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EFFECT OF ROTATION IN LUBRICATION PROBLEMS: A NEW FUNDAMENTAL SOLUTION

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Summary

An extended theory of hydrodynamic lubrication is derived from the Reynolds equation; the theory takes into account the effects of uniform rotation about an axis that lies across the fluid film. The pressure and load capacity of bearing systems are obtained when the film thickness is a linear or an exponential function of the coordinate along the bearing length. While the load capacity increases with increasing coefficient of fluid viscosity for a plane inclined slider or an exponentially inclined slider using the classical theory, it is independent of the fluid viscosity using the extended theory when rotation is small.

1. Introduction

"Hydrodynamic lubrication" is a process by which two surfaces moving at some relative velocity with respect to each other are separated by a fluid film in which forces are generated by the relative motion only. A two-dimensional theory of lubrication was first developed by Reynolds [1], who showed that the variation in the lubricant pressure in the bearing is described by a partial differential equation (the Reynolds equation) and that if the lubricant layer is to transmit pressure between a shaft and a journal the layer

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must have varying thickness; otherwise, the stresses in the lubricant cannot balance the load of the shaft [2].

To extend the applicability of the theory and to enlarge its scope, several generalizations and extensions of the Reynolds equation have been attempted and the resulting theories worked out. Banerjee *et al.* [3] derived an extended Reynolds equation which takes into account the effects of uniform rotation about an axis across the fluid film and derived quantitative details for the "short bearing" case when the extent of rotation is small. Rotation introduces a number of new elements into a hydrodynamic problem. Some of the consequences are unexpected. For example, the role of viscosity is inverted. The effect of rotation can be traced to certain general theorems, relating to vorticity, in the dynamics of rotating fluids. Rotation induces a component of vorticity in the direction of rotation, and the effects arising from this are predominant: for large Taylor numbers the stream lines become closely wound spirals with motions principally confined to planes transverse to the direction of rotation. Therefore, derivation of an extended Reynolds equation, in which the effects of uniform rotation about an axis which lies across the fluid film are considered, would be useful because derived results are experimentally verifiable and most real physical systems are affected by rotation. The investigations of Banerjee *et al.* [3] are limited because of the simplifying assumption that the extent of rotation is small. More importantly, a certain class of fundamental solutions of this extended problem was omitted in their work [3].

An extended theory of hydrodynamic lubrication is formulated in the present paper: this theory takes into account the effects of uniform rotation about an axis that lies across the fluid film. An extended version of the Reynolds equation is derived for arbitrary amounts of uniform rotation which depends on the rotation number M (the square root of the conventional Taylor number) in addition to density, viscosity, film thickness and surface and transverse velocities. Certain fundamental solutions are presented which are *not* possible with the classical theory. The pressure and load capacity of the bearing systems are obtained where the film thickness is a linear or an exponential function of the coordinate along the bearing length. An important qualitative result is that whereas the load capacity increases with increasing coefficient of fluid viscosity for a plane inclined slider or an exponentially inclined slider by the classical theory, it is independent of the fluid viscosity in the present context when the extent of rotation is small.

2. The governing equations of hydrodynamic lubrication in a rotating frame of reference

Consider a layer of fluid kept rotating at a constant rate. Let Ω denote the angular velocity of rotation about the z axis. The hydrodynamic equations of momentum and continuity in the usual tensor component notation are

$$\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j}$$

$$= \rho X_i - \frac{\partial}{\partial x_i}\left\{p - \left(\frac{1}{2}\rho\,\epsilon_{ijk}\Omega_i r_k\right)^2\right\} + \frac{\partial}{\partial x_j}\left\{\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\mu\frac{\partial u_k}{\partial x_k}\right\} +$$

$$+ 2\rho\epsilon_{ijk}u_j\Omega_k \tag{1}$$

and

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_j)}{\partial x_j} = 0 \tag{2}$$

If we apply standard assumptions of lubrication theory [4], eqns. (1) reduce to

$$0 = -\frac{\partial P}{\partial x} + \mu\frac{\partial^2 u}{\partial z^2} + 2\rho\Omega v \tag{3}$$

$$0 = -\frac{\partial P}{\partial y} + \mu\frac{\partial^2 v}{\partial z^2} - 2\rho\Omega u \tag{4}$$

and

$$0 = -\frac{\partial P}{\partial z} \tag{5}$$

while eqn. (2) for steady flow gives

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \tag{6}$$

where

$$P = p - (\tfrac{1}{2}\rho\epsilon_{ijk}\Omega_i r_k)^2 \tag{7}$$

denotes the modified pressure.

Equations (3) - (7) give the governing hydrodynamical equations of momentum and continuity for the problem of steady lubrication.

3. Derivation of the extended Reynolds equation

From eqns. (3) - (5) the governing equations for u and v are obtained as follows:

$$\frac{\partial^4 u}{\partial z^4} + \left(\frac{2\rho\Omega}{\mu}\right)^2 u = -\frac{2\rho\Omega}{\mu^2}\frac{\partial P}{\partial y} \tag{8}$$

and

$$\frac{\partial^4 v}{\partial z^4} + \left(\frac{2\rho\Omega}{\mu}\right)^2 v = \frac{2\rho\Omega}{\mu^2}\frac{\partial P}{\partial x} \tag{9}$$

The boundary conditions on u and v are given by

$$\begin{aligned} &u = U_0 && \text{at } z = 0 \\ &u = U_h && \text{at } z = h \\ &\mu \frac{\partial^2 u}{\partial z^2} = \frac{\partial P}{\partial x} && \text{at } z = 0 \text{ and } z = h \end{aligned} \tag{10}$$

and

$$\begin{aligned} &v = 0 && \text{at } z = 0 \text{ and } z = h \\ &\mu \frac{\partial^2 v}{\partial z^2} = \frac{\partial P}{\partial y} + 2\rho\Omega U_0 && \text{at } z = 0 \\ &\mu \frac{\partial^2 v}{\partial z^2} = \frac{\partial P}{\partial y} + 2\rho\Omega U_h && \text{at } z = h \end{aligned} \tag{11}$$

Using the non-dimensional quantities defined by

$$\begin{aligned} &\bar{x} = \frac{x}{h_c} && \bar{u} = \frac{u}{q_c} && \bar{\mu} = \frac{\mu}{\mu_c} \\ &\bar{y} = \frac{y}{h_c} && \bar{v} = \frac{v}{q_c} && \bar{P} = \frac{h_c P}{\mu_c q_c} \\ &\bar{z} = \frac{z}{h_c} && \bar{w} = \frac{w}{q_c} && M = \frac{2\Omega h_c^2 \rho_c}{\mu_c} \\ &\bar{h} = \frac{h}{h_c} && \bar{\rho} = \frac{\rho}{\rho_c} && \end{aligned} \tag{12}$$

and dropping the bars for convenience, eqns. (8) - (11) and (6) respectively reduce to

$$\frac{\partial^4 u}{\partial z^4} + \frac{M^2 \rho^2}{\mu^2} u = -\frac{M\rho}{\mu^2} \frac{\partial P}{\partial y} \tag{13}$$

$$\frac{\partial^4 v}{\partial z^4} + \frac{M^2 \rho^2}{\mu^2} v = \frac{M\rho}{\mu^2} \frac{\partial P}{\partial x} \tag{14}$$

$$\begin{aligned} &u = \frac{U_0}{q_c} = U_1 && \text{at } z = 0 \\ &u = \frac{U_h}{q_c} = U_2 && \text{at } z = h \\ &\mu \frac{\partial^2 u}{\partial z^2} = \frac{\partial P}{\partial x} && \text{at } z = 0 \text{ and } z = h \end{aligned} \tag{15}$$

$$v = 0 \qquad \text{at } z = 0 \text{ and } z = h$$

$$\mu \frac{\partial^2 v}{\partial z^2} = \frac{\partial P}{\partial y} + M\rho U_1 \qquad \text{at } z = 0 \tag{16}$$

$$\mu \frac{\partial^2 v}{\partial z^2} = \frac{\partial P}{\partial y} + M\rho U_2 \qquad \text{at } z = h$$

and

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \tag{17}$$

Solutions for u and v which satisfy the relevant equations and boundary conditions are given by

$$u = \exp(\lambda z)\{c_1 \cos(\lambda z) + c_2 \sin(\lambda z)\} + \\ + \exp(-\lambda z)\{c_3 \cos(\lambda z) + c_4 \sin(\lambda z)\} - \frac{1}{M\rho}\frac{\partial P}{\partial y} \tag{18}$$

