggestions.

Appendix A. Proof of Lemma 4.1

Let $w(t) = x(t) + y(t)$, and $\eta > 0$ be a constant. Then we have

$$\begin{aligned}\frac{\mathrm{d}w}{\mathrm{d}t} + \eta w &= (r+\eta-qE)x - \frac{rx^2}{K} + (s+\eta)y - \frac{sy^2}{L} \\ &= \frac{K}{4r}(r+\eta-qE)^2 - \frac{r}{K}\left\{x - \frac{K}{2r}(r+\eta-qE)\right\}^2 + \frac{L}{4s}(s+\eta)^2 \\ &\quad - \frac{s}{L}\left\{y - \frac{L}{2s}(s+\eta)\right\}^2 \\ &\leqslant \frac{K}{4r}(r+\eta-qE)^2 + \frac{L}{4s}(s+\eta)^2 = \mu.\end{aligned}$$

By the theory of differential inequality [2], we have

$$0 < w(x(t), y(t)) \leqslant \frac{\mu}{\eta}(1 - e^{-\eta t}) + w(x(0), y(0))e^{-\eta t},$$

and when $t \to \infty$, $0 < w \leqslant (\mu/\eta)$, proving the lemma. □

Appendix B. Derivation of Eq. (6.7)

According to the Pontryagin's maximum Principle, the adjoint variables λ_1 and λ_2 must satisfy

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial x}, \quad \frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial y} \tag{B.1}$$

The above set of equations can be rewritten as

$$\frac{d\lambda_1}{dt} = -e^{-\delta t} pqE - \lambda_1 \left(r - \frac{2rx}{K} - \sigma_1 - qE \right) - \lambda_2 \sigma_1, \tag{B.2}$$

$$\frac{d\lambda_2}{dt} = -\lambda_1 \sigma_2 - \lambda_2 \left(s - \frac{2sy}{L} - \sigma_2 \right). \tag{B.3}$$

Considering the interior equilibrium $P^*(x^*, y^*)$ and Eq. (6.6), Eq. (B.3) can be written as

$$\frac{d\lambda_2}{dt} - A_1 \lambda_2 = -e^{-\delta t} A_2, \tag{B.4}$$

where A_1 and A_2 are defined in Eq. (6.9).

A solution of Eq. (B.4) is given by

$$\lambda_2(t) = \frac{A_2}{A_1 + \delta} e^{-\delta t}. \tag{B.5}$$

Similarly, considering the interior equilibrium, Eq. (B.2) can be written as

$$\frac{d\lambda_1}{dt} - B_1 \lambda_1 = -e^{-\delta t} B_2, \tag{B.6}$$

whose solution is given by

$$\lambda_1(t) = \frac{B_2}{B_1 + \delta} e^{-\delta t}, \tag{B.7}$$

where B_1 and B_2 are defined in Eq. (6.9).

From (6.6) and (B.7), we get the desired Eq. (6.7) which is an equation to the singular path.

