Perturbative Summations

Davison E. Soper
University of Oregon

Kanpur, November 2002

Overview

- Four kinds of partial summations of perturbation theory.
    - Scale change (renormalization group)
    - Transverse momentum distributions
    - Small x behavior (BFKL)
    - Large x behavior (threshold summation)
- Does threshold sum affect the LHC jet cross section.
- Transverse momentum summation
    - **b** space and the ABC formula
    - Estimating predictive power
    - What to do with large **b**
    - Main steps in the derivation

Beyond fixed order

Simply calculating Feynman diagrams at a fixed order of perturbation theory is not enough.

Use the factorization property of QCD

$$\frac{d\sigma}{dE_T dy} \approx$$
$$\sum_{a,b} \int_{x_A}^{1} d\xi_A \int_{x_B}^{1} d\xi_B \; f_{a/A}(\xi_A,\mu) \; f_{b/B}(\xi_B,\mu) \; \frac{d\hat{\sigma}^{ab}(\mu)}{dE_T dy}.$$

Sum an infinite number of important contributions

- $\sum C_n \left[\alpha_s \, \log(\mu^2/\mu^2_{\rm data})\right]^n + \cdots$
- $\sum C_n \left[\alpha_s \, \log^2(k_T^2/Q^2)\right]^n + \cdots$
- $\sum C_n \left[\alpha_s \, \log(1/x)\right]^n + \cdots$
- $\sum C_n \left[\alpha_s \, \log^2(1-x)\right]^n + \cdots$

Evolution

$$\frac{d\sigma}{dE_T dy} \approx$$

$$\sum_{a,b} \int_{x_A}^{1} d\xi_A \int_{x_B}^{1} d\xi_B \; f_{a/A}(\xi_A, \mu) \; f_{b/B}(\xi_B, \mu) \; \frac{d\hat{\sigma}^{ab}(\mu)}{dE_T dy}.$$

Parton scale set to $\mu = E_T$, say, but parton densities determined from data at much lower scales $\mu_{\rm data}$.

The parton evolution equation sums a series

$$\sum C_n \left[\alpha_s \log(\mu^2/\mu_{\rm data}^2)\right]^n + \cdots$$

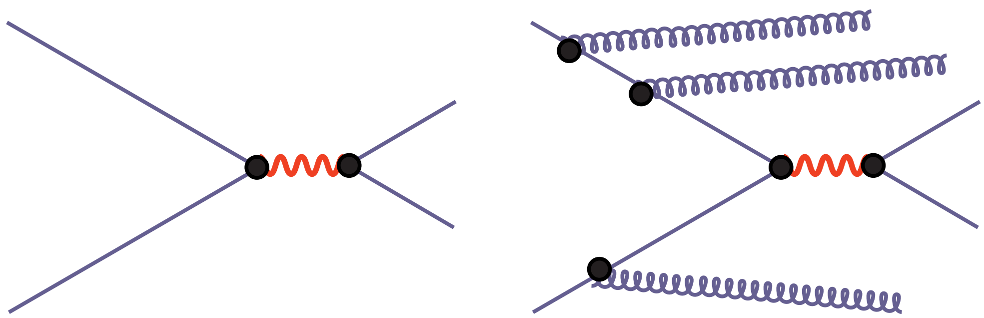
- This is normally done with the next-to-leading order evolution kernel.
- Fractional errors: $\alpha_s(\mu_{\rm data})^3 \log(\mu^2/\mu_{\rm data}^2)$.
- We have partial information about the kernel at N^3LO.
- Inaccuracy in evolution will probably not be a limiting factor in predictions.

Transverse momentum distributions

If we produce a heavy particle (say a W boson) in Run II we may want to know the distribution of its transverse momentum k_T.

• The hard scale of the process is the W-boson mass M.

• At leading order, $d\sigma/d^2\vec{k}_T \propto \delta(\vec{k}_T)$.



• At higher orders, bremsstrahlung gives nonzero $\vec{k}_T$.

• For $k_T^2 \ll M^2$ there are large logarithms and we need to sum the most important contributions.

$$\int_0^{P_T^2} dk_T^2 \, \frac{d\sigma}{dk_T^2} \sim \sigma_0 \, \sum C_n \left[\alpha_s \, \log^2(P_T^2/M^2)\right]^n + \cdots$$

• Methods for treating this have long been known.

• The key idea, "Fourier transform," was invented by Parisi and Petronzio.

• There has been some debate about what methods are most convenient.

Small x summation

- Some of Run II physics will involve "small x."
- Define

$$x_A = \frac{\sum k_n^+}{p_A^+} \qquad x_B = \frac{\sum k_n^-}{p_B^-}$$

where the sum runs over particles in measured jets, *etc.*

- x_A and x_B can be among the measured variables.
- If not, other variables typically play the same role.
- In many processes of interest, $x_A\sqrt{s/2} \gg 1$ GeV but $x_A \ll 1$.