and

$$v = \exp(\lambda z)\{d_1 \cos(\lambda z) + d_2 \sin(\lambda z)\} + \\ + \exp(-\lambda z)\{d_3 \cos(\lambda z) + d_4 \sin(\lambda z)\} + \frac{1}{M\rho}\frac{\partial P}{\partial x} \tag{19}$$

where

$$\lambda = \left(\frac{M\rho}{2\mu}\right)^{1/2} \tag{20}$$

$$c_1 = \frac{1}{D}\{a_1 \sinh(\lambda h)\cos(\lambda h) - a_2 \sin(\lambda h)\cosh(\lambda h)\} \tag{21}$$

$$c_2 = \frac{1}{D}\{a_1 \cosh(\lambda h)\sin(\lambda h) + a_2 \cos(\lambda h)\sinh(\lambda h)\} \tag{22}$$

$$c_3 = U_1 + \frac{1}{M\rho}\frac{\partial P}{\partial y} - c_1 \tag{23}$$

$$c_4 = -\frac{1}{M\rho}\frac{\partial P}{\partial x} + c_2 \tag{24}$$

$$a_1 = \frac{1}{2M\rho}\left[M\rho U_2 + \frac{\partial P}{\partial y} + \exp(-\lambda h)\left\{\frac{\partial P}{\partial x}\sin(\lambda h) - \left(M\rho U_1 + \frac{\partial P}{\partial y}\right)\cos(\lambda h)\right\}\right] \tag{25}$$

$$a_2 = \frac{1}{2M\rho}\left[\frac{\partial P}{\partial x} - \exp(-\lambda h)\left\{\left(M\rho U_1 + \frac{\partial P}{\partial y}\right)\sin(\lambda h) + \frac{\partial P}{\partial x}\cos(\lambda h)\right\}\right] \tag{26}$$

$$D = \cos^2(\lambda h)\sinh^2(\lambda h) + \cosh^2(\lambda h)\sin^2(\lambda h) \tag{27}$$

$$d_1 = -\frac{1}{D}\{a_1 \sin(\lambda h)\cosh(\lambda h) + a_2 \sinh(\lambda h)\cos(\lambda h)\} \tag{28}$$

$$d_2 = \frac{1}{D}\{a_1 \cos(\lambda h)\sinh(\lambda h) - a_2 \cosh(\lambda h)\sin(\lambda h)\} \tag{29}$$

$$d_3 = -\left(\frac{1}{M\rho}\frac{\partial P}{\partial x} + d_1\right) \tag{30}$$

and

$$d_4 = -\left(U_1 + \frac{1}{M\rho}\frac{\partial P}{\partial y}\right) + d_2 \tag{31}$$

Expanding u and v in terms of non-negative integral powers of M, we obtain the following:

$$u = \left\{\frac{1}{2\mu}\frac{\partial P}{\partial x}z(z-h) + \frac{h-z}{h}U_1 + \frac{z}{h}U_2\right\} - \\ - \left\{\frac{\rho}{24\mu^2}\frac{\partial P}{\partial y}z(z^3 - 2z^2h + h^3)\right\}M + \ldots \tag{32a}$$

and

$$v = \frac{1}{2\mu}\frac{\partial P}{\partial y}z(z-h) + \left[\frac{\rho}{24\mu^2}\frac{\partial P}{\partial x}z(z^3 - 2z^2h + h^3) + \\ + \frac{\rho}{6\mu h}z\{(U_1 - U_2)z^2 + 3U_1 zh - (2U_1 + U_2)h^2\}\right]M + \ldots \tag{32b}$$

The expansions for u and v reproduce the classical Reynolds values and the values obtained by Banerjee *et al.* [3] for u and v for no rotation and small rotation respectively.