References

[1] D.K. Bhattacharya, S. Begum, Bionomic equilibrium of two species system, Math. Biosci. 135 (2) (1996) 111–127.
[2] G. Birkhoff, G.S. Rota, Ordinary Differential Equations, Ginn, Boston, 1882.
[3] D.N. Burges, A. Graham, Introduction to Control Theory Including Optimal Control, Wiley, New York, 1980.
[4] J. Chattopadhyay, A. Mukhopadhyay, P.K. Tapaswi, A resourced based competitive system in three species fishery, Nonlinear Stud. 3 (1996) 73–83.
[5] C.W. Clark, Profit maximization and the extinction of animal species, Journal of Political Economy 81 (1973) 950–961.
[6] C.W. Clark, Mathematical Bioeconomics: The Optimal Management of Renewable Resources, Wiley, New York, 1976.
[7] C.W. Clark, Mathematical models in the economics of renewable resources, SIAM Rev. 21 (1979) 81–99.
[8] C.W. Clark, Bioeconomic Modelling and Fisheries Management, Wiley Interscience, New York, 1985.
[9] C.W. Clark, Mathematical Bioeconomics: The Optimal Management of Renewable Resources, Wiley, New York, 1990.
[10] C.W. Clark, F.H. Clarke, G.R. Mundro, The optimal exploitation of renewable resource stocks: problems of irreversible investment, Econometrica 47 (1) (1979) 25–47.
[11] C.W. Clark, J.D. De Pree, A simple linear model for the optimal exploitation of renewable resources, Appl. Math. Optim. 5 (1979) 181–196.
[12] K.S. Chaudhuri, A bioeconomic model of harvesting a multispecies fishery, Ecol. Model. 32 (1986) 267–279.
[13] K.S. Chaudhuri, Dynamic optimisation of combined harvesting of a two species fishery, Ecol. Model. 41 (1988) 17–25.
[14] B. Dubey, P. Chandra, P. Sinha, A resource dependent fishery model with optimal harvesting policy, J. Biol. Systems 10 (2002) 1–13.
[15] M. Fan, K. Wang, Optimal harvesting policy for single population with periodic coefficients, Math. Biosci. 152 (1998) 165–177.
[16] S. Ganguli, K.S. Chaudhuri, Regulation of a single-species fishery by taxation, Ecol. Model. 82 (1995) 51–60.
[17] B.S. Goh, Management and Analysis of Biological Populations, Elsevier, Amsterdam, 1980.
[18] F.B. Hanson, D. Ryan, Optimal harvesting with both population and price dynamics, Math. Biosci. 148 (1998) 129–146.
[19] Y. Kitabatake, A dynamic predator–prey model for fishery resources: a case of Lake Kasumigaura, Environ. Planning A 14 (1982) 225–235.
[20] J. La Salle, S. Lefschetz, Stability by Liapunov's Direct Method with Applications, Academic Press, New York, London, 1961.
[21] A. Leung, A. Wang, Analysis of models for commercial fishing: mathematical and economical aspects, Econometrica 44 (2) (1976) 295–303.
[22] M. Mesterton-Gibbons, On the optimal policy for combined harvesting of independent species, Nat. Res. Model. 2 (1987) 109–134.
[23] M. Mesterton-Gibbons, On the optimal policy for combined harvesting of predator-prey, Nat. Res. Model. 3 (1988) 63–90.
[24] M. Mesterton-Gibbons, A technique for finding optimal two species harvesting policies, Ecol. Model. 92 (1996) 235–244.
[25] A. Mukhopadhyay, J. Chattopadhyay, P.K. Tapaswi, Selective harvesting in a two species fishery model, Ecol. Model. 94 (1997) 243–253.
[26] T. Pradhan, K.S. Chaudhuri, Bioeconomic modelling of a single-species fishery with Gompertz law of growth, J. Biol. Systems 6 (4) (1998) 393–409.
[27] T. Pradhan, K.S. Chaudhuri, A dynamic reaction model of a two species fishery with taxation as a control instrument: a capital theoretic analysis, Ecol. Model. 121 (1999) 1–16.

[28] T. Pradhan, K.S. Chaudhuri, Bioeconomic harvesting of a schooling fish species: a dynamic reaction model, Korean J. Comput. Appl. Math. 6 (1) (1999) 127–141.
[29] T. Pradhan, K.S. Chaudhuri, Bioeconomic modelling of selective harvesting in an inshore–offshore fishery, Differential Equations Dyn. Systems 7 (3) (1999) 305–320.
[30] D.L. Ragozin Jr., G. Brown, Harvest policies and nonmarket valuation in a predator–prey system, J. Environ. Econom. Manage. 12 (1985) 155–168.
[31] X. Song, L. Chen, Optimal harvesting and stability for a two-species competitive system with stage structure, Math. Biosci. 170 (2) (2001) 173–186.
[32] X. Zhang, L. Chen, A.U. Neumann, The stage structured predator–prey model and optimal harvesting policy, Math. Biosci. 168 (2) (2000) 201–210.