- Consider $x_A \ll 1$ and, for simplicity, suppress all notation concerning x_B and hadron B.

$$\sigma = \int_{x_A}^1 d\xi_A\ f_{a/A}(\xi_A)\ \hat{\sigma}(x_A/\xi_A)$$

- Potentially large logs occur in perturbation theory:

$$\Delta\hat{\sigma} \propto \sigma_0\ \alpha_s^n\ \log^{n-1}(x_A/\xi_A)$$

- One knows something about summing such logs (BFKL).
- The small x logs are interesting.

Is small x summation typically needed for the LHC?

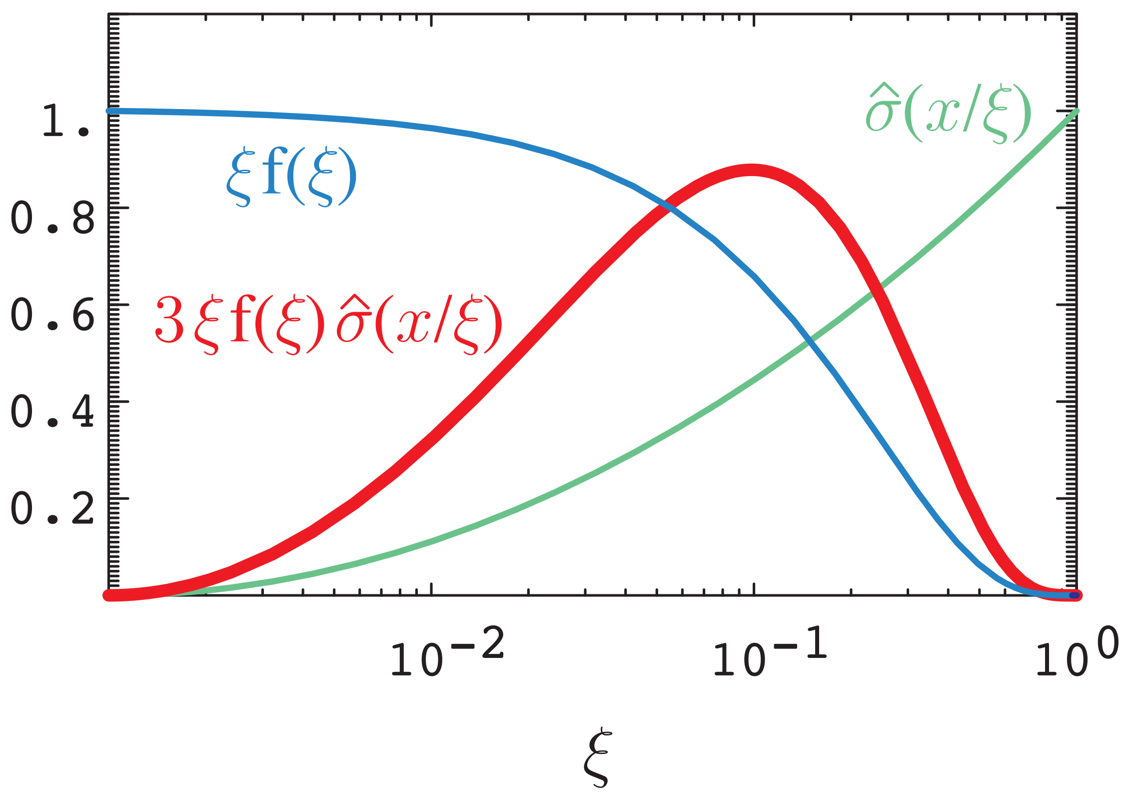
That is, if one doesn't worry about small x summation in a process not designed to look for "small x" effects, will one make a big mistake?

$$\Delta\sigma = \sigma_0\ \alpha_s^n \int_x^1 \frac{d\xi}{\xi}\ \xi f(\xi)\quad \log^{n-1}(x/\xi)$$

Case 1: $\xi\, f(\xi) \propto 1$ for small ξ. Then $\Delta\sigma \propto \log^n(x)$.

- Say $x = 10^{-3}$, $n = 3$, $\xi\, f(\xi) = (1-\xi)^4$. Then

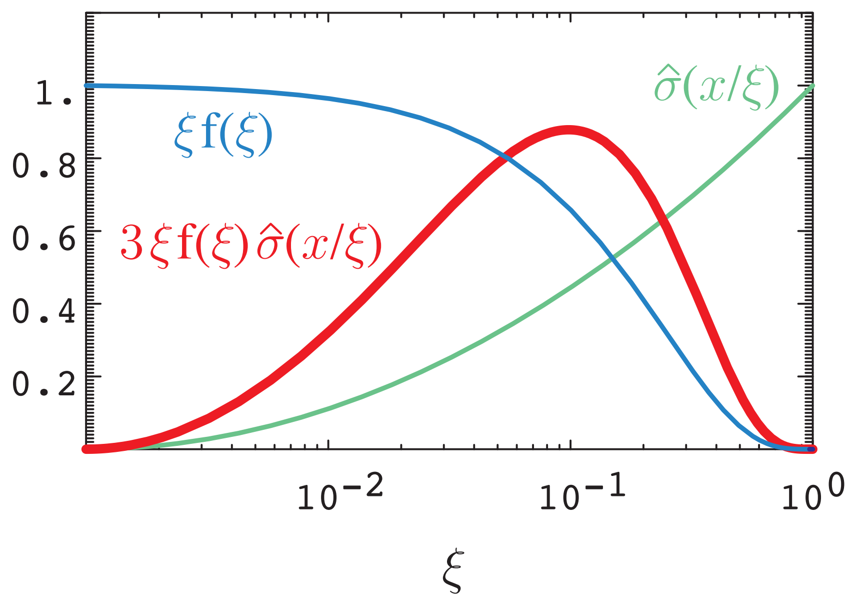
$$\int_x^1 \frac{d\xi}{\xi}\ \frac{(1-\xi)^4}{(1-x)^4}\ \log^2(x/\xi) \approx 44.$$