The substitution of expressions for u and v from eqns. (18) and (19) in the equation of continuity (eqn. (17)) gives

$$\frac{\partial(\rho w)}{\partial z} = -\frac{\partial}{\partial x}\left[-\frac{1}{M}\frac{\partial P}{\partial y} + \frac{\exp(-\lambda z)}{M}\left\{\left(M\rho U_1 + \frac{\partial P}{\partial y}\right)\cos(\lambda z) - \frac{\partial P}{\partial x}\sin(\lambda z)\right\} + \\ + 2\rho\{c_1 \cos(\lambda z)\sinh(\lambda z) + c_2 \sin(\lambda z)\cosh(\lambda z)\}\right] - \\ - \frac{\partial}{\partial y}\left[\frac{1}{M}\frac{\partial P}{\partial x} - \frac{\exp(-\lambda z)}{M}\left\{\left(M\rho U_1 + \frac{\partial P}{\partial y}\right)\sin(\lambda z) + \frac{\partial P}{\partial x}\cos(\lambda z)\right\} + \\ + 2\rho\{d_1 \cos(\lambda z)\sinh(\lambda z) + d_2 \sin(\lambda z)\cosh(\lambda z)\}\right] \tag{33}$$

Integrating both sides of eqn. (33) with respect to z with the conditions

$$
\begin{aligned}
w &= \frac{w_0}{q_c} = w_1 \qquad \text{at } z = 0 \\
w &= \frac{w_h}{q_c} = w_2 \qquad \text{at } z = h
\end{aligned}
\tag{34}
$$

gives

$$
\begin{aligned}
&\rho(w_2 - w_1) \\
&= -\int_0^h \frac{\partial}{\partial x}\left[-\frac{1}{M}\frac{\partial P}{\partial y} + \frac{\exp(-\lambda z)}{M}\left\{\left(M\rho U_1 + \frac{\partial P}{\partial y}\right)\cos(\lambda z) - \frac{\partial P}{\partial x}\sin(\lambda z)\right\} + \right. \\
&\left. \quad + 2\rho\{c_1 \cos(\lambda z)\sinh(\lambda z) + c_2 \sin(\lambda z)\cosh(\lambda z)\}\right] dz - \\
&\quad - \int_0^h \frac{\partial}{\partial y}\left[\frac{1}{M}\frac{\partial P}{\partial x} - \frac{\exp(-\lambda z)}{M}\left\{\left(M\rho U_1 + \frac{\partial P}{\partial y}\right)\sin(\lambda z) + \frac{\partial P}{\partial x}\cos(\lambda z)\right\} + \right. \\
&\left. \quad + 2\rho\{d_1 \cos(\lambda z)\sinh(\lambda z) + d_2 \sin(\lambda z)\cosh(\lambda z)\}\right] dz
\end{aligned}
\tag{35}
$$

The upper limit h in the last equation is a function of the coordinates x and y. Integrating before differentiation, which is permissible in the present case, gives

$$
\begin{aligned}
&\frac{\partial}{\partial x}\left\{\rho\psi_1(h)\frac{\partial P}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{\rho\psi_1(h)\frac{\partial P}{\partial y}\right\} + \frac{\partial}{\partial x}\left\{\rho\psi_2(h)\frac{\partial P}{\partial y}\right\} - \frac{\partial}{\partial y}\left\{\rho\psi_2(h)\frac{\partial P}{\partial x}\right\} \\
&= -\frac{\partial}{\partial x}\left[\frac{\rho}{2}(U_1 + U_2)\{h + M\rho\psi_2(h)\}\right] - \frac{\partial}{\partial y}\left\{\frac{M\rho^2}{2}(U_1 + U_2)\psi_1(h)\right\} - \\
&\quad - \rho(w_2 - w_1) + U_2 \frac{\partial}{\partial x}(\rho h)
\end{aligned}
\tag{36}
$$

where

$$
\psi_1(h) = -\frac{1}{\lambda M\rho}\,\frac{\sinh(\lambda h) - \sin(\lambda h)}{\cosh(\lambda h) + \cos(\lambda h)} \tag{37}
$$

and

$$
\psi_2(h) = -\frac{h}{M\rho} + \frac{1}{\lambda M\rho}\,\frac{\sinh(\lambda h) + \sin(\lambda h)}{\cosh(\lambda h) + \cos(\lambda h)} \tag{38}
$$

Equation (36) is the extended generalized Reynolds equation which reproduces the classical Reynolds equation and the extended Reynolds equation obtained by Banerjee *et al.* [3] for no rotation and small rotation respectively.

In most practical cases, the bearing is stationary and only the runner in the thrust bearings and the shaft in the journal bearings are moving. This implies

$$
\begin{aligned}
U_1 &= U \\
U_2 &= 0
\end{aligned}
\tag{39}
$$

and then eqn. (36) becomes

$$\frac{\partial}{\partial x}\left\{\rho\psi_1(h)\frac{\partial P}{\partial x}\right\}+\frac{\partial}{\partial y}\left\{\rho\psi_1(h)\frac{\partial P}{\partial y}\right\}+\frac{\partial}{\partial x}\left\{\rho\psi_2(h)\frac{\partial P}{\partial y}\right\}-\frac{\partial}{\partial y}\left\{\rho\psi_2(h)\frac{\partial P}{\partial x}\right\}$$
$$=-\frac{\partial}{\partial x}\left[\frac{\rho U}{2}\{h+M\rho\psi_2(h)\}\right]-\frac{\partial}{\partial y}\left\{\frac{M\rho^2U}{2}\psi_1(h)\right\}-\rho(w_2-w_1) \quad (40)$$

which is the same for both thrust and journal bearings with U as the sliding velocity of either runner or journal. For pure sliding,

$$w_2=w_1 \quad (41)$$

and eqn. (40) reduces to

$$\frac{\partial}{\partial x}\left\{\rho\psi_1(h)\frac{\partial P}{\partial x}\right\}+\frac{\partial}{\partial y}\left\{\rho\psi_1(h)\frac{\partial P}{\partial y}\right\}+\frac{\partial}{\partial x}\left\{\rho\psi_2(h)\frac{\partial P}{\partial y}\right\}-\frac{\partial}{\partial y}\left\{\rho\psi_2(h)\frac{\partial P}{\partial x}\right\}$$
$$=-\frac{\partial}{\partial x}\left[\frac{\rho U}{2}\{h+M\rho\psi_2(h)\}\right]-\frac{\partial}{\partial y}\left\{\frac{M\rho^2U}{2}\psi_1(h)\right\} \quad (42)$$

It is reasonable to assume that the lubricant is incompressible, the density is constant and the viscosity and sliding velocity do not change: then ρ, μ and U should be treated as constants in eqn. (42).

4. New fundamental solutions of the extended Reynolds equation

Fundamental solutions of the extended Reynolds equation (eqn. (36)) in one dimension are possible. Such solutions are not possible with the classical Reynolds theory. Two situations, *i.e.* when the film thickness is a linear or an exponential function of the coordinate along the bearing length, are considered.

4.1. Plane inclined slider

The plane inclined pad is the most common form of lubricated slider bearing. As an example of the application of the extended Reynolds equation the pressure and load capacity for such a configuration are determined. Let

$$U=+U$$
$$h=h(y) \quad (43)$$
$$P=P(y)$$

Equation (42) then gives

$$\frac{\mathrm{d}}{\mathrm{d}y}\left\{\rho\psi_1(h)\frac{\mathrm{d}P}{\mathrm{d}y}\right\}=-\frac{\mathrm{d}}{\mathrm{d}y}\left\{\frac{M\rho^2U}{2}\psi_1(h)\right\} \quad (44)$$

Integrating eqn. (44) with respect to y and using the condition

$$\frac{\mathrm{d}P}{\mathrm{d}y} = 0 \qquad \text{at } h = h^* \tag{45}$$

gives

$$\frac{\mathrm{d}P}{\mathrm{d}y} = \frac{M\rho U}{2}\left\{\frac{\psi_1(h^*)}{\psi_1(h)} - 1\right\} \tag{46}$$

The film thickness can be expressed at any point as

$$h = h_o\left(1 + \frac{ny}{L}\right) \tag{47}$$

where

$$n = \frac{h_i}{h_o} - 1 \tag{48}$$

Using eqn. (47), eqn. (46) can be written as

$$\frac{\mathrm{d}P}{\mathrm{d}h} = \frac{L}{nh_o}\frac{M\rho U}{2}\left\{\frac{\psi_1(h^*)}{\psi_1(h)} - 1\right\} \tag{49}$$

Integrating eqn. (49) with respect to h gives

$$P = \frac{L}{nh_o}\frac{M\rho U}{2}\left\{\psi_1(h^*)\int\frac{\mathrm{d}h}{\psi_1(h)} - h\right\} + A \tag{50}$$

where A is a constant of integration. h^* is the value of h where $\mathrm{d}P/\mathrm{d}y = 0$, where the pressure is at a maximum. There are two unknowns ($\psi_1(h^*)$ and A) in eqn. (50); these must be found by the introduction of two boundary conditions:

$$\begin{aligned} &P = 0 \text{ at } y = 0 \text{ or } P = 0 \text{ at } h = h_o \\ &P = 0 \text{ at } y = L \text{ or } P = 0 \text{ at } h = h_o(1+n) \end{aligned} \tag{51}$$

Pressures are expressed as gauge pressures, *i.e.* $P = 0$ represents ambient pressure. Substitution of these two conditions gives

$$\psi_1(h^*) = \frac{nh_o}{\phi_1\{h_o(1+n)\} - \phi_1(h_o)} \tag{52}$$

and

$$A = \frac{LM\rho U[\phi_1\{h_o(1+n)\} - (1+n)\phi_1(h_o)]}{2n[\phi_1\{h_o(1+n)\} - \phi_1(h_o)]} \tag{53}$$

where

$$\phi_1(h) = \int\frac{\mathrm{d}h}{\psi_1(h)} = -M\rho\lambda\int\frac{\cosh(\lambda h) + \cos(\lambda h)}{\sinh(\lambda h) - \sin(\lambda h)}\,\mathrm{d}h \tag{54}$$

under the present circumstances.

Substituting the above values of $\psi_1(h^*)$ and A in eqn. (50) gives

$$P = -\frac{M\rho U}{2}\left(y - \frac{L[\phi_1\{h_o(1+ny/L)\} - \phi_1(h_o)]}{\phi_1\{h_o(1+n)\} - \phi_1(h_o)}\right) \tag{55}$$

A further integration of the pressure gives the normal load capacity per unit length of the bearing system:

$$W = \int_0^L p \, dy = \frac{L}{nh_o} \int_{h_o}^{h_o(1+n)} p \, dh \tag{56}$$

Using eqn. (55), the load capacity of the plane inclined slider is given by

$$W = M\rho UL^2 \left(-\frac{1}{4} + \frac{-nh_o\phi_1(h_o) + \phi_2\{h_o(1+n)\} - \phi_2(h_o)}{2nh_o[\phi_1\{h_o(1+n)\} - \phi_1(h_o)]} \right) \tag{57}$$

where

$$\phi_2(h) = \int \phi_1(h) \, dh \tag{58}$$

If the extent of rotation is small, so that it is reasonable to ignore second and higher powers of M compared with the first power of M, the expressions for pressure and load capacity given by eqn. (55) and eqns. (57) and (58) respectively can be approximated by

$$P = -\frac{M\rho U}{2} \left[y + \frac{L(n+1)^2}{n(n+2)} \left\{ \frac{1}{(1+ny/L)^2} - 1 \right\} \right] \tag{59}$$

and

$$W = \frac{M\rho UL^2}{4} \frac{n}{n+2} \tag{60}$$

4.2. *Exponentially inclined slider*

In this case

$$U = -U$$

$$h = h(y) \tag{61}$$

$$P = P(y)$$

and eqn. (42) then gives

$$\frac{d}{dy}\left\{ \rho\psi_1(h) \frac{dP}{dy} \right\} = \frac{d}{dy}\left\{ \frac{M\rho^2 U}{2} \psi_1(h) \right\} \tag{62}$$

Integrating eqn. (62) with respect to y and using the condition

$$\frac{dP}{dy} = 0 \qquad \text{at } h = h^* \tag{63}$$

gives

$$\frac{dP}{dy} = \frac{M\rho U}{2} \left\{ 1 - \frac{\psi_1(h^*)}{\psi_1(h)} \right\} \tag{64}$$

The film thickness can be expressed at any point as

$$h = h_o \exp(-\alpha y) \tag{65}$$

Using eqn. (65), eqn. (64) can be written as

$$\frac{dP}{dh} = -\frac{1}{\alpha h}\frac{M\rho U}{2}\left\{1 - \frac{\psi_1(h^*)}{\psi_1(h)}\right\} \tag{66}$$

Integrating eqn. (66) with respect to h gives the following:

$$P = -\frac{M\rho U}{2\alpha}\left\{\log_e h - \psi_1(h^*)\int\frac{dh}{h\psi_1(h)}\right\} + A \tag{67}$$

where A is a constant of integration. Using the boundary conditions

$$\begin{aligned} &P = 0 \ \text{ at } \ y = 0 \ \text{ or } \ P = 0 \ \text{ at } \ h = h_o \\ &P = 0 \ \text{ at } \ y = -L \ \text{ or } \ P = 0 \ \text{ at } h = h_o \exp(\alpha L) \end{aligned} \tag{68}$$

we obtain

$$\psi_1(h^*) = -\frac{\alpha L}{\phi_3(h_o) - \phi_3\{h_o \exp(\alpha L)\}} \tag{69}$$

and

$$A = \frac{M\rho U}{2\alpha}\frac{[\phi_3(h_o) - \phi_3\{h_o \exp(\alpha L)\}]\log_e h_o + \alpha L\phi_3(h_o)}{\phi_3(h_o) - \phi_3\{h_o \exp(\alpha U)\}} \tag{70}$$

where

$$\phi_3(h) = \int\frac{dh}{h\psi_1(h)} \tag{71}$$

Substituting the values of $\psi(h^*)$ and A in eqn. (67) gives

$$P = \frac{M\rho U}{2}\left(y + \frac{L[\phi_3\{h_o \exp(-\alpha y)\} - \phi_3(h_o)]}{\phi_3\{h_o \exp(\alpha L)\} - \phi_3(h_o)}\right) \tag{72}$$

A further integration of the pressure gives the normal load capacity per unit length of the bearing system:

$$W = \int_{-L}^{0} P\,dy = -\frac{1}{\alpha}\int_{h_o\exp(\alpha L)}^{h_o}\frac{P}{h}\,dh \tag{73}$$

Using eqn. (72), we obtain the load capacity of the exponentially inclined slider:

$$W = \frac{M\rho UL^2}{4}\left(1 + \frac{2}{\alpha L}\left[\log_e h_o - \frac{\alpha L\phi_3(h_o) + \phi_4(h_o) - \phi_4\{h_o \exp(\alpha L)\}}{\phi_3\{h_o \exp(\alpha L)\} - \phi_3(h_o)}\right]\right) \tag{74}$$

where

$$\phi_4(h) = \int\frac{\phi_3(h)\,dh}{h} \tag{75}$$

When the extent of rotation is small, the expressions for pressure and load capacity given by eqn. (72) and eqns. (74) and (75) respectively may be approximated by

$$P = \frac{M\rho U}{2}\left\{y + L\,\frac{\exp(3\alpha y) - 1}{\exp(-3\alpha L) - 1}\right\} \tag{76}$$

and

$$W = \frac{M\rho UL}{12\alpha}\,\frac{6\alpha L + (2 + 3\alpha L)\{\exp(-3\alpha L) - 1\}}{1 - \exp(-3\alpha L)} \tag{77}$$

5. Conclusions

An important qualitative result is that while the load capacity increases with increasing coefficient of fluid viscosity for a plane inclined slider or an exponentially inclined slider by the classical theory, it is independent of the fluid viscosity in the present context for both of the bearing systems [5, 6] when the extent of rotation is small. This may also indicate that low viscosity fluids could be successfully utilized as lubricants. This aspect will be further investigated.

Nomenclature

h	film thickness
$\bar{h}$	dimensionless film thickness
h^*	film thickness at the point of maximum pressure
h_c	characteristic film thickness
h_i	inlet film thickness
h_o	outlet film thickness for the slider
L	dimensionless length of the bearing
M	rotation number
n	$h_i/h_o - 1$
p	pressure
P	modified pressure
$\bar{P}$	dimensionless pressure
q_c	characteristic velocity
r_k	kth component of the radius vector r
t	time
u	fluid velocity in the x direction
$\bar{u}$	dimensionless fluid velocity in the x direction
U	dimensionless surface velocity in the x direction
v	fluid velocity in the y direction
$\bar{v}$	dimensionless fluid velocity in the y direction
w	fluid velocity in the z direction
$\bar{w}$	dimensionless fluid velocity in the z direction
W	load capacity
x	coordinate along the span of the bearing system
X_i	external force vector component

y coordinate along the length of the bearing system
z coordinate across the fluid film
α exponential coefficient for the film thickness
ϵ_{ijk} alternating tensor component
μ viscosity of the fluid
$\bar{\mu}$ dimensionless viscosity of the fluid
μ_c characteristic viscosity of the fluid
ρ density of the fluid
$\bar{\rho}$ dimensionless density of the fluid
ρ_c characteristic density of the fluid
Ω angular velocity of rotation
Ω_k kth component of the angular velocity of rotation

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