EFFECTS OF CONSISTENCY VARIATION OF POWER LAW LUBRICANTS IN SQUEEZE FILMS

J. B. SHUKLA and K. R. PRASAD
Department of Mathematics, Indian Institute of Technology, Kanpur, Kanpur 208016 (India)

PEEUYSH CHANDRA
The Mehta Research Institute of Mathematics and Mathematical Physics, 26 Dilkusha, New Katra, Allahabad 211002 (India)

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Summary

The characteristics of squeeze film bearings with power law lubricants have been investigated by considering the effects of consistency variation. Various bearing geometries have been considered with rigid surfaces as well as with compliant layers. With stiff solids, a high consistency layer adjacent to the bearing surface increases the load capacity and time of squeezing and this increase is enhanced by the pseudoplastic behaviour of the lubricant. For squeeze film bearings with compliant layers, the film thickness increases with load, compliancy and conformity of the surfaces even with power law lubricants. It also increases with the consistency of the layer adjacent to the bearing surface.

1. Introduction

Many lubricants such as silicone fluids behave as non-newtonian power law fluids at low shear rates [1 - 3]. The characteristics of these lubricants have been studied in different systems, such as squeeze films [4 - 6], externally pressurized bearings [7 - 9], journal bearings [10 - 14] and elastohydrodynamic rollers [3]. Although the effects of viscosity variation on the behaviour of lubricated systems using newtonian lubricants have been investigated [15 - 19], little attention has been given to study the effects of consistency variation on bearing characteristics using non-newtonian lubricants [20, 21].

The effects of consistency variation on the characteristics of squeeze film stiff bearings have thus been investigated using power law fluids as lubricants. The squeeze film between two compliant surfaces has also been discussed.
2. Basic equations

Consider the flow of a power law fluid in a thin clearance (slit) of film thickness $2h(x)$ as shown in Fig. 1. Under the usual assumptions of lubrication theory the equation governing the flow of the fluid is given by

$$\frac{\partial}{\partial y} \left( m \left| \frac{\partial u}{\partial y} \right|^{n-1} \frac{\partial u}{\partial y} \right) = \frac{dp}{dx} \tag{1}$$

where $u(x, y)$ is the velocity, $p(x, y)$ is the fluid pressure, $m(x, y)$ is the consistency of the fluid and $n$ is the flow behaviour index of the fluid.

Since $u$ decreases as $y$ increases for $y > 0$, $\partial u/\partial y$ is negative and eqn. (1) can be rewritten as

$$\frac{\partial}{\partial y} \left\{ m \left( - \frac{\partial u}{\partial y} \right)^n \right\} = \frac{dp}{dx} \quad y > 0 \tag{2}$$

Solving eqn. (2) and using the boundary conditions

$$\frac{\partial u}{\partial y} = 0 \quad \text{at} \quad y = 0 \tag{3}$$

$$u = 0 \quad \text{at} \quad y = h$$

gives

$$u = \left( - \frac{dp}{dx} \right)^{1/n} \int_0^h \left( \frac{y}{m} \right)^{1/n} dy \tag{4}$$

The total volume flux $Q$ across the width $b$ of the slit is given by

$$Q = 2b \int_0^h u \, dy$$

which on using eqn. (4) gives

$$Q = 2bF \left( - \frac{dp}{dx} \right)^{1/n} \tag{5}$$

Fig. 1. The flow of a power law fluid through a slit.
where

\[ F = \int_{0}^{h} y \left( \frac{y}{m} \right)^{1/n} \, dy \]  

Equation (5) determines the flux of the fluid in the slit for any given consistency function \( m(x, y) \).

Owing to various effects, e.g. consolidation, reaction, absorption and thermal effects, the consistency of the lubricant in the central layer may be different from that of the peripheral layer and the consistency function \( m(x, y) \) may be assumed to be given by

\[
\begin{align*}
  m(x, y) &= m_1 \quad 0 \leq y < h - a \\
  m(x, y) &= km_1 \quad h - a \leq y \leq h
\end{align*}
\]

where \( a \) is the thickness of the peripheral layer and \( k \) is a constant. Using eqns. (5) - (7) the final expression for flow flux can be obtained as

\[ Q = 2b \left( -\frac{dp}{dx} \right)^{1/n} \frac{n}{2n + 1} \frac{h^{(2n+1)/n}}{m_1^{1/n}} F_0 \]

where

\[ F_0 = 1 - (1 - k^{-1/n}) \left\{ 1 - \left( 1 - \frac{a}{h} \right)^{(2n+1)/n} \right\} \]

3. Squeezing between stiff solids

Equation (8) is used to study the lubrication of squeeze film bearings with stiff or compliant surfaces.

3.1. Parallel plates

Consider the squeeze film between two parallel plates with film thickness \( 2h \); each plate approaches the other symmetrically with velocity \( V \) as shown in Fig. 2. The flow flux is given by eqn. (8) which can also be obtained by using the equation of continuity as [22, 23]

\[ Q = 2b V x \]
Thus, from eqns. (8) and (10), the equation governing the pressure is
\[
\frac{dp}{dx} = -m_1 \left( \frac{2n+1}{n} \frac{Vx}{F_0} \right) \left( \frac{1}{h} \right)^{2n+1}
\] (11)

Integrating eqn. (11) and using \( p = 0 \) at \( x = l \) gives
\[
p = m_1 \left( \frac{2n+1}{n} \frac{V}{F_0} \right) \left( \frac{1}{h} \right)^{2n+1} \left( \frac{l^{n+1} - x^{n+1}}{n+1} \right)
\] (12)

where 2l is the length of the bearing and \( F_0 \) is given by eqn. (9).

The load capacity \( W_{p2} \) is given by
\[
W_{p2} = 2b \int_0^l p \, dx
\]

which on using eqn. (12) gives
\[
W_{p2} = \frac{2bm_1}{n+2} \left( \frac{2n+1}{n} \frac{V}{F_0} \right) \left( \frac{1}{h} \right)^{2n+1} l^{n+2}
\] (13)

The squeezing time \( t_{p2} \) for the plates to approach from an initial film thickness \( 2h_1 \) to a final thickness \( 2h_2 \) is obtained by putting \( dh/dt = -V \) in eqn. (13) and integrating:
\[
t_{p2} = \frac{2n+1}{n} l (n+2)/n \left\{ \frac{2bm_1}{(n+2)W_{p2}} \right\}^{1/n} \int_{h_1}^{h_2} \frac{1}{F_0 h^{(n+1)/n}} \, dh
\] (14)

For \( k = 1 \), the case of no consistency variation, \( F_0 = 1 \) and the expressions for load capacity \( W_{p1} \) and time \( t_{p1} \) of approach can be obtained from eqns. (13) and (14) as
\[
W_{p1} = \frac{2bm_1}{n+2} \left( \frac{2n+1}{n} \frac{V}{h} \right)^{n+1} l^{n+2}
\] (15)
\[
t_{p1} = \frac{2n+1}{n+1} l (n+2)/n \left\{ \frac{2bm_1}{(n+2)W_{p1}} \right\}^{1/n} \left( \frac{1}{h_2^{1+1/n} - h_1^{1+1/n}} \right)
\] (16)

To assess the effects of the peripheral layer and consistency variation on the characteristics of a squeeze film, we assume that the peripheral layer thickness \( \alpha = \delta h \), where \( \delta \ll 1 \), is a constant. Then, from eqns. (13) - (16),
\[
\frac{W_{p2}}{W_{p1}} = \frac{1}{F_0^n}
\] (17)
\[
\frac{t_{p2}}{t_{p1}} = \frac{1}{F_0}
\] (18)

where \( F_0 \) is defined by eqn. (9) with \( \alpha = \delta h \).

The ratios given by eqns. (17) and (18) are plotted for various values of \( \delta, k \) and \( n \) in Figs. 3 and 4. As \( k \), which is greater than unity, increases, these ratios increase showing that the effect of the high consistency peripheral
layer adjacent to the surface is to increase the load capacity and squeezing time. Also, the load capacity and squeezing time increase as δ increases and this increase is enhanced owing to the pseudoplastic behaviour of the fluid.

3.2. Cylindrical surfaces

Consider the case of the squeeze film between two cylindrical surfaces using a power law lubricant, as shown in Fig. 5. The film thickness is given by

\[ h = h_0 + \frac{x^2}{2R_0} \]  

(19)

where \( 2h_0 \) is the minimum film thickness and \( R_0 \) is the radius of curvature of the approaching cylinders. The load capacity \( W_{e2} \) and squeezing time \( t_{e2} \) are

![Fig. 5. Squeezing between cylindrical surfaces.](image-url)
where 2l is the length of the projected film and $F_0$ and $h$ are given by eqns. (9) and (19) respectively.

For $k = 1$, eqns. (20) and (21) take the following forms:

$$W_{c1} = 2b m_1 \left( \frac{2n+1}{n} V \right)^n \int_0^1 \frac{x^{n+1}}{h^{2n+1}} \, dx$$  \hspace{1cm} (22)

$$t_{c1} = \frac{2n+1}{n} \left( \frac{2b m_1}{W_{c1}} \right)^{1/n} \int_0^1 \left( \int_0^{h^{2n+1}} \frac{x^{n+1}}{h^{2n+1}} \, dx \right)^{1/n} \, dh_0$$  \hspace{1cm} (23)

For $a = \delta h$, the ratios $W_{c2}/W_{c1}$ and $t_{c2}/t_{c1}$ also take the same form as in eqns. (17) and (18) and hence the existence of a high consistency peripheral layer for $k > 1$ increases the load capacity and time of squeezing in this case also.

To assess the interaction of geometry of the surface and the non-newtonian behaviour of the lubricant on load capacity and time of squeezing, eqns. (20) and (21) are compared with the corresponding eqns. (15) and (16) for the case of parallel plates having the same projected area and the same minimum film thickness. In such a case, from eqns. (15), (16), (20) and (21),

$$\frac{W_{c2}}{W_{p1}} = (n+2) \int_0^1 \frac{X^{n+1}}{F_{11}^{n/(1+\alpha x^2)} x^{2n+1}} \, dx$$  \hspace{1cm} (24)

$$\frac{t_{c2}}{t_{p1}} = \frac{(n+1)(n+2)^{1/n}}{n} \frac{F_{11}^{1/(n+1)}}{H_1 H_2} \left\{ \frac{1}{F_{12} n (H_0 + \alpha_1 X^2)^{2n+1}} \right\} \, dX$$  \hspace{1cm} (25)

where in eqn. (25) $W_{c2} = W_{p1}$ and

$$F_{11} = 1 - (1 - k^{-1/n}) \left\{ 1 - \left( 1 - \frac{\bar{a}}{1 - \alpha X^2} \right)^{(2n+1)/n} \right\}$$

$$F_{12} = 1 - (1 - k^{-1/n}) \left\{ 1 - \left( 1 - \frac{\bar{a}_1}{H_0 + \alpha_1 X^2} \right)^{(2n+1)/n} \right\}$$

$$X = \frac{x}{l} \quad H_1 = \frac{h_1}{h_2} \quad H_0 = \frac{h_0}{h_2}$$

$$\bar{a} = \frac{a}{h_0} \quad \bar{a}_1 = \frac{a}{h_2} \quad \alpha = \frac{l^2}{2R_0 h_0} \quad \alpha_1 = \frac{l^2}{2R_0 h_2}$$
Equations (24) and (25) are plotted for various values of $\alpha, \alpha_1, \bar{a}, \bar{a}_1, k$ and $n$ in Figs. 6 - 9. These ratios increase as $k$ increases and this increase is enhanced by the pseudoplastic behaviour of the fluid for $k > 1$. From Figs.

Fig. 6. Variation in $W_{c2}/W_{p1}$ with $k$ for various values of $n$ ($\alpha = 0.01; \bar{a} = 0.1$): $\cdots$, $n = 0.5$; $\cdots$, $n = 1.0$; $\cdots$, $n = 1.5$.

Fig. 7. Variation in $W_{c2}/W_{p1}$ with $\alpha$ for various values of $k$ and $\bar{a}$ ($n = 0.5$): $\cdots$, $\bar{a} = 0.1$; $\cdots$, $\bar{a} = 0.05$.

Fig. 8. Variation in $t_{c2}/t_{p1}$ with $k$ for various values of $n$ ($\bar{a}_1 = 0.1; \alpha_1 = 0.01; H_1 = 2.0$): $\cdots$, $n = 0.5$; $\cdots$, $n = 1.0$; $\cdots$, $n = 1.5$.

Fig. 9. Variation in $t_{c2}/t_{p1}$ with $\alpha_1$ for various values of $k$ and $\bar{a}_1$ ($n = 0.5; H_2 = 2.0$): $\cdots$, $\bar{a}_1 = 0.1$; $\cdots$, $\bar{a}_1 = 0.05$. 
7 and 9 the effect of the curvature of the bearing surface is to decrease the load capacity and the squeezing time.

3.3. Journal bearing

Consider the case of a squeeze film in a journal bearing when the journal and bearing are approaching symmetrically with a velocity \( V \) as shown in Fig. 10. The film thickness \( 2h \) is given by

\[
2h = c(1 - \epsilon \cos \theta)
\]

where \( c = R_1 - r_1 \) is the clearance and \( \epsilon = e/c \) is the eccentricity ratio.

The flow flux \( Q \) can be obtained from eqn. (8) by replacing \( x \) by \( r_1 \theta \).

This flux can also be obtained by using the equation of continuity as \([23]\)

\[
Q = 2b V r_1 \sin \theta
\]

which on using eqn. (8) gives

\[
\frac{dp}{d\theta} = -r_1^{n+1} m_1 V^n \sin^n \theta \frac{1}{F_0^n h^{2n+1}} \left( \frac{2n + 1}{n} \right)
\]

where \( F_0 \) and \( h \) are defined in eqns. (9) and (26) respectively.

Integrating eqn. (28) and using the boundary condition

\[
p = 0 \quad \text{at} \quad \theta = \pi/2
\]

gives the pressure distribution as

\[
p(\theta) = \frac{\pi/2}{\theta} \left[ r_1^{n+1} m_1 \left( \frac{2n + 1}{n} V \sin \theta \right)^n \left\{ \frac{2}{c(1 - \epsilon \cos \theta)} \right\}^{2n+1} \frac{1}{F_0^n} \right] d\theta
\]

The load capacity \( W_{12} \) is given by

\[
W_{12} = 2b r_1 \int_0^{\pi/2} p \cos \theta \, d\theta
\]

Fig. 10. Squeezing in a journal bearing.
which on using eqn. (30) gives

\[ W_{j2} = B_1 V^n I_1 \]  

(31)

where

\[ I_1 = \int_0^{\pi/2} \frac{\sin^{n+1} \theta}{F_{21}(1 - \varepsilon \cos \theta)^{2n+1}} \, d\theta \]

\[ F_{21} = 1 - (1 - k^{-1/n}) \left\{ 1 - \left( 1 - \frac{\bar{a}_2}{1 - \varepsilon \cos \theta} \right)^{(2n+1)/n} \right\} \]

\[ \bar{a}_2 = \frac{2a}{c} \quad B_1 = 2b \pi \frac{n+2}{n} \left( \frac{2}{c} \right)^{2n+1} \left( \frac{2n+1}{n} \right) \]

The squeezing time \( t_{j2} \) for the surfaces to approach from the initial concentric position (\( \varepsilon = 0 \)) to a final eccentric position (\( \varepsilon = \varepsilon_1 \)) is given by

\[ t_{j2} = \frac{c}{2} \left( \frac{B_1}{W_{j2}} \right)^{1/n} \int_0^{\varepsilon_1} I_1^{1/n} \, d\varepsilon \]  

(32)

For \( k = 1 \) the expressions for load capacity \( W_{j1} \) and squeezing time \( t_{j1} \) are written as

\[ W_{j1} = B_1 V^n \int_0^{\pi/2} \frac{\sin^{n+1} \theta}{(1 - \varepsilon \cos \theta)^{2n+1}} \, d\theta \]  

(33)

\[ t_{j1} = \frac{c}{2} \left( \frac{B_1}{W_{j1}} \right)^{1/n} \int_0^{\varepsilon_1} \int_0^{\pi/2} \frac{\sin^{n+1} \theta}{\left( 1 - \varepsilon \cos \theta \right)^{2n+1}} \, d\theta \, d\varepsilon \]  

(34)

When \( a = \delta h \), the ratios \( W_{j2}/W_{j1} \) and \( t_{j2}/t_{j1} \) are of the same form as in eqns. (17) and (18) and give the same results.

To assess the interaction of the geometry of the surface and the non-Newtonian behaviour of the lubricant on load capacity and time of squeezing, the expressions given by eqns. (31) and (32) are compared with the corresponding eqns. (15) and (16) for parallel plates having the same projected area and the same minimum film thickness. Thus taking

\[ l = r \quad h = \frac{c}{2} (1 - \varepsilon) \quad h_2 = \frac{c}{2} (1 - \varepsilon_1) \quad h_1 = \frac{c}{2} \]

in eqns. (15) and (16) and using eqns. (31) and (32),

\[ \frac{W_{j2}}{W_{p1}} = (1 - \varepsilon)^{2n+1} (n + 2) \frac{I_1}{I_1} \]  

(35)

\[ \frac{t_{j2}}{t_{p1}} = (n + 2)^{1/n} \frac{n + 1}{n} \int_0^{\varepsilon_1} I_1^{1/n} \, d\varepsilon / \left\{ \left( \frac{1}{1 - \varepsilon_1} \right)^{1+1/n} - 1 \right\} \]  

(36)

where \( W_{j2} = W_{p1} \) in eqn. (36).
Equations (35) and (36) are plotted for various values of $\epsilon, \epsilon_1, \bar{a}_1, k$ and $n$ in Figs. 11 - 14. From Figs. 11 and 13 these ratios increase as $k$ increases and the increase is enhanced by the pseudoplastic behaviour of

Fig. 11. Variation in $W_{j2}/W_{p1}$ with $k$ for various values of $n$ ($\bar{a} = 0.05; \epsilon = 0.3$): ---, $n = 0.5$; ---, $n = 1.0$; ---, $n = 1.5$.

Fig. 12. Variation in $W_{j2}/W_{p1}$ with $\epsilon$ for various values of $k$ and $\bar{a}$ ($n = 0.5$): ---, $\bar{a} = 0.1$; ---, $\bar{a} = 0.05$.

Fig. 13. Variation in $t_{j2}/t_{p1}$ with $k$ for various values of $n$ ($\bar{a}_1 = 0.05; \epsilon_1 = 0.3$): ---, $n = 0.5$; ---, $n = 1.0$; ---, $n = 1.5$.

Fig. 14. Variation in $t_{j2}/t_{p1}$ with $\epsilon_1$ for various values of $k$ and $\bar{a}_1$ ($n = 0.5$): ---, $\bar{a}_1 = 0.1$; ---, $\bar{a}_1 = 0.05$. 

the fluid for \( k > 1 \). From Figs. 12 and 14 the effect of the eccentricity ratio is to decrease the load capacity and the squeezing time.

### 3.4. Parallel circular plates

Consider squeezing between two parallel circular plates of film thickness \( 2h \), approaching each other symmetrically with a velocity \( V \), as shown in Fig. 15. The flow flux is given by eqn. (8) by replacing \( b \) by \( 2\pi r \). This flux can also be obtained by using the equation of continuity as [22]

\[
Q = 4\pi r^2 V
\]

which on using eqn. (8) gives

\[
\frac{dp}{dr} = -m_1 \left( \frac{2n + 1}{2n} \frac{V r^n}{F_0} \left( \frac{1}{h} \right)^{2n+1} \right)
\]

where \( F_0 \) is defined by eqn. (9).

Integrating eqn. (38) and using \( p = 0 \) at \( r = R \), where \( R \) is the radius of the approaching surfaces, gives

\[
p = m_1 \left( \frac{2n + 1}{2n} \frac{V}{F_0} \right)^n \frac{1}{h^{2n+1}} \frac{R^{n+1} - r^{n+1}}{n+1}
\]

The load capacity \( W_{pc2} \) is defined by

\[
W_{pc2} = \int_0^R 2\pi r p \, dr
\]

which on using eqn. (39) gives

\[
W_{pc2} = \frac{\pi m_1}{n+3} \left( \frac{2n + 1}{2n} \frac{V}{F_0} \right)^n \frac{1}{h^{2n+1}} R^{n+3}
\]

The elapsed time \( t_{pc2} \) to reduce the film from an initial film thickness \( 2h_1 \) to a final thickness \( 2h_2 \) is

\[
t_{pc2} = \frac{2n + 1}{2n} R^{(n+3)/n} \left( \frac{\pi m_1}{n+3} \frac{1}{W_{pc2}} \right)^{1/n} \frac{h_1}{h_2^{1/n}} \int_{h_2}^{h_1} \frac{1}{F_0 h (2n+1)/n} \, dh
\]

Fig. 15. Squeeze film between parallel circular plates.
For \( k = 1 \), the expressions for load capacity \( W_{pc1} \) and squeezing time \( t_{pc1} \) are written as

\[
W_{pc1} = \pi m_1 \left( \frac{2n+1}{2n} \right)^n \left( \frac{1}{n+3} \right) R^{n+3} h^{2n+1} \quad (42)
\]

\[
t_{pc1} = \frac{2n+1}{2(n+1)} R^{(n+3)/n} \left( \frac{\pi m_1}{n+3} \right)^{1/n} \left( \frac{1}{W_{pc1}} \right)^{1/n} \left( \frac{1}{h_{2(n+1)n}} - \frac{1}{h_{1(n+1)n}} \right) \quad (43)
\]

For \( a = \delta h \), the ratios \( W_{pc2}/W_{pc1} \) and \( t_{pc2}/t_{pc1} \) are of the same form as in eqns. (17) and (18) and therefore the same results are also valid.

### 3.5. Spherical surfaces

Consider the case of a symmetrical squeeze film between two spherical surfaces as shown in Fig. 16. The film thickness \( 2h \) is given by

\[
h = h_0 + \frac{r^2}{2R_0} \quad (44)
\]

where \( 2h_0 \) is the minimum film thickness and \( R_0 \) is the radius of curvature of the approaching surfaces.

Following the same procedure as in Section 3.4, we can write the load capacity \( W_{s2} \) and the squeezing time \( t_{s2} \) as

\[
W_{s2} = \pi m_1 \left( \frac{2n+1}{2n} \right)^n \int_0^R \frac{r^{n+2}}{F_0^* h^{2n+1}} \, dr \quad (45)
\]

\[
t_{s2} = \frac{2n+1}{2n} \left( \frac{\pi m_1}{W_{s2}} \right)^{1/n} \int_{h_2}^{h_1} \left( \int_0^R \frac{r^{n+2}}{F_0^* h^{2n+1}} \, dr \right)^{1/n} \, dh_0 \quad (46)
\]

where \( R \) is the radius of the projected circular area and \( F_0 \) and \( h \) are defined in eqns. (9) and (44) respectively.

For \( k = 1 \), eqns. (45) and (46) take the form

\[
W_{s1} = \pi \left( \frac{2n+1}{2n} \right)^n m_1 \int_0^R \frac{r^{n+2}}{h^{2n+1}} \, dr \quad (47)
\]

\[
t_{s1} = \frac{2n+1}{2n} \left( \frac{\pi m_1}{W_{s1}} \right)^{1/n} \int_{h_2}^{h_1} \left( \int_0^R \frac{r^{n+2}}{h^{2n+1}} \, dr \right)^{1/n} \, dh_0 \quad (48)
\]

Fig. 16. Squeezing between spherical surfaces.
For $a = \delta h$ the ratios $W_{s2}/W_{s1}$ and $t_{s2}/t_{s1}$ take the same form as in eqns. (17) and (18) and hence the same results are obtained.

To assess the effects of the interaction of the geometry of the surface and the non-newtonian behaviour of the lubricant on load capacity and time of squeezing, eqns. (45) and (46) are compared with the corresponding eqns. (40) and (41) for parallel circular plates having the same projected area and the same minimum film thickness. From eqns. (40), (41), (45) and (46),

$$\frac{W_{s2}}{W_{pc1}} = \frac{(n+3)\int_0^1 \frac{X^{n+2}}{F_{s1}(1+\beta X^2)^{2n+1}} \, dX}{\int_0^1 \frac{X^{n+2}}{F_{pc1}(1+\beta X^2)^{2n+1}} \, dX}$$

(49)

$$\frac{t_{s2}}{t_{pc1}} = \frac{(n+1)(n+3)^{1/n}}{n} \frac{H_1^{(n+1)/n}}{H_1^{(n+1)/n} - 1} \times$$

$$\times \int_1^{H_1} \left\{ \int_0^1 \frac{X^{n+2}}{F_{s2}(H_0 + \beta_1 X^2)^{2n+1}} \, dX \right\}^{1/n} \, dH_0$$

(50)

where $W_{s2} = W_{pc1}$ in eqn. (50) and

$$F_{s1} = 1 - (1 - k^{-1/n}) \left\{ 1 - \left( 1 - \frac{\bar{a}}{1 + \beta X^2} \right)^{(2n+1)/n} \right\}$$

$$F_{s2} = 1 - (1 - k^{-1/n}) \left\{ 1 - \left( \frac{\bar{a}}{H_0 + \beta_1 X^2} \right)^{(2n+1)/n} \right\}$$

$$X = \frac{r}{R} \quad H_1 = \frac{h_1}{h_2} \quad H_0 = \frac{h_0}{h_2}$$

$$\bar{a} = \frac{a}{h_0} \quad \bar{a}_1 = \frac{a}{h_2}$$

$$\beta = \frac{R^2}{2R_0 h_0} \quad \beta_1 = \frac{R^2}{2R_0 h_2}$$

Equations (49) and (50) are of the same form as eqns. (24) and (25) and hence the same results are obtained as with cylindrical surfaces.

3.6. Spherical bearing

Consider squeezing between two eccentric spherical surfaces of radii $r_1$ and $R_1$ which are approaching each other with velocity $V$, as shown in Fig. 17. The film thickness $2h$ is given by

$$2h = c(1 - \epsilon \cos \theta)$$

(51)

where $c = R_1 - r_1$ is the clearance and $\epsilon = e/c$ is the eccentricity ratio.

The flux in this case can be obtained by substituting $b = 2\pi r_1 \sin \theta$ in eqn. (8) and also by using the equation of continuity as [23]

$$Q = 2\pi V r_1^2 \sin^2 \theta$$

(52)
which on using eqn. (8) gives

$$\frac{dp}{d\theta} = -r_1^{n+1}m_1V^n\sin^n\theta \frac{1}{F_0^n h^{2n+1}} \left( \frac{2n+1}{2n} \right)^n$$

where \(F_0\) and \(h\) are defined in eqns. (9) and (51) respectively.

Integrating eqn. (53) and using the boundary condition

\[ p = 0 \text{ at } \theta = \pi/2 \]

gives the pressure distribution as

$$p(\theta) = \int_0^{\pi/2} \left[ r_1^{n+1}m_1 \left( \frac{2n+1}{2n} V \sin \theta \right)^n \left( \frac{2}{c(1 - \epsilon \cos \theta)} \right)^{2n+1} \frac{1}{F_0^n} \right] d\theta$$

The load capacity \(W_{sp2}\) is given by

$$W_{sp2} = 2\pi r_1^2 \int_0^{\pi/2} p \sin \theta \cos \theta \, d\theta$$

which on using eqn. (54) gives

$$W_{sp2} = B_2 V^n I_2$$

where

$$I_2 = \int_0^{\pi/2} \frac{\sin^{n+2}\theta}{F_{21}^n (1 - \epsilon \cos \theta)^{2n+1}} \, d\theta$$

$$B_2 = \pi m_1 r_1^{n+3} \left( \frac{2n+1}{2n} \right)^n \left( \frac{2}{c} \right)^{2n+1}$$

and \(F_{21}\) is defined in eqn. (31).
The squeezing time $t_{sp2}$ for the surfaces to approach from the initial concentric position ($\epsilon = 0$) to a final eccentric position ($\epsilon = \epsilon_1$) is given by

$$t_{sp2} = \frac{c}{2} \left( \frac{B_2}{W_{sp2}} \right)^{1/n} \int_0^{\epsilon_1} I_2^{1/n} d\epsilon$$

(56)

For $k = 1$ the expressions for load capacity $W_{sp1}$ and squeezing time $t_{sp1}$ are written as

$$W_{sp1} = B_2 V^n \int_0^{\pi/2} \frac{\sin^{n+2\theta}}{(1 - \epsilon \cos \theta)^{2n+1}} d\theta$$

(57)

$$t_{sp1} = \frac{c}{2} \left( \frac{B_2}{W_{sp1}} \right)^{1/n} \int_0^{\pi/2} \int_0^{\epsilon_1} \frac{\sin^{n+2\theta}}{(1 - \epsilon \cos \theta)^{2n+1}} d\theta \right)^{1/n} d\epsilon$$

(58)

When $a = \delta h$, the ratios $W_{sp2}/W_{sp1}$ and $t_{sp2}/t_{sp1}$ are of the same form as in eqns. (17) and (18) and give the same results.

To assess the interaction of the geometry of the surface and the non-newtonian behaviour of the lubricant on load capacity and time of squeezing the expressions given by eqns. (55) and (56) are compared with the corresponding eqns. (40) and (41) for circular plates of the same projected area and minimum film thickness. Thus, taking

$$h = \frac{c}{2} (1 - \epsilon) \quad h_2 = \frac{c}{2} (1 - \epsilon_1) \quad h_1 = \frac{c}{2}$$

in eqns. (42) and (43) and using eqns. (55) and (56),

$$\frac{W_{sp2}}{W_{pc1}} = (n + 3)(1 - \epsilon)^{2n+1} I_2$$

(59)

$$\frac{t_{sp2}}{t_{pc1}} = (n + 3)^{1/n} \int_0^{\epsilon_1} I_2^{1/n} d\epsilon \left\{ \frac{1}{1 - \epsilon_1} \right\}^{1+1/n} - 1$$

(60)

where $W_{sp2} = W_{pc1}$ in eqn. (60).

Equations (59) and (60) are of the same form as eqns. (35) and (36) and hence give the same results as for journal bearings.

4. Squeeze film between compliant solids

The analysis of squeeze films presented above is applicable to stiff solids. When the surfaces have low elastic moduli, as in the case of rubber, the lubrication mechanism is modified owing to the large elastic deformation caused by pressures which are low enough to leave the viscosity unaltered [24]. Though some investigations have been carried out using newtonian lubricants [24 - 32], little attention has been given to non-newtonian
lubricants [33]. The effects of consistency variation and non-newtonian behaviour on the characteristics of squeeze films between compliant solids can be studied following the procedure suggested by Fein [31].

In contact between compliant cylinders the elastic effects can be taken into account by assuming the length $l_d$ of the deformed contact zone under the applied load $W$ to be the same as in dry contact [34], i.e.

$$l_d = 2 \left( \frac{W}{\pi \frac{R_2}{E}} \right)^{1/2}$$

$$R_2 = \frac{r_2}{2}$$

$$\frac{1}{E} = \frac{1 - \nu^2}{E_1}$$

where $r_2$ is the radius of each cylinder, $\nu$ the Poisson ratio and $E_1$ the Young's modulus of the compliant surfaces. Similarly, in elastic contact between two spherical surfaces, the radius $r_d$ of the deformed contact zone under the applied load $W$ can be written as [31, 32]

$$r_d = 1.15 \left( \frac{WR_2}{E} \right)^{1/3}$$

where

$$R_2 = \frac{r_2}{2}$$

and $r_2$ is the radius of each sphere.

4.1. Rectangular plates model (cylinders)

If we consider squeezing between two compliant cylinders, the deformed zone may be approximated in the form of rectangular plates, as shown in Fig. 18. The load capacity $W_E$ can be written from eqns. (13) and (61) as

![Fig. 18. Squeezing between compliant cylinders.](image)
\[ W_{E_2} = \frac{2b m_1}{n + 2} \left( \frac{2n + 1}{n} \frac{V}{F_0} \right)^n \frac{1}{h} \left( \frac{1}{h} \right)^{2n+1} l_d^{n+2} \]  

(63)

where \( F_0 \) is defined in eqn. (9).

The time \( t_{E_2} \) of squeezing for the surfaces to approach from an initial position \( 2h_1 \) to a final position \( 2h_2 \) is given by eqns. (14) and (61) as

\[ t_{E_2} = \frac{2n + 1}{n} \left( \frac{2b m_1}{n + 2} \right)^{1/n} W_{E_2}^{1/2} \left( \frac{4R_2}{\pi E} \right)^{(n + 2)/2n} \int_{h_1}^{h_2} \frac{1}{F_0 h^{(2n+1)/n}} dh \]  

(64)

To assess the interaction of the non-newtonian behaviour of the lubricant and the elasticity of the compliant surfaces the ratio \( t_{E_2}/t_2 \) can be written from eqns. (14) and (64) as

\[ \frac{t_{E_2}}{t_2} = \left( \frac{2}{l} \right)^{1 + 2/n} \left( \frac{W_{E_2} R_2}{\pi E} \right)^{1/2 + 1/n} \]  

(65)

From eqn. (65), the squeezing time increases as the elastic modulus \( E \) decreases and as the radius \( R_2 \) of curvature increases. From this equation this ratio is greater for \( n = 0.5 \) than for \( n = 1.5 \). Hence, the above-mentioned increase is enhanced owing to the pseudoplastic behaviour of the fluid. For \( a = \delta h \), eqn. (64) can be integrated and the squeezing time \( t_{E_2} \) and film thickness \( h_2 \) written as follows for \( h_2/h_1 \ll 1 \):

\[ t_{E_2} = \left( \frac{2b m_1}{n + 2} \right)^{1/n} \frac{2n + 1}{n + 1} W_{E_2}^{1/2} \left( \frac{4R_2}{\pi E} \right)^{(n + 2)/2n} \frac{1}{F_E} \frac{1}{h_2^{(n + 1)/n}} \]  

(66)

\[ h_2 = \left( \frac{2n + 1}{n + 1} \right)^{n/(n+1)} \left( \frac{1}{n + 2} \right)^{1/(n+1)} \left( \frac{4W_{E_2} R_2}{\pi E} \right)^{(2n+2)/n} \times \]  

(67)

\[ \times \left( \frac{8b m_1 R_2}{\pi E} \right)^{1/(n+1)} \]

where

\[ F_E = 1 - (1 - k^{-1/n}) \left[ 1 - \left( 1 - \delta \right)^{(2n+1)/n} \right] \]  

(68)

From eqns. (66) and (67) the time of approach and film thickness increase as the compliant surface becomes more elastic (i.e. \( E \) decreases). These quantities also increase owing to the radius of curvature and the load as the elastic deformation gives a larger load-bearing area. From eqn. (68), \( F_E \) decreases as \( k \) and \( \delta \) increase. Thus from eqns. (66) and (67) the squeezing time and film thickness increase as the thickness and consistency of the peripheral layer fluid increase for \( k > 1 \).

For no consistency variation, i.e. \( k = 1 \), the corresponding expressions for the load capacity \( W_{E_1} \), the squeezing time \( t_{E_1} \) and the film thickness \( h_2 \) can be written from eqns. (63), (66) and (67) respectively as follows for \( h_2/h_1 \ll 1 \):
which are of the same form as obtained by Chandra [33]. From these equations, for \( n = 1 \), the same results as discussed by Fein [31] are obtained.

4.2. Parallel circular plates model (spheres)

We consider squeezing between two compliant spheres, the deformed zone of which may be approximated in the form of two parallel circular plates as shown in Fig. 19. The load capacity \( W_{E2} \) can be written from eqns. (40) and (62) as

\[
W_{E2} = \frac{\pi m_1}{n+3} \left( \frac{2n+1}{2n} \frac{V}{F_0} \right)^n \left( \frac{1}{h} \right)^{2n+3} \tag{72}
\]

where \( F_0 \) is defined in eqn. (9).

The time \( t_{E2} \) of squeezing for the surfaces to approach from an initial position \( 2h_1 \) to the final position \( 2h_2 \) is given from eqns. (66) and (62) as

\[
t_{E2} = \frac{2n+1}{2n} \left( \frac{\pi m_1}{n+3} \right)^{1/n} \frac{1}{W_{E2}^{1/3}} \left( \frac{R_2}{E} \right)^{(n+3)/n} (1.15)^{(n+3)/n} \int_{h_2}^{h_1} \frac{dh}{F_0 h^{(2n+1)/n}} \tag{73}
\]

To assess the interaction of the non-newtonian behaviour of the lubricant and the compliance of the surfaces, the ratio \( t_{E2}/t_{pc2} \) can be written from eqns. (73) and (41) as

\[
\frac{t_{E2}}{t_{pc2}} = \left( \frac{1.15}{l} \right)^{1+3/n} \left( \frac{W_{E2} R_2}{\pi E} \right)^{1/3+1/n} \tag{74}
\]

Fig. 19. Squeezing between compliant spheres.
Equation (74) is of the same form as eqn. (65) and hence the same conclusions are obtained. For \( a = \delta h \), eqn. (73) can be integrated and the squeezing time \( t_{E2} \) and film thickness \( h_2 \) can be written as follows for \( h_2/h_1 < 1 \):

\[
t_{E2} = \frac{2n + 1}{2(n + 1)} \left( \frac{\pi m_1}{n + 3} \right)^{1/n} W_{E2}^{1/3} \left( \frac{R_2}{E} \right)^{(n + 3)/3n} \frac{1}{F_E h_2^{(n + 1)/n}}
\]

\[
h_2 = (1.15)^{(n + 3)/(n + 1)} \left\{ \frac{2n + 1}{2(n + 1)} \frac{1}{F_E t_{E2}} \right\}^{n/(n + 1)} \left( \frac{1}{n + 3} \right)^{1/(n + 1)}
\]

\[
\times \left( \frac{W_{E2} R_2}{E} \right)^{n/(3n + 3)} \left( \frac{\pi m_1 R_2}{E} \right)^{1/(n + 1)}
\]

(75)

From eqns. (75) and (76), the squeezing time and the film thickness increase with \( k, \delta \) and \( W_{E2} \).

For no consistency variation, i.e. \( k = 1 \), the load capacity \( W_{E1} \), squeezing time \( t_{E1} \) and film thickness \( h_2 \) can be written from eqns. (72) and (73) as follows for \( h_2/h_1 < 1 \):

\[
W_{E1} = \frac{\pi m_1}{n + 3} \left( \frac{2n + 1}{2n} \right)^{n/2} \left( \frac{1}{h} \right)^{2n + 1} r_d^{n + 3}
\]

(77)

\[
t_{E1} = \frac{2n + 1}{2(n + 1)} \left( \frac{\pi m_1}{n + 3} \right)^{1/n} W_{E1}^{1/3} \left( \frac{R_2}{E} \right)^{(n + 3)/3n} \frac{1}{h_2^{(n + 1)/n}}
\]

\[
h_2 = (1.15)^{(n + 3)/(n + 1)} \left\{ \frac{2n + 1}{2(n + 1)} \frac{1}{t_{E1}} \right\}^{n/(n + 1)} \left( \frac{1}{n + 3} \right)^{1/(n + 1)} \left( \frac{W_{E1} R_2}{E} \right)^{n/(3n + 3)}
\]

\[
\times \left( \frac{\pi m_1 R_2}{E} \right)^{1/(n + 1)}
\]

(78)

which are of the same form as obtained by Chandra [33]. From these equations, for \( n = 1 \), the same results as discussed by Fein [31] are obtained.

5. Conclusion

The characteristics of various squeeze film bearings with a power law lubricant were investigated by considering the effects of consistency variation.

With stiff solids the effect of a high consistency layer adjacent to the bearing surface is to increase the load capacity and squeezing time. These parameters also increase as the thickness of the high consistency peripheral layer increases. This increase is enhanced by the pseudoplastic behaviour of the fluid. The effect of bearing curvature with variable film thickness is to decrease the load capacity and time of squeezing compared with the case of constant film thickness with the same projected area and the same minimum film thickness.
For cylindrical and spherical surfaces with a compliant layer, the film thickness increases owing to the compliance of the surface and the load and conformity of the surfaces. It also increases because of the high consistency layer present at the bearing surface and this increase is enhanced by the pseudoplastic behaviour of the fluid.

Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>thickness of the peripheral layer</td>
</tr>
<tr>
<td>(b)</td>
<td>width of the bearing or length of the journal</td>
</tr>
<tr>
<td>(c)</td>
<td>clearance width in journal bearing or spherical bearing</td>
</tr>
<tr>
<td>(E)</td>
<td>Young's moduli of the materials for compliant solids</td>
</tr>
<tr>
<td>(2h)</td>
<td>film thickness</td>
</tr>
<tr>
<td>(2h_0)</td>
<td>minimum film thickness between cylindrical surfaces</td>
</tr>
<tr>
<td>(2h_1)</td>
<td>initial film thickness</td>
</tr>
<tr>
<td>(k)</td>
<td>consistency ratio of the peripheral layer to the middle layer</td>
</tr>
<tr>
<td>(l_d)</td>
<td>length of the deformed zone for compliant cylinders</td>
</tr>
<tr>
<td>(m_1)</td>
<td>consistency of the central layer</td>
</tr>
<tr>
<td>(n)</td>
<td>flow behaviour index</td>
</tr>
<tr>
<td>(P)</td>
<td>hydrodynamic pressure</td>
</tr>
<tr>
<td>(Q)</td>
<td>flow flux</td>
</tr>
<tr>
<td>(r)</td>
<td>radial coordinate</td>
</tr>
<tr>
<td>(r_1)</td>
<td>radius of the journal or radius of the spheres</td>
</tr>
<tr>
<td>(r_2)</td>
<td>radius of the cylinders or spheres for compliant solids</td>
</tr>
<tr>
<td>(r_d)</td>
<td>radius of the deformed contact zone of spherical surfaces</td>
</tr>
<tr>
<td>(R)</td>
<td>radius of the circular plates</td>
</tr>
<tr>
<td>(R_0)</td>
<td>radius of the cylindrical surface or spherical surface</td>
</tr>
<tr>
<td>(R_1)</td>
<td>radius of the bearing</td>
</tr>
<tr>
<td>(u, v)</td>
<td>velocity components in the (x, y) directions</td>
</tr>
<tr>
<td>(V)</td>
<td>normal velocities of the approaching surfaces</td>
</tr>
<tr>
<td>(W)</td>
<td>load capacity</td>
</tr>
<tr>
<td>(x, y)</td>
<td>coordinate system</td>
</tr>
<tr>
<td>(\nu)</td>
<td>Poisson's ratio of the compliant surfaces</td>
</tr>
</tbody>
</table>

References

5 J. B. Shukla, Load capacity and time relation for squeeze films in conical bearings, *Wear*, 7 (1964) 368.