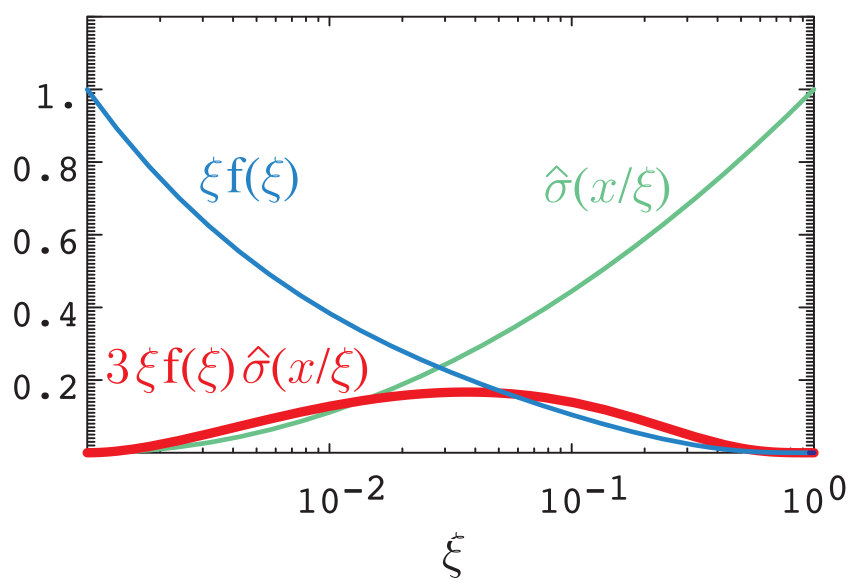


Case 2: $\xi\, f(\xi) \propto \xi^{-A}$ for small ξ with $A > 0$. Then $\Delta\sigma$ is finite for $x \to 0$.

- Say $x = 10^{-3}$, $n = 3$, $\xi\, f(\xi) = \xi^{-0.4}(1-\xi)^4$. Then

$$\int_x^1 \frac{d\xi}{\xi}\ \frac{\xi^{-0.4}(1-\xi)^4}{x^{-0.4}(1-x)^4}\ \log^2(x/\xi) \approx 9.7.$$





Conclusion: Small x summation is less needed than you might think.

$x \to 1$ summation

Instead of the small x problem,

$$\Delta\sigma = \sigma_0\ \alpha_s^n \int_x^1 \frac{d\xi}{\xi}\ \xi f(\xi) \quad \log^{n-1}(x/\xi)$$

we can have a large x problem,

$$\Delta\sigma = \sigma_0\ \alpha_s^n \int_x^1 \frac{d\xi}{\xi}\ \xi f(\xi)\ \left[\frac{\log^{2n-1}(1-x/\xi)}{1-x/\xi}\right]_+ .$$

• Using the + prescription, this is

$$\Delta\sigma = \sigma_0\ \alpha_s^n \int_x^\infty d\xi\ \left\{f(\xi) - \frac{x^2}{\xi^2} f(x)\right\} \frac{\log^{2n-1}(1-x/\xi)}{1-x/\xi}.$$

• For $x \to 1$ the range of the integral over ξ in the first term is severely limited because $f(\xi) = 0$ for $\xi > 1$, while there is no such restriction in the second term.
• This gives $\Delta\sigma \propto \log^{2n}(1-x)$.

Is $x \to 1$ summation typically needed for LHC?

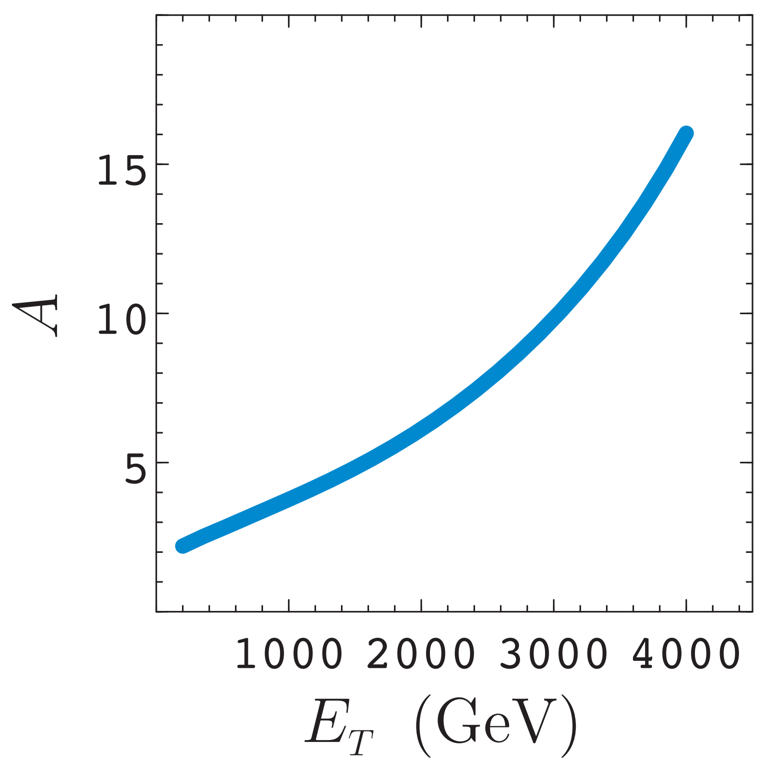
- $\Delta\sigma \propto \log^n(1-x)$ is not important for the LHC.
- What is important is that if $\xi\, f(\xi) \propto \xi^{-A}$ near $\xi = x$, and if $A \gg 1$, then

$$\Delta\sigma \propto \log^n(A).$$

For a measure of A for the jet cross section, define

$$A = \frac{d \log\left(E_T^3\, d\sigma/dE_T\right)}{d\log E_T}$$

Here it is



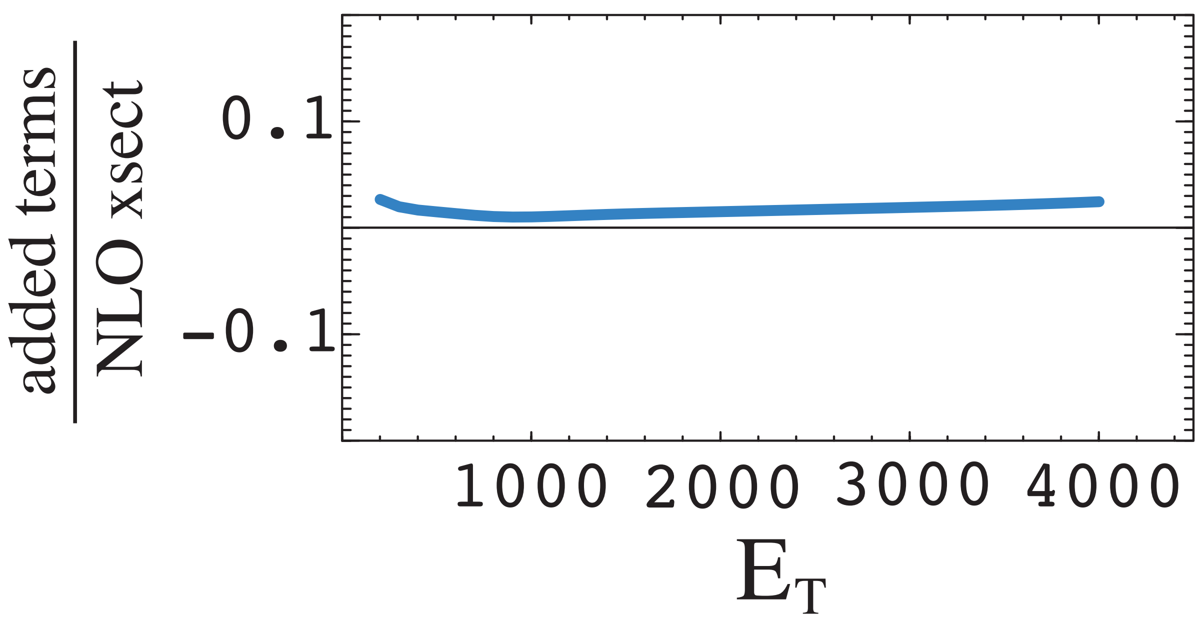
Conclusion: $x \to 1$ summation is more needed than you might think.

Result of an actual calculation

Owens and Kidonakis have calculated the contributions proportional to

$$\sigma_0\ \alpha_s^{2} \left[\frac{\log^{3}(1-x/\xi)}{1-x/\xi}\right]_+ \quad \text{and} \quad \sigma_0\ \alpha_s^{2} \left[\frac{\log^{2}(1-x/\xi)}{1-x/\xi}\right]_+$$

in the summation of threshold logs. For LHC, they find



The effect could have been large, but it is small.

These results are courtesy of J. F. Owens. The analysis for jets at Fermilab is published in N. Kidonakis and J. F. Owens, Phys. Rev. D **63**, 054019 (2001) and uses N. Kidonakis, G. Oderda and G. Sterman, Nucl. Phys. B **525**, 299 (1998); E. Laenen, G. Oderda and G. Sterman, Phys. Lett. B **438**, 173 (1998).

Comments the transverse momentum summation

I consider $p + \bar{p} \to Z + X$. We look at the transverse momentum distribution of the Z when $p_T^2 \ll M_Z^2$:

$$F(\mathbf{p}_T) = \left.\frac{d\sigma}{d^2\mathbf{p}_T\, dy}\right|_{y=0}$$

Then

$$F(\mathbf{p}_T) = \sum_{a,b} \int dx_A\, f_{a/A}(x_A) \int dx_B\, f_{b/B}(x_B)\ \hat{F}_{ab}(\mathbf{p}_T, x_A, x_B)$$

We calculate $\hat{F}$ in perturbation theory. Let

$$L = \log(M_Z^2/p_T^2) \qquad \alpha_s = \alpha_s(p_T)$$

Then

$$\begin{aligned}\hat{F} = \alpha_s & \left[\frac{1}{p_T^2}\,(C_{11}L + C_{10}) + Y_1\right] \\ & + \alpha_s^2 \left[\frac{1}{p_T^2}\left(C_{23}L^3 + C_{22}L^2 + C_{21}L + C_{20}\right) + Y_2\right] \\ & + \alpha_s^3 \left[\frac{1}{p_T^2}\left(C_{35}L^5 + C_{32}L^4 + \cdots + C_{30}\right) + Y_3\right] \\ & + \cdots\end{aligned}$$

Here $Y_J = Y_J(p_T^2, x_A, x_B)$ with $p_T^2 Y_J \to 0$ as $p_T^2 \to 0$.

- The first row is not useful unless $\alpha_s L^2 \ll 1$.
- The sum of the first column is not useful unless $\alpha_s L^2 \lesssim 1$.

The ABC's of the $\mathbf{p}_T$ distribution

Following Parisi & Petronzio, we write the singular parts of F as a Fourier transform. Then, after a lot of analysis we arrive at

$$
\begin{aligned}
&F(\mathbf{p}_T) = \\
&\sum_{a,b,j} \int dx_A\, f_{a/A}(x_A) \int dx_B\, f_{b/B}(x_B) \int d^2\mathbf{b}\ \exp(i\,\mathbf{p}_T \cdot \mathbf{b}) \\
&\times \exp\left(- \int_{1/b^2}^{M_Z^2} \frac{d\mu^2}{\mu^2} \left[A(\alpha_s(\mu)) \log\left(\frac{M_Z^2}{\mu^2}\right) + B(\alpha_s(\mu)) \right] \right) \\
&\times \tilde{e}_j^2\ C_{ja}\big(x_A, \alpha_s(1/b)\big) C_{\bar{j}b}\big(x_B, \alpha_s(1/b)\big) \\
&+ Y\big(p_t, \alpha_s(p_T)\big)
\end{aligned}
$$

• The Y term is evaluated perturbatively evaluating F and subtracting the parts with $1/p_T^2$ singularities.
• The A, B, C functions have perturbative expansions.
• For large b^2, we will need to put in something non-perturbative for A, B, C. I will return to this question.

See J. C. Collins and D. E. Soper, Nucl Phys. B193 (1981) 381 and J. C. Collins, D. E. Soper, and G. Sterman, Nucl. Phys. B250 (1985) 199, plus subsequent works by G. Sterman and collaborators.

Counting logs in the Sudakov exponent

- Just going to $\mathbf{b}$ space is a big help. Consider the exponent $S(b^2)$.

$$F(\mathbf{p}_T) - Y\big(p_t, \alpha_s(p_T)\big) =$$

$$\sum_{a,b,j} \int dx_A\, f_{a/A}(x_A) \int dx_B\, f_{b/B}(x_B) \int d^2\mathbf{b}\ \exp(i\,\mathbf{p}_T \cdot \mathbf{b})$$

$$\times \exp(-S(b)) \times \tilde{e}_j^2\ C_{ja}\big(x_A, \alpha_s(1/b)\big) C_{\bar{j}b}\big(x_B, \alpha_s(1/b)\big).$$

Let

$$L = \log(M_Z^2\, b^2) \qquad\qquad \alpha_s = \alpha_s(1/b)$$

Then

$$\begin{aligned} S(b) = \alpha_s &\left(C_{12} L^2 + C_{11} L + C_{10}\right) \\ &+ \alpha_s^2 \left(C_{23} L^3 + C_{22} L^2 + C_{21} L + C_{20}\right) \\ &+ \alpha_s^3 \left(C_{34} L^4 + C_{33} L^3 + \cdots + C_{30}\right) \\ &+ \cdots \end{aligned}$$

- In $S(b)$ there is only one log per loop, plus 1.
- *I.e.* half of the logs are gone.

$$\begin{aligned} S(b) = {} & L\, f(\alpha_s L) \\ & + g_0(\alpha_s L) + \alpha_s\, g_1(\alpha_s L) + \alpha_s^2\, g_2(\alpha_s L) + \cdots . \end{aligned}$$

- f gives the leading log series.
- g_0 gives the next-to-leading logs.
- Knowing a few g_j is useful if $\alpha_s L \lesssim 1$.

It's better than that

In the ABC formula,

$$S(b) = \int_{1/b^2}^{M_Z^2} \frac{d\mu^2}{\mu^2} \left[A(\alpha_s(\mu)) \log\left(\frac{M_Z^2}{\mu^2}\right) + B(\alpha_s(\mu)) \right]$$

Again let

$$L = \log(M_Z^2\, b^2) \qquad\qquad \alpha_s = \alpha_s(1/b)$$

The functions A and B have perturbative expansions starting at order α_s.

• Suppose that we know enough terms in the β function to evaluate $\alpha_s(\mu)$ as accurately as we want.

• Suppose that L is big but $\alpha_s^N L \lesssim 1$.

• If we know $2N$ terms in A and N terms in B, then the remainder is proportional to

$$C\alpha_s^{2N+1} L^2 + C' \alpha_s^{N+1} L = \alpha_s \left[C(\alpha_s^N L)^2 + C'(\alpha_s^N L) \right] \lesssim \alpha_s$$

• That is, there is real content in the formula for $S(b^2)$ beyond just knowing that there is only one log per loop, plus 1.

• This extra information gives us useful error estimates.

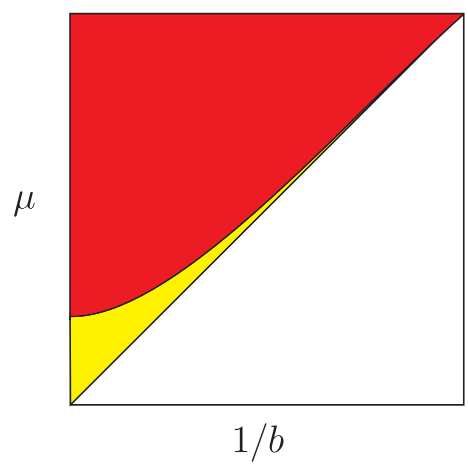
What about large b?

This is a matter of taste, but here is one sensible procedure.

$$
\begin{aligned}
&F(\mathbf{p}_T) = \\
&\sum_{a,b,j} \int dx_A\, f_{a/A}(x_A) \int dx_B\, f_{b/B}(x_B) \int d^2\mathbf{b}\;\exp(i\,\mathbf{p}_T\cdot\mathbf{b}) \\
&\times \exp\left(-\int_{1/b_*^2}^{M_Z^2} \frac{d\mu^2}{\mu^2} \left[A(\alpha_s(\mu)) \log\left(\frac{M_Z^2}{\mu^2}\right) + B(\alpha_s(\mu)) \right] \right) \\
&\times \tilde{e}_j^2\; C_{ja}\big(x_A, \alpha_s(1/b_*)\big) C_{\bar{j}b}\big(x_B, \alpha_s(1/b_*)\big)\big) \\
&\times \exp\left(-g_1(b) \log\left(\frac{M_Z^2}{Q_0^2}\right) - g_{j/A}(x_A, b) - g_{\bar{j}/B}(x_B, b) \right) \\
&+ Y\big(p_t, \alpha_s(p_T)\big)
\end{aligned}
$$

where

$$
b_* = \frac{b}{\sqrt{1 + b^2/b_{\rm max}^2}}.
$$

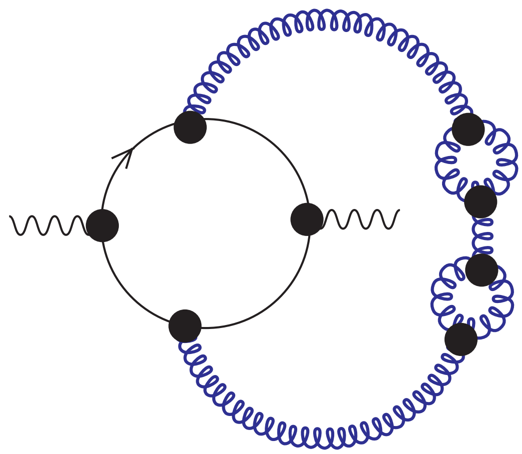


- Then $\alpha_s(\mu)$ and $\alpha_s(b_*)$ are perturbative.
- The functions g are nonperturbative.
- The structure of the non-perturbative part follows from the original structure of the formula.
- The g functions should vanish for $b \to 0$.
- Fit them to experiment.

More about large b

• Should we be distressed that if $g(b) \propto b^2$ then there is some influence of "non-perturbative" physics even for $b < b_{\rm max}$?

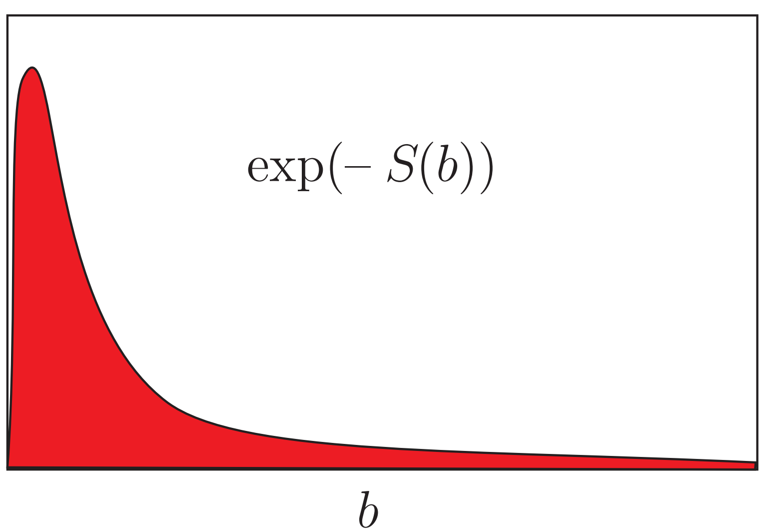
• No. There are always power suppressed effects that one usually ignores, but sometimes estimates with "renormalon" techniques.



• Wouldn't it be nice to get rid of the nonperturbative uncertainty by working directly in k_T space?

• You would lose the special structure that exists in b space.

• $S(b)$ heavily damps large b is "M_Z" is large.



How the theory works

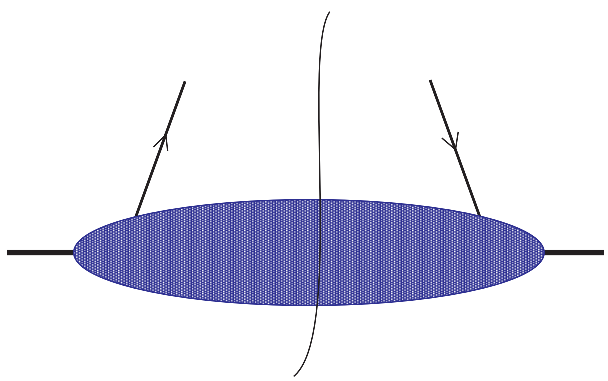
• The real derivation is for e^+e^-.
• The axial gauge used is not so well defined. It might be better to use Feynman gauge with

$$\psi(x)\ \exp(\int_{-\infty}^{0} d\lambda\, n\cdot A(x^\mu+\lambda n^\mu))$$

in the operator definition of $\mathcal{P}$ below.

Study the *unintegrated* parton distribution functions,

$$\mathcal{P}(x, \mathbf{k}_T, \zeta)$$



• x is the momentum fraction of the observed parton.
• $\mathbf{k}_T$ is its transverse momentum.
• We work in the gauge $A\cdot n = 0$, where n will point in the 3-direction in the frame in which the Z boson has $P_z = 0$, with $n^2 = -1$.

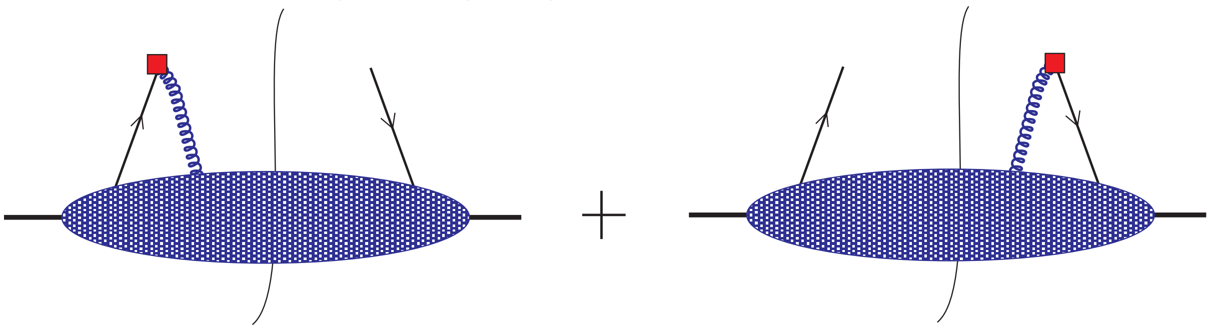
$$\zeta = (2P\cdot n)^2$$

• The use of something that is not boost invariant in the definition is crucial. Look for $\log^2(k_T^2/\zeta)$.

Differentiate with respect to the gauge vector

$$\frac{\partial}{\partial \log \zeta} \mathcal{P}(x, \mathbf{k}_T, \zeta)$$

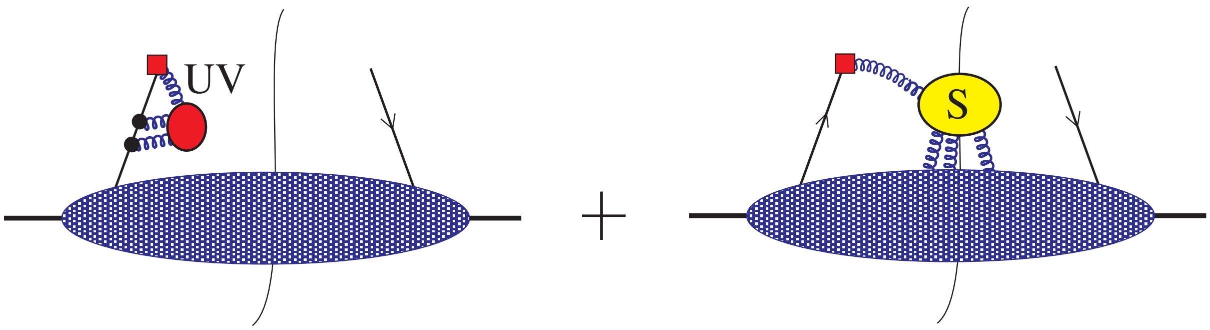
Evaluate this using the gauge invariance of the theory. Get



with a special rule for the square vertex.

Examine leading regions

$$\frac{\partial}{\partial \log \zeta} \mathcal{P}(x, \mathbf{k}_T, \zeta)$$



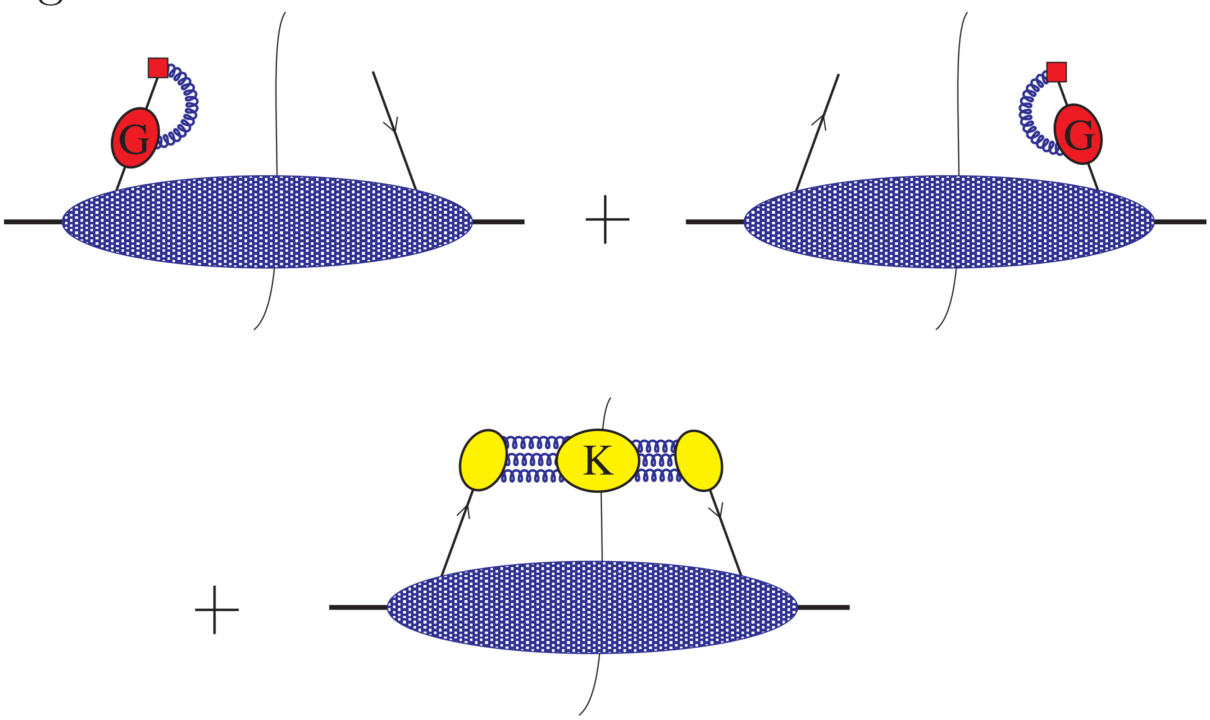
- The gluon that attaches to the square vertex can be hard (all of its momentum components big).
- The gluon that attaches to the square vertex can be soft (all of its momentum components small).
- The blue subgraph contains lots of collinear partons.
- The structure of the square vertex suppresses a collinear gluon attachment.

Use soft gluon approximations

For $k^+ \gg k^-, k_T^j$ and all components of q^μ small,

$$(\not{k}+\not{q})\not{\epsilon}\not{k} \approx (\not{k}+\not{q})\gamma^+\not{k}\epsilon^- = (\not{k}+\not{q})\gamma^+ q^- \not{k}\frac{\epsilon^-}{q^-} \approx (\not{k}+\not{q})\not{q}\not{k}\frac{\epsilon^-}{q^-}.$$

This gives



$$\frac{\partial}{\partial \log \zeta} \mathcal{P}(x, \mathbf{k}_T, \zeta) \approx \int d\mathbf{l}_T \left[\delta(\mathbf{l}_T) G(\zeta) + K(\mathbf{l}_T) \right] \times \mathcal{P}(x, \mathbf{k}_T + \mathbf{l}_T, \zeta).$$

- Fourier transforming to $\mathbf{b}$ simplifies this.
- We get a differential equation.
- The solution is an exponential.
- Use renormalization group to get structure of $G + K$.

- This leads to the ABC formula.