QCD2002

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Free Fermions on the transverse lattice

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(Continuation of the talk given by Prof. A. Harindranath)

HAMILTONIAN WITH FORWARD - BACKWARD DERIVATIVES

In the free limit the fermionic Hamiltonian becