

Mathematical modeling and analysis of the depletion of dissolved oxygen in eutrophied water bodies affected by organic pollutants

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Abstract

In this paper, an ecological type nonlinear mathematical model is proposed to study the simultaneous effect of water pollution and eutrophication on the concentration of dissolved oxygen (DO) in a water body. It is assumed that the organic pollutants and the nutrients are discharged into water body from outside with constant rates. The system is modeled by considering the variables such as cumulative concentration of organic pollutants, the densities of bacteria, nutrients, algae, detritus and the concentration of DO. The analysis of the model shows that the decrease in the concentration of DO due to simultaneous effect of water pollution and eutrophication is much more than when only single effect is present in the water body, thus leading to more uncertainty about the survival of DO-dependent species.

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1. Introduction

It is well known that in real situations, water bodies, such as lakes, are polluted by various kinds of organic and inorganic pollutants coming from household and industrial sources. Further, they are also eutrophied by nutrients discharged by water runoff from agricultural fields. When household and industrial wastes are discharged into water, organic matters and nutrients present in them are uptaken by bacteria and other biological species such as algae using dissolved oxygen (DO) in the interacting processes [3,7,11,32]. As these processes are part of a food chain in the water body, the level of DO decreases due to various interactive biochemical and biodegradation processes [18,19,25,28]. Moreover, when a water body is also eutrophied, a much more complex interacting phenomenon takes place involving algae, macrophytes, detritus, DO, etc. [5,9,13,22,24,29,30].

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It may be noted here that the input of the DO in a water body is mainly due to atmospheric diffusion through the water surface and to a certain extent due to its production by photosynthesis by algae and emerging macrophytes in a eutrophied water body. However, the oxygen from these processes does not result in an increase in the concentration of DO due to obstruction from algal bloom, etc. [6,27,31,35]. Further when algae die and sink to the bottom of the water body, detritus is formed. The detritus while decaying uses DO and hence reducing the concentration of DO in the bottom of water body (hypolimnion). Thus, the concentration of oxygen in the water body decreases due to organic pollutants as well as by eutrophication affecting various aquatic species dependent on DO [27].

The study of effect of the organic pollutants on DO in a water body started with the well-known linear model of Streeter and Phelps [28]. Since then this model has been generalized by many workers including Dobbins [7], O'Connor [25], Beck and Young [3] by considering various aspects. It is pointed out here that these models are only linear and do not take into account the nonlinear processes that involve interactions of DO with organic pollutants, bacteria and other biological species, as well as of nutrients with algae and zooplankton, etc. in a eutrophied water body. Also some investigators have studied the effect of discharge of nutrients in water bodies such as a lake causing eutrophication without considering the effect of water pollution [1,2,4,8,14–17,21,23,24,26,29,30,33,34]. In particular, Voinov and Tonkikh [34] have presented a eutrophication model in an unpolluted lake assuming that the nutrient is supplied only by detritus of algae and macrophytes and have not considered the discharge of nutrients by water runoff from agricultural fields. Jayaweera and Asaeda [16] have studied biomanipulation in shallow eutrophic lakes by using mathematical models involving phytoplankton, zooplankton, detritus, bacteria and fish population but they have not considered either the supply of nutrients by water runoff or the effect of organic pollutants on DO.

Further, some ecological type nonlinear models involving fast and slow biodegradable pollutants have been proposed which take into account implicitly the depletion of DO by bacteria in a water body [11,19]. However, simultaneous effect of organic pollutants and eutrophication caused by nutrients has not been considered in these models. Keeping in view these considerations, we propose a nonlinear model to study the simultaneous effect of water pollution caused by organic pollutants and eutrophication when nutrients are supplied to the water body from outside by water runoff from agricultural fields, using ecological concepts.

2. Mathematical model

We consider a water body where organic pollutants are discharged in the form of wastes from household and industrial sources and the nutrients are supplied by water runoff from agricultural fields. Let T be a cumulative concentration of organic pollutants, B be the density of bacteria, n be the cumulative concentration of various nutrients, a be the density of algae, S be the density of detritus and C be the concentration of the DO. It is assumed that the cumulative discharge of organic pollutants into the water body is Q (a constant) and its rate of decrease due to biochemical and other factors is proportional to its cumulative concentration T . It is further assumed that the cumulative rate of depletion of T due to bacteria is given by monod type of interaction (i.e. $TB/(K_{12} + K_{11}T)$). As bacteria wholly depend on organic pollutants, the growth rate of bacteria is proportional to this monod type interaction term. The depletion rate of bacteria due to natural factors is assumed to be proportional to B while that due to crowding is proportional to B^2 . We also assume that the cumulative rate of discharge of nutrients into the water body is q from outside, which is depleted with rate $(\alpha_2 n)$ due to natural factors. It is further assumed that the growth rate of nutrient by detritus is $(\pi\delta S)$ and the depletion of nutrients by algae is given by monod type of interaction. The growth rate of algae is assumed to be wholly dependent on the nutrient and is therefore proportional to this monod type interaction. The natural depletion rate of algae is assumed to be proportional to its density a , and its depletion rate due to crowding is proportional to a^2 . Since some part of natural depletions of bacteria and algae is converted into detritus, the growth rate of detritus is assumed to be proportional to B and a and its natural depletion rate is assumed to be proportional to the density of detritus, S . We consider that the rate of growth of DO by various sources is q_c (assumed a constant) and its natural depletion rate is proportional to its concentration C . It is also considered that the rate of growth of DO by algae is proportional to density of algae a , and the depletion rate of DO caused by eutrophication is proportional to the detritus concentration S . It is further considered that the depletion of DO is proportional to the terms $TB/(K_{12} + K_{11}T)$ and $\alpha_1 B$ (representing growth and depletion of bacteria, respectively).

Keeping in view the above considerations this problem is governed by the following differential equations:

$$\begin{aligned}
 \frac{dT}{dt} &= Q - \alpha_0 T - \frac{K_1 T B}{K_{12} + K_{11} T}, \\
 \frac{dB}{dt} &= \frac{\lambda_1 K_1 T B}{K_{12} + K_{11} T} - \alpha_1 B - \lambda_{10} B^2, \\
 \frac{dn}{dt} &= q + \pi \delta S - \alpha_2 n - \frac{\beta_1 n a}{\beta_{12} + \beta_{11} n}, \\
 \frac{da}{dt} &= \frac{\theta_1 \beta_1 n a}{\beta_{12} + \beta_{11} n} - \alpha_3 a - \beta_{10} a^2, \\
 \frac{dS}{dt} &= \pi_1 \alpha_1 B + \pi_2 \alpha_3 a - \delta S, \\
 \frac{dC}{dt} &= q_c - \alpha_4 C - \lambda_{12} \frac{K_1 T B}{K_{12} + K_{11} T} - \lambda_{11} \alpha_1 B + \lambda_{22} a - \delta_1 S,
 \end{aligned}
 \tag{2.1}$$

where $T(0) \geq 0, B(0) \geq 0, n(0) \geq 0, a(0) \geq 0, S(0) \geq 0, C(0) \geq 0$.

Here α_0, α_2 and α_4 are natural depletion rate coefficients of organic pollutants, nutrients and DO, which are constants and positive. The coefficient α_1 represents the natural depletion rate as well as predating rate of bacterial population whereas α_3 represents the similar coefficient for algae, both are assumed to be positive constants. The positive constants λ_{10} and β_{10} are coefficients representing crowding (flaking off coefficients, [28]) of bacteria and algae, respectively, with respect to the aquatic habitat. Further as π, π_1 and π_2 are fractional proportionality constants, we have $0 < \pi, \pi_1, \pi_2 < 1$.

It may be pointed out here that for feasibility of the model (2.1), the growth rate of bacteria should be positive. Hence from the second and fourth equation of model (2.1), it follows that

$$\lambda_1 K_1 - K_{11} \alpha_1 > 0, \tag{2.2}$$

$$\theta_1 \beta_1 - \beta_{11} \alpha_3 > 0. \tag{2.3}$$

3. Equilibrium analysis

Model (2.1) has the following four nonnegative equilibria. They are listed below:

1. $E_1(Q/\alpha_0, 0, q/\alpha_2, 0, 0, q_c/\alpha_4)$, which always exists.
2. $E_2(Q/\alpha_0, 0, n_2^*, a_2^*, S_2^*, C_2^*)$ exists, provided

$$(\theta_1 \beta_1 - \beta_{11} \alpha_3) q - \beta_{12} \alpha_2 \alpha_3 > 0, \tag{3.1}$$

and

$$q_c + \lambda_{22} a_2^* - \delta_1 S_2^* > 0. \tag{3.2}$$

3. $E_3(T_3^*, B_3^*, n_3^*, 0, S_3^*, C_3^*)$ exists, provided

$$(\lambda_1 K_1 - K_{11} \alpha_1) Q - K_{12} \alpha_0 \alpha_1 > 0, \tag{3.3}$$

and

$$q_c - \lambda_{12} \frac{K_1 T_3^* B_3^*}{K_{12} + K_{11} T_3^*} - \lambda_{11} \alpha_1 B_3^* - \delta_1 S_3^* > 0. \tag{3.4}$$

4. $E_4(T^*, B^*, n^*, a^*, S^*, C^*)$ exists, provided

$$(\lambda_1 K_1 - K_{11} \alpha_1) Q - K_{12} \alpha_0 \alpha_1 > 0, \tag{3.5}$$

$$(\theta_1 \beta_1 - \beta_{11} \alpha_3) (q + \pi \pi_1 \alpha_1 B^*) - \beta_{12} \alpha_2 \alpha_3 > 0, \tag{3.6}$$

and

$$q_c - \lambda_{12} \frac{K_1 T^* B^*}{K_{12} + K_{11} T^*} - \lambda_{11} \alpha_1 B^* + \lambda_{22} a^* - \delta_1 S^* > 0. \tag{3.7}$$

Proof. The equilibria $E_i (i = 1, 2, 3, 4)$ of model (2.1) are obtained by solving the following set of algebraic equations:

$$Q - \alpha_0 T - \frac{K_1 T B}{K_{12} + K_{11} T} = 0, \tag{3.8}$$

$$\frac{\lambda_1 K_1 T B}{K_{12} + K_{11} T} - \alpha_1 B - \lambda_{10} B^2 = 0, \tag{3.9}$$

$$q + \pi \delta S - \alpha_2 n - \frac{\beta_1 n a}{\beta_{12} + \beta_{11} n} = 0, \tag{3.10}$$

$$\frac{\theta_1 \beta_1 n a}{\beta_{12} + \beta_{11} n} - \alpha_3 a - \beta_{10} a^2 = 0, \tag{3.11}$$

$$\pi_1 \alpha_1 B + \pi_2 \alpha_3 a - \delta S = 0, \tag{3.12}$$

$$q_c - \alpha_4 C - \lambda_{12} \frac{K_1 T B}{K_{12} + K_{11} T} - \lambda_{11} \alpha_1 B + \lambda_{22} a - \delta_1 S = 0. \tag{3.13}$$

The equilibrium $E_1(Q/\alpha_0, 0, q/\alpha_2, 0, 0, q_c/\alpha_4)$ exists obviously. We show the existence of others as follows:

Existence of E_2 : For the equilibrium $E_2(Q/\alpha_0, 0, n_2^*, a_2^*, S_2^*, C_2^*)$, the values of n_2^* and a_2^* are given by following algebraic equations:

$$a = (q - \alpha_2 n)(\beta_{12} + \beta_{11} n) / \{(\beta_1 - \pi \pi_2 \alpha_3 \beta_{11}) n - \pi \pi_2 \alpha_3 \beta_{12}\} \tag{3.14}$$

and

$$a = \frac{1}{\beta_{10}} \left(\frac{\theta_1 \beta_1 n}{\beta_{12} + \beta_{11} n} - \alpha_3 \right). \tag{3.15}$$

For the isocline given by (3.14), we note the following:

(i) $a = 0$ at $n = q/\alpha_2$ (ii) $a = 0$ at $n = -\beta_{12}/\beta_{11}$ (iii) $n = n_a = \pi \pi_2 \beta_{12} \alpha_3 / (\beta_1 - \pi \pi_2 \beta_{11} \alpha_3)$ is an asymptote, (iv) da/dn is always negative provided the point $n = q/\alpha_2$ lies right to the asymptote $n = n_a$. Thus, a decreases as n increases.

Similarly from Eq. (3.15), we note that $a = 0$ for $n = n_1 = \beta_{12} \alpha_3 / (\theta_1 \beta_1 - \beta_{11} \alpha_3)$, which is positive in view of (2.3). Also, we get $a = -\alpha_3/\beta_{10}$ for $n = 0$ and $da/dn = \theta_1 \beta_1 \beta_{12} / \beta_{10} (\beta_{12} + \beta_{11} n)^2$, which is positive.

It may be noted here that $n_1 > \pi \pi_2 \beta_{12} \alpha_3 / (\beta_1 - \pi \pi_2 \beta_{11} \alpha_3)$ as $0 < \theta_1, \pi, \pi_2 < 1$.

From the above analysis we note that the two isoclines (3.14) and (3.15) intersect at a unique point (n_2^*, a_2^*) in the interior of first quadrant [see Fig. 1] provided the point $n = q/\alpha_2$ lies to the right of the point $n = n_1$, i.e.

$$(\theta_1 \beta_1 - \beta_{11} \alpha_3) q - \beta_{12} \alpha_2 \alpha_3 > 0. \tag{3.16}$$

Using this value of a_2^* in Eqs. (3.12) and (3.13), we get

$$S_2^* = \frac{\pi_2 \alpha_3}{\delta} a_2^* \tag{3.17}$$

and

$$C_2^* = \frac{1}{\alpha_4} [q_c + \lambda_{22} a_2^* - \delta_1 S_2^*], \tag{3.18}$$

which is positive provided the right-hand side of Eq. (3.18) is positive.

Thus $E_2(Q/\alpha_0, 0, n_2^*, a_2^*, S_2^*, C_2^*)$ exists provided conditions (3.1) and (3.2) are satisfied.

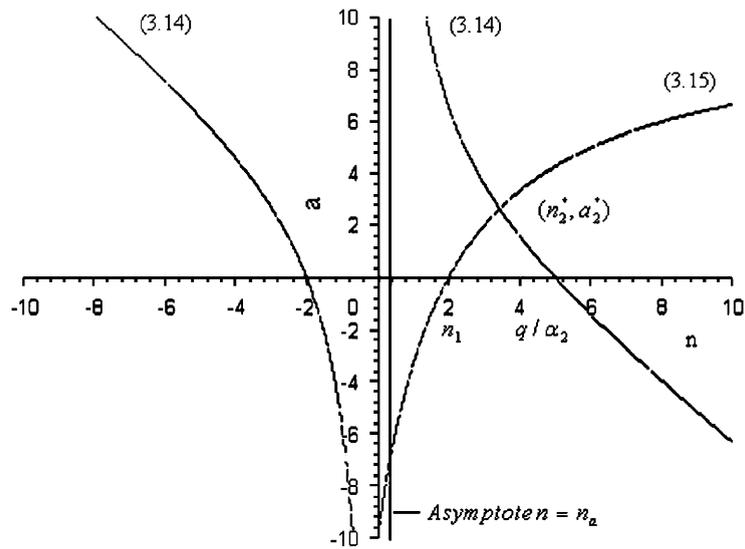


Fig. 1. Existence of (n_2^*, a_2^*) .

Existence of E_3 : For the equilibrium $E_3 (T_3^*, B_3^*, n_3^*, 0, S_3^*, C_3^*)$, the values T_3^* and B_3^* are given by the following algebraic equations:

$$B = \left(\frac{Q - \alpha_0 T}{K_1} \right) \left(K_{11} + \frac{K_{12}}{T} \right) \tag{3.19}$$

and

$$B = \frac{1}{\lambda_{10}} \left[\frac{\lambda_1 K_1 T}{K_{12} + K_{11} T} - \alpha_1 \right]. \tag{3.20}$$

For the isocline given by (3.19), it is easy to note the following:

- (i) $B = 0$ for $T = Q/\alpha_0$, (ii) $B \rightarrow \infty$ as $T \rightarrow 0$, (iii) $dB/dT < 0$. Thus B decreases as T increases.

From the isocline given by (3.20) it is noted that (i) $B < 0$ at $T = 0$, (ii) $B = 0$ for $T = K_{12}\alpha_1/(\lambda_1 K_1 - K_{11}\alpha_1)$, which is positive in view of (2.2), (iii) $dB/dT > 0$, and hence B increases for all $T > 0$.

Thus two isoclines (3.19) and (3.20) intersect at a unique point (T_3^*, B_3^*) in the interior of first quadrant [see Fig. 2] provided the point $T = Q/\alpha_0$ lies to the right of the point $T = K_{12}\alpha_1/(\lambda_1 K_1 - K_{11}\alpha_1)$. Thus we get condition (3.3).

Using this value of B_3^* , we get $S_3^* = (\pi_1\alpha_1/\delta)B_3^*$ and $n_3^* = (1/\alpha_2)(q + \pi\pi_1\alpha_1 B_3^*)$, which are positive.

Using T_3^*, B_3^* and S_3^* in Eq. (3.13), we get

$$C_3^* = \frac{1}{\alpha_4} \left[q_c - \lambda_{12} \frac{K_1 T_3^* B_3^*}{K_{12} + K_{11} T_3^*} - \lambda_{11}\alpha_1 B_3^* - \delta_1 S_3^* \right], \tag{3.21}$$

which is positive provided the right-hand side of (3.21) is positive.

Hence the equilibrium $E_3 (T_3^*, B_3^*, n_3^*, 0, S_3^*, C_3^*)$ exists, provided conditions (3.3) and (3.4) are satisfied.

Existence of E_4 : For the equilibrium $E_4(T^*, B^*, n^*, a^*, S^*, C^*)$, T^* and B^* are obtained by solving Eqs. (3.8) and (3.9) and hence have same values as T_3^* and B_3^* . The values of n^* and a^* are obtained by solving the following algebraic equations:

$$a = (q + \pi\pi_1\alpha_1 B^* - \alpha_2 n)(\beta_{12} + \beta_{11}n)/\{(\beta_1 - \pi\pi_2\alpha_3\beta_{11})n - \pi\pi_2\alpha_3\beta_{12}\} \tag{3.22}$$

and

$$a = \frac{1}{\beta_{10}} \left(\frac{\theta_1\beta_1 n}{\beta_{12} + \beta_{11}n} - \alpha_3 \right) \tag{3.23}$$

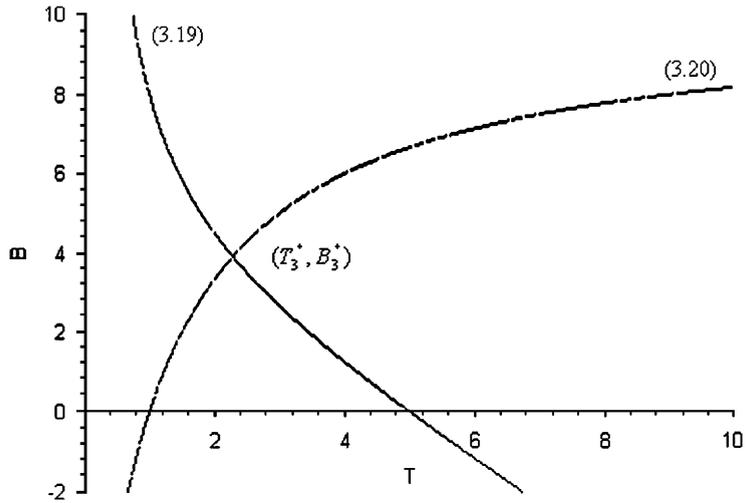


Fig. 2. Existence of (T_3^*, B_3^*) .

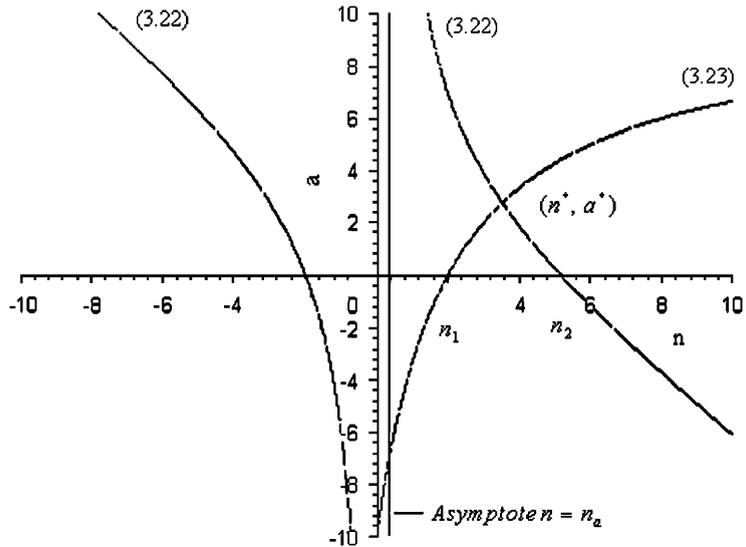


Fig. 3. Existence of (n^*, a^*) .

For the isocline given by (3.22), we note the following:

(i) $a = 0$ at $n = (q + \pi\pi_1\alpha_1 B^*)/\alpha_2$, (ii) $a = 0$ at $n = -\beta_{12}/\beta_{11}$, (iii) $n = n_a = \pi\pi_2\beta_{12}\alpha_3/(\beta_{11} - \pi\pi_2\beta_{11}\alpha_3)$ is an asymptote and (iv) da/dn is always negative provided $n = (q + \pi\pi_1\alpha_1 B^*)/\alpha_2$ lies right to the asymptote $n = n_a$.

Thus, a decreases as n increases.

Eq. (3.23) is same as (3.15), hence its behavior is also the same.

From the above analysis we note that the two isoclines (3.22) and (3.23) intersect at a unique point (n^*, a^*) in the interior of first quadrant [see Fig. 3] provided the point $n = n_2 = (q + \pi\pi_1\alpha_1 B^*)/\alpha_2$ lies to the right of the point $n = n_1$, i.e.

$$(\theta_1\beta_1 - \beta_{11}\alpha_3)(q + \pi\pi_1\alpha_1 B^*) - \beta_{12}\alpha_2\alpha_3 > 0. \tag{3.24}$$

Using these values of T^* , B^* and a^* in Eqs. (3.12) and (3.13), we get

$$S^* = \frac{1}{\delta} [\pi_1\alpha_1 B^* + \pi_2\alpha_3 a^*] \tag{3.25}$$

and

$$C^* = \frac{1}{\alpha_4} \left[q_c - \lambda_{12} \frac{K_1 T^* B^*}{K_{12} + K_{11} T^*} - \lambda_{11} \alpha_1 B^* + \lambda_{22} a^* - \delta_1 S^* \right], \tag{3.26}$$

which is positive provided condition (3.7) is satisfied.

Hence the equilibrium $E_4(T^*, B^*, n^*, a^*, S^*, C^*)$ exists, provided conditions (3.5)–(3.7) are satisfied. \square

Remark. Using Eqs. (3.8) and (3.9) it is noted that $dB^*/dQ > 0$. Thus B^* increases as the rate of discharge of organic pollutants increases.

From Eqs. (3.22) and (3.23), we note that $\partial n^*/\partial q > 0$ and $\partial n^*/\partial Q > 0$ as $n^* > \beta_{12}\alpha_3/(\theta_1\beta_1 - \beta_{11}\alpha_3)$.

From Eqs. (3.23) and (3.25), we get $\partial a^*/\partial Q > 0$, $\partial a^*/\partial q > 0$, $\partial S^*/\partial Q > 0$ and $\partial S^*/\partial q > 0$. Thus we note that all the derivatives with respect to q and Q are positive.

Hence as the cumulative rate of discharge of organic pollutants, Q , into the water body increases T^*, B^*, n^*, a^*, S^* increase. Further, when the rate of discharge of cumulative nutrients, q , increases, then also n^*, a^*, S^* increase.

From Eq. (3.26), we also note that

$$\frac{\partial C^*}{\partial Q} = \frac{1}{\alpha_4} \left[\left\{ \lambda_{22} - \frac{\pi_2 \alpha_3 \delta_1}{\delta} \right\} \frac{\partial a^*}{\partial Q} - \frac{\lambda_{12} K_1 K_{12} B^*}{(K_{12} + K_{11} T^*)^2} \frac{dT^*}{dQ} - \left(\frac{\lambda_{12} K_1 T^*}{K_{12} + K_{11} T^*} + \lambda_{11} \alpha_1 + \frac{\pi_1 \delta_1 \alpha_1}{\delta} \right) \frac{dB^*}{dQ} \right]$$

and

$$\frac{\partial C^*}{\partial q} = \frac{1}{\alpha_4} \left[\lambda_{22} - \frac{\pi_2 \alpha_3 \delta_1}{\delta} \right] \frac{\partial a^*}{\partial q}.$$

Here we note that most of the algae float on the surface of the water body. Thus the oxygen formed by algae during photosynthesis will go in the atmosphere and will not affect the level of DO below the water surface, so λ_{22} is very small. Hence the expression $[\lambda_{22} - \pi_2 \alpha_3 \delta_1 / \delta]$ is likely to be negative. Thus, $\partial C^*/\partial Q$ and $\partial C^*/\partial q$ are negative in a realistic situation.

From the above analysis, we conclude that the simultaneous effect of water pollution and eutrophication in the water body causes decrease in the concentration of DO much more in comparison to the case if only a single process is present. This result has also been shown in Fig. 12 (see Section 5 on simulation).

4. Stability analysis

In the following, we discuss the local and nonlinear stability of the nonzero equilibria of model (2.1). The results are stated in the form of the following theorem.

Theorem 1. *The equilibrium $E_i (i = 1, 2 \text{ or } 3)$ is unstable whenever E_{i+1} exists. The equilibrium E_4 is locally asymptotically stable without any condition.*

The proof is given in Appendix A.

In the following, we prove that E_4 is nonlinearly stable [20]. For this we need the following lemma, which is stated without proof, following Freedman and So [10] and Hale [12].

Lemma 1. *The set Ω is a region of attraction for all solution initiating in the positive octant.*

$$\Omega : \left\{ 0 \leq T \leq \frac{Q}{\alpha_0}, 0 \leq B \leq R_B, 0 \leq n + a + S \leq \frac{q + \pi_1 \alpha_1 R_B}{\delta_m}, 0 \leq C \leq \frac{q_c \delta_m + \lambda_{22} (q + \pi_1 \alpha_1 R_B)}{\delta_m \alpha_4} \right\},$$

where

$$R_B = \frac{(\lambda_1 K_1 - K_{11} \alpha_1) Q - K_{12} \alpha_0 \alpha_1}{\lambda_{10} (K_{12} \alpha_0 + K_{11} Q)} \quad \text{and} \quad \delta_m = \text{Min}\{\alpha_2, (1 - \pi) \delta, (1 - \pi_2) \alpha_3\}.$$

Theorem 2. *The equilibrium E_4 is nonlinearly stable in Ω provided the following conditions are satisfied:*

$$\left[\frac{K_1 K_{12} Q}{(K_{11} Q + K_{12} \alpha_0)(K_{12} + K_{11} T^*)} \right]^2 T^* < \frac{2 \lambda_{10} \alpha_0}{3 \lambda_1}, \tag{4.1}$$

$$\left[\frac{\beta_1 \beta_{12} q}{(\beta_{11} q + \beta_{12} \alpha_2)(\beta_{12} + \beta_{11} n^*)} \right]^2 n^* < \frac{2 \beta_{10} \alpha_2}{3 \theta_1}, \tag{4.2}$$

$$\frac{\pi^2}{\alpha_2} < \frac{1}{6} \text{Min} \left\{ \frac{\lambda_{10} T^*}{\lambda_1 \pi_1^2 \alpha_1^2}, \frac{\beta_{10} n^*}{\theta_1 \pi_2^2 \alpha_3^2} \right\}. \tag{4.3}$$

The proof is given in Appendix B.

We note from the above that if $\lambda_{10} = 0$ or $\beta_{10} = 0$, the above conditions are never satisfied. This shows that crowding coefficients of bacterial as well as algal population densities (i.e. flaking off terms) have a stabilizing effect on the system.

The above theorems imply that the simultaneous effect of organic pollutants and nutrients have detrimental effect in decreasing the level of DO in a water body and the amount of decrease is more in comparison to the case when only a single phenomenon is present.

5. Simulation analysis

To check the feasibility of our analysis regarding the existence of E_4 and the corresponding stability conditions, we conduct some numerical computation of model (2.1) by Runge–Kutta method and we choose the following values of the parameters in model (2.1):

$$\begin{aligned} Q = 5.0, \quad \alpha_0 = 1.0, \quad K_1 = 1.0, \quad K_{11} = 1.0, \quad K_{12} = 1.0, \quad \lambda_1 = 1.0, \quad \alpha_1 = 0.5, \quad \lambda_{10} = 0.05, \\ q = 5.0, \quad \pi = 0.1, \quad \delta = 1.0, \quad \alpha_2 = 1.0, \quad \beta_1 = 1.0, \quad \theta_1 = 1, \quad \alpha_3 = 0.5, \quad \beta_{10} = 0.05, \\ \beta_{11} = 1.0, \quad \beta_{12} = 2.0, \quad \pi_1 = 0.9, \quad \pi_2 = 0.9, \\ q_c = 10.0, \quad \alpha_4 = 1.0, \quad \lambda_{12} = 0.25, \quad \delta_1 = 2.0, \quad \lambda_{11} = 0.25, \quad \lambda_{22} = 0.25. \end{aligned} \tag{5.1}$$

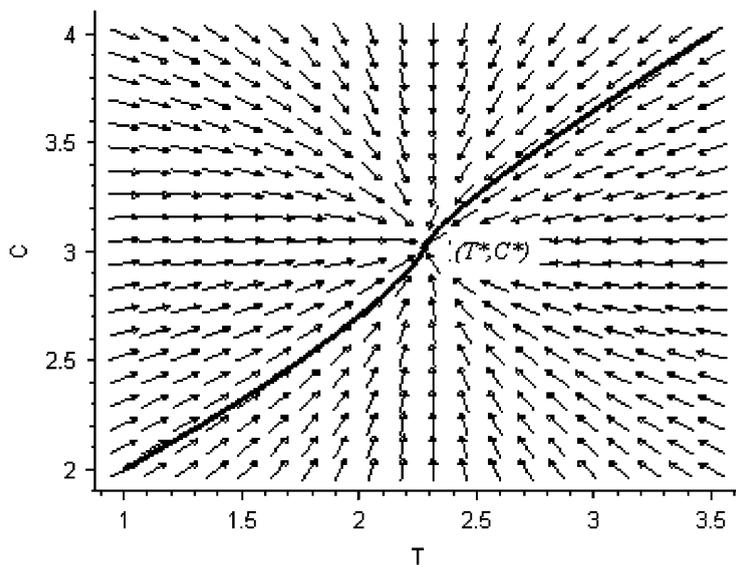


Fig. 4. Nonlinear stability of (T^*, C^*) .

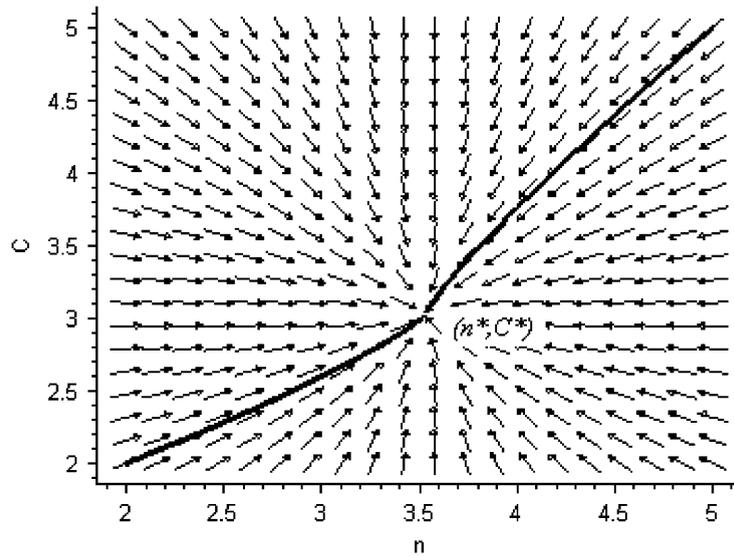


Fig. 5. Nonlinear stability of (n^*, C^*) .

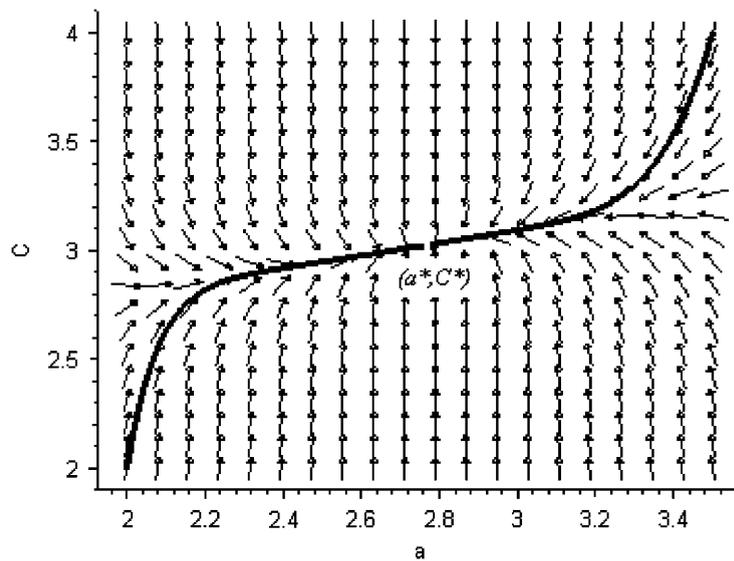


Fig. 6. Nonlinear stability of (a^*, C^*) .

It is found that for the above set of parameters, conditions for the existence of interior equilibrium $E_4(T^*, B^*, n^*, a^*, S^*, C^*)$ are satisfied and E_4 is given by

$$T^* = 2.28276, \quad B^* = 3.907568, \quad n^* = 3.532167, \quad a^* = 2.769561, \quad S^* = 3.004708, \quad C^* = 3.026773.$$

The eigenvalues of the Jacobian matrix M corresponding to this equilibrium E_4 are obtained as, $-1.0, -1.076407, -0.481572, -0.251047, -1.03421 + 0.09851i, -1.03421 - 0.09851i$, which are either negative or have negative real parts. Hence E_4 is locally stable.

It is pointed out here that for the above set of parameters, the conditions for nonlinear stability (4.1)–(4.3) are also satisfied.

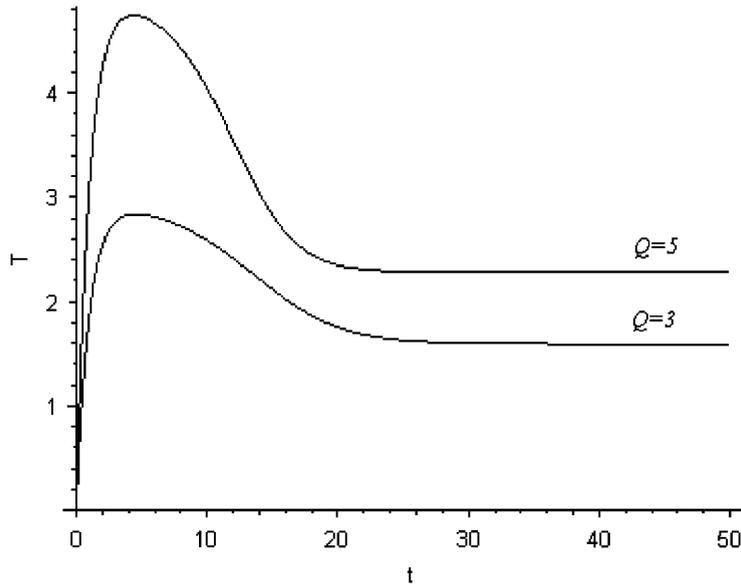


Fig. 7. Variation of T w.r.t. t for different Q .

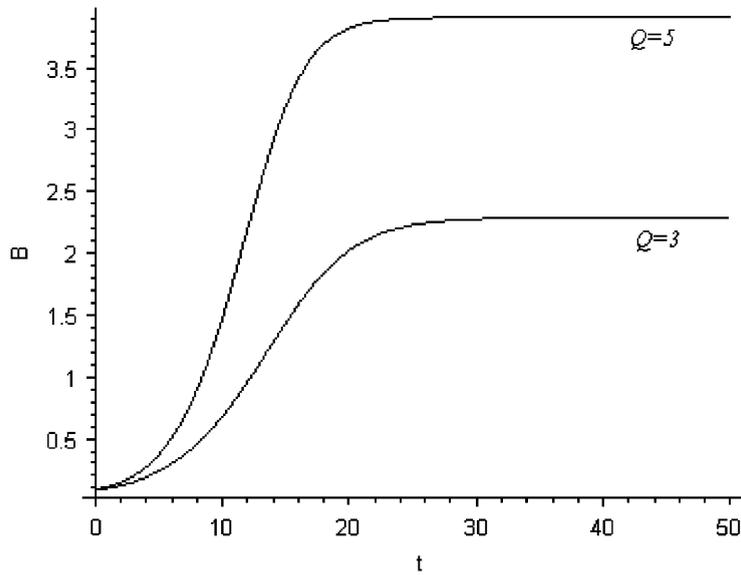


Fig. 8. Variation of B w.r.t. t for different Q .

Further, for the above set of parameters, computer-generated graphs of T with C , n with C and a with C are shown in Fig. 4, Fig. 5 and Fig. 6, respectively, which indicates the global stability of (T^*, C^*) in T - C plane, (n^*, C^*) in n - C plane and (a^*, C^*) in a - C plane, respectively.

In Figs. 7 and 8, we see the effect of the rate of input of organic pollutants Q on T and B , by keeping other parameters constant (as given in (5.1)). We observe here that as Q increases, both T and B increases.

In Figs. 9–11, we see the effect of the rate of input of organic pollutants Q and the rate of input of nutrients q on n , a and S , by keeping other parameters constant (as given in (5.1)). We observe here that as Q or q increases, n , a and S increase. In Fig. 10, we also note that if the rate of discharge of nutrients becomes zero, then the density of algal

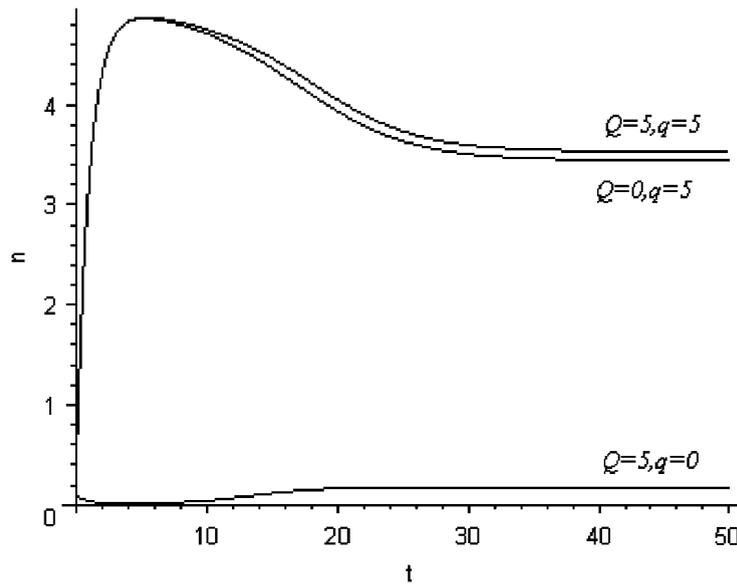


Fig. 9. Variation of n w.r.t. t for different Q and q .

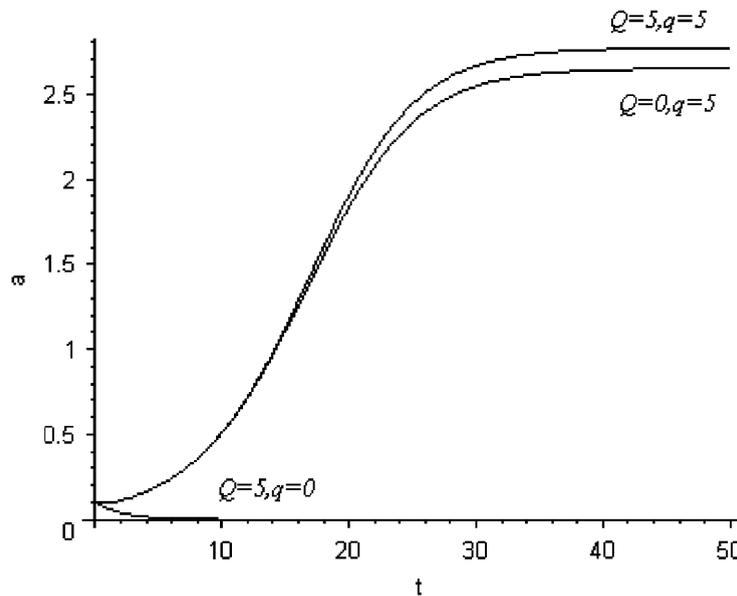


Fig. 10. Variation of a w.r.t. t for different Q and q .

population tends to zero. This result is expected because of the fact that the nutrients then formed from detritus will not be sufficient for the growth of algal population.

In Fig. 12, we see the effect of the rate of input of organic pollutants Q and the rate of input of nutrients q on C , by keeping other parameters constant (as given in (5.1)). We observe here that as Q or q increases, the level of DO decreases. In this figure we also see that if the rates of input of organic pollutants and nutrients are nonzero then the level of DO will be less in comparison to the case of a single discharge.

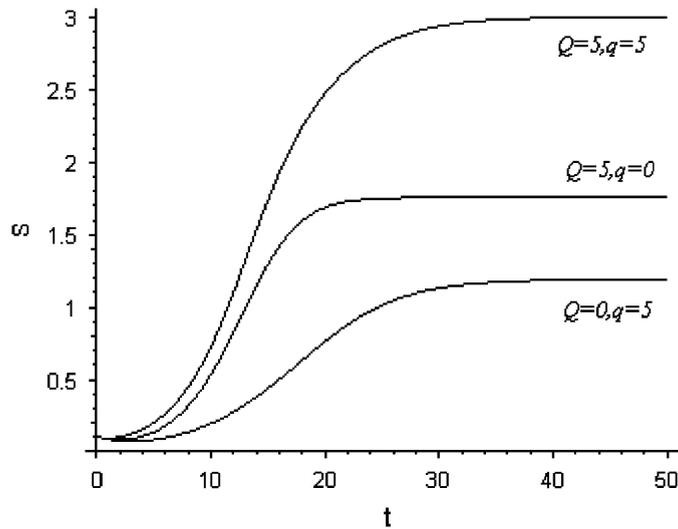


Fig. 11. Variation of S w.r.t. t for different Q and q .

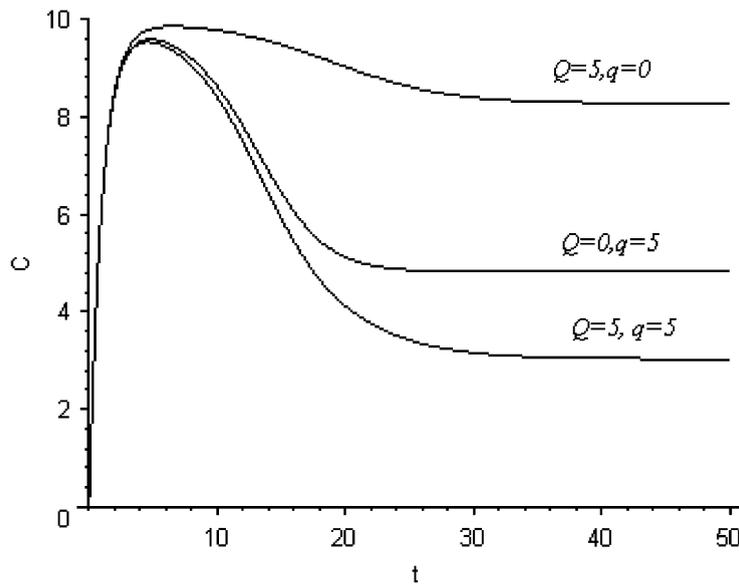


Fig. 12. Variation of C w.r.t. t for different Q and q .

6. Conclusion

In this paper, we have modeled and analyzed the simultaneous effect of water pollution (organic pollutants) and eutrophication on the concentration of DO in a water body such as a lake. It has been assumed that discharge rates of organic pollutants and nutrients by water runoff from agricultural fields are constant. The model has been analyzed by using stability theory of differential equations and numerical simulation by considering its various feasible equilibria. The conditions for existence and stability corresponding to each equilibria have been determined. It has been shown both by analysis and simulation that the simultaneous effect of water pollution and eutrophication is much more in decreasing the concentration of DO in comparison to the case when only a single phenomenon is present in the water body. Thus, it may be conjectured that the survival of species in a water body, which is polluted as well as eutrophied, is much more uncertain.

Appendix A. Local stability analysis

By computing eigenvalues corresponding to each equilibrium of the corresponding Jacobian matrix, it is easy to note that the equilibrium $E_i (i = 1, 2, 3)$ is unstable whenever E_{i+1} exists.

Proof of Theorem 1. Here we consider the stability of the nontrivial equilibrium $E_4(T^*, B^*, n^*, a^*, S^*, C^*)$. The characteristic equation for the Jacobian matrix M (evaluated at E_4) is given by

$$(\lambda + \alpha_4)(\lambda^2 + p_1\lambda + p_2)(\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3) = 0, \tag{A.1}$$

where

$$p_1 = \left(\alpha_0 + \frac{K_1 K_{12} B^*}{(K_{12} + K_{11} T^*)^2} \right) + \lambda_{10} B^*, \quad p_2 = \lambda_{10} B^* \left(\alpha_0 + \frac{K_1 K_{12} B^*}{(K_{12} + K_{11} T^*)^2} \right) + \frac{\lambda_1 K_1^2 K_{12} T^* B^*}{(K_{12} + K_{11} T^*)^3},$$

$$A_1 = \left(\alpha_2 + \frac{\beta_1 \beta_{12} a^*}{(\beta_{12} + \beta_{11} n^*)^2} \right) + \beta_{10} a^* + \delta, \tag{A.2}$$

$$A_2 = (\beta_{10} a^* + \delta) \left(\alpha_2 + \frac{\beta_1 \beta_{12} a^*}{(\beta_{12} + \beta_{11} n^*)^2} \right) + \beta_{10} \delta a^* + \frac{\beta_1 n^*}{(\beta_{12} + \beta_{11} n^*)} \frac{\theta_1 \beta_1 \beta_{12} a^*}{(\beta_{12} + \beta_{11} n^*)^2}, \tag{A.3}$$

$$A_3 = \beta_{10} \delta a^* \left(\alpha_2 + \frac{\beta_1 \beta_{12} a^*}{(\beta_{12} + \beta_{11} n^*)^2} \right) + \frac{\delta \beta_1 n^*}{(\beta_{12} + \beta_{11} n^*)} \frac{\theta_1 \beta_1 \beta_{12} a^*}{(\beta_{12} + \beta_{11} n^*)^2} - \frac{\pi \pi_2 \alpha_3 \delta \theta_1 \beta_1 \beta_{12} a^*}{(\beta_{12} + \beta_{11} n^*)^2}. \tag{A.4}$$

Keeping in mind that $n^* > \pi \pi_2 \beta_{12} \alpha_3 / (\beta_1 - \pi \pi_2 \beta_{11} \alpha_3)$, we can easily show that all the conditions of the Routh–Hurwitz criterion for the polynomial $\lambda^3 + A_1 \lambda^2 + A_2 \lambda + A_3 = 0$, i.e. $A_1 > 0$; $A_3 > 0$ and $A_1 A_2 > A_3$ are satisfied. The roots of the polynomial $\lambda^2 + p_1 \lambda + p_2 = 0$ will be either negative or have negative real part. Hence all the eigenvalues of the Jacobian matrix are either negative or have negative real parts. Hence E_4 is locally asymptotically stable.

Appendix B. Proof of Theorem 2

To prove this theorem, we consider the following positive definite function:

$$V = \frac{1}{2}(T - T^*)^2 + m_1 \left(B - B^* - B^* \ln \frac{B}{B^*} \right) + \frac{1}{2} m_2 (n - n^*)^2 + m_3 \left(a - a^* - a^* \ln \frac{a}{a^*} \right) + \frac{1}{2} m_4 (S - S^*)^2 + \frac{1}{2} m_5 (C - C^*)^2. \tag{B.1}$$

Now using system (2.1) we get

$$\begin{aligned} \frac{dV}{dt} = & - \frac{K_1 K_{12} B}{(K_{12} + K_{11} T)(K_{12} + K_{11} T^*)} (T - T^*)^2 - m_2 \frac{\beta_1 \beta_{12} a}{(\beta_{12} + \beta_{11} n)(\beta_{12} + \beta_{11} n^*)} (n - n^*)^2 \\ & - \alpha_0 (T - T^*)^2 - m_1 \lambda_{10} (B - B^*)^2 - m_2 \alpha_2 (n - n^*)^2 - m_3 \beta_{10} (a - a^*)^2 \\ & - m_4 \delta (S - S^*)^2 - m_5 \alpha_4 (C - C^*)^2 \\ & + \frac{(T - T^*)(B - B^*)}{(K_{12} + K_{11} T^*)(K_{12} + K_{11} T^*)} \left[-K_1 K_{12} T^* - K_1 K_{11} T T^* + m_1 \lambda_1 K_1 K_{12} \right] \end{aligned}$$

$$\begin{aligned}
& + \frac{(n - n^*)(a - a^*)}{(\beta_{12} + \beta_{11}n^*)(\beta_{12} + \beta_{11}n^*)} [-m_2\beta_1\beta_{12}n^* - m_2\beta_1\beta_{11}nn^* + m_3\theta_1\beta_1\beta_{12}] \\
& + (n - n^*)(S - S^*)m_2\pi\delta + (B - B^*)(S - S^*)[m_4\pi_1\alpha_1] \\
& + (a - a^*)(S - S^*)[m_4\pi_2\alpha_3] + (a - a^*)(C - C^*)[m_5\lambda_{22}] + (S - S^*)(C - C^*)[-m_5\delta_1] \\
& + (T - T^*)(C - C^*) \left[-m_5 \frac{\lambda_{12}K_1K_{12}B}{(K_{12} + K_{11}T)(K_{12} + K_{11}T^*)} \right] \\
& - m_5(B - B^*)(C - C^*) \left[\frac{\lambda_{12}K_1T^*}{K_{12} + K_{11}T^*} + \lambda_{11}\alpha_1 \right]. \tag{B.2}
\end{aligned}$$

Now choosing $m_1 = T^*/\lambda_1$, $m_2 = 1$, $m_3 = n^*/\theta_1$, we note that dV/dt can be made negative definite inside Ω if

$$\begin{aligned}
\left[\frac{K_1K_{12}Q}{(K_{11}Q + K_{12}\alpha_0)(K_{12} + K_{11}T^*)} \right]^2 T^* &< \frac{2}{3} \frac{\lambda_{10}\alpha_0}{\lambda_1}, \\
\left[\frac{\beta_1\beta_{12}q}{(\beta_{11}q + \beta_{12}\alpha_2)(\beta_{12} + \beta_{11}n^*)} \right]^2 n^* &< \frac{2}{3} \frac{\beta_{10}\alpha_2}{\theta_1}, \\
\frac{\pi^2}{\alpha_2} &< \frac{1}{6} \text{Min} \left\{ \frac{\lambda_{10}T^*}{\lambda_1\pi_1^2\alpha_1^2}, \frac{\beta_{10}n^*}{\theta_1\pi_2^2\alpha_3^2} \right\}
\end{aligned}$$

by making an appropriate choice for m_4 and m_5 .

References

- [1] E.M. Arnold, Aspects of a zooplankton, phytoplankton and phosphorus system, *Ecol. Modelling* 5 (1978) 293–300.
- [2] E.M. Arnold, D.A. Voss, Numerical behavior of zooplankton, phytoplankton and phosphorus System, *Ecol. Modelling* 13 (1981) 183–193.
- [3] M.B. Beck, P.C. Young, A dynamic model for DO–BOD relationship in a non-tidal stream, *Water Res.* 9 (1975) 769–776.
- [4] A.M. Beeton, W.T. Edmondson, The eutrophication problem, *J. Fish. Res. Board Canada* 29 (1972) 673–682.
- [5] S. Busenberg, S. Kishore Kumar, P. Austin, G. Wake, The dynamics of a plankton-nutrient interaction, *Bull. Math. Biol.* 52 (4) (1990) 677–696.
- [6] J. Dachs, S.J. Eisenreich, R.M. Hoff, Influence of eutrophication on air–water exchange, vertical fluxes and phytoplankton concentration of persistence organic pollutants, *Environ. Sci. Technol.* 34 (2000) 1095–1102.
- [7] W.E. Dobbins, BOD and oxygen relationship in streams. *J. Sanit. Eng. Div. Proc. Am. Soc. Civ. Eng.* 90 (1964) 53–78.
- [8] A.M. Edwards, J. Brindley, Zooplankton mortality and the dynamical behavior of plankton population models, *Bull. Math. Biol.* 61 (1999) 303–339.
- [9] U. Franke, K. Hutter, K. Johnk, A physical–biological coupled model for algal dynamics in lakes, *Bull. Math. Biol.* 61 (1999) 239–272.
- [10] H.I. Freedman, J.W.H. So, Global stability and persistence of simple food chains, *Math. Biosci.* 76 (1985) 69–86.
- [11] W.E. Gates, J.T. Marler, J.D. Westfield, The application of bacterial process kinetics in stream simulation and stream analysis. *Water Res.* 3 (1969) 663–686.
- [12] J.K. Hale, *Ordinary Differential Equations*, Wiley-Interscience Press, New York, 1969.
- [13] T.G. Hallam, Structural sensitivity of grazing formulations in nutrient controlled plankton models. *J. Math. Biol.* 5 (1978) 269–280.
- [14] A.D. Hasler, Eutrophication of lakes by domestic drainage. *Ecology.* 28 (1947) 383–395.
- [15] G.E. Hutchinson, Eutrophication, past and present, in: G.A. Rohlich (Ed.), *Eutrophication: causes, consequence, correctives*, National Academy of Sciences, Washington, D.C., 1969, pp. 17–26.
- [16] M. Jayaweera, T. Asaeda, Modeling of biomanipulation in shallow, eutrophic lakes: an application to lake Bleiswijkse Zoom, the Netherlands, *Ecol. Modelling* 85 (1996) 113–127.
- [17] S.E. Jorgenson, A eutrophication model for a lake. *Ecol. Modelling* 4 (1976) 147–165.
- [18] S.E. Jorgenson, *Fundamental of Ecological Modeling*, Elsevier Science Publishers, Amsterdam, The Netherlands, 1988.
- [19] R.A. Kelly, Conceptual ecological model of the Delaware estuary, in: B.C. Patten (Ed.), *Systems Analysis and Simulation in Ecology*, vol. IV, Academic Press, New York, 1976.
- [20] J. LaSalle, S. Lefschetz, *Stability by Liapunov's Direct Method with Applications*, Academic Press, New York, 1961.

- [21] J.W.G. Lund, . Eutrophication. *Proc. Roy. Soc. London B* 180 (1972) 371–382.
- [22] A.J. McDonel, Oxygen budgets in macrophyte impacted streams, *Water Res.* 16 (1982) 1037–1046.
- [23] F. Nishimura, M. Watanabe, R. Takahashi, T. Akase, Runoff characteristics of nutrients from citrus fruit grove and its effects on water area, *Water Sci. Technol.* 45 (12) (2002) 37–44.
- [24] N. Nyholm, A simulation model for phytoplankton growth cycling in eutrophic shallow lakes, . *Ecol. Modelling* 4 (1978) 279–310.
- [25] D.J. O’Connor, The temporal and spatial distribution of dissolved oxygen in streams. *Water Resour. Res.* 3 (1967) 65–79.
- [26] J.J. Pauer, M.T. Auer, Nitrification in the water column and sediments of a hypereutrophic lake and adjoining river system, *Water Res.* 34 (4) (2000) 1247–1254.
- [27] J.H.C. Peeters, P.H.C. Eilers, The relationship between light intensity photosynthesis. A simple mathematical model, *Hydrobiol. Bull.* 12 (1978) 134–136.
- [28] S. Rinaldi, R. Soncini-Sessa, H. Stehfest, H. Tamura, *Modeling and Control of River Quality*, McGraw-Hill, UK, 1979, p. 380.
- [29] I.R. Smith, A simple theory of algal deposition. *Freshwater Biol.* (1982) 445–449.
- [30] J.H. Steele, B.W. Frost, The structure of plankton communities. *Phil. Trans. R. Soc. London B* 280 (1977) 485–534.
- [31] M. Straskraba, A. Gnauck, *Freshwater ecosystems modeling and simulation*, Developments in Environmental Modeling, vol. 8, Elsevier, Amsterdam, 1985.
- [32] R.V. Thomann, Bioaccumulation model of organic chemical distribution in aquatic food chains, *Environ. Sci. Technol.* 18 (1989) 67–71.
- [33] A.A. Voinov, Yu. M. Svirezhev, A minimal model of eutrophication in freshwater ecosystems, *Ecol. Modelling* 23 (1984) 277–292.
- [34] A.A. Voinov, A.P. Tonkikh, Qualitative model of eutrophication in macrophyte lakes, *Ecol. Modelling* 35 (1987) 211–226.
- [35] F.-L. Xu, S. Tao, R.W. Dawson, P.-G. Li, J. Cao, Lake ecosystem health assessment indicators and methods, *Water Res.* 35 (13) (2001) 3157–3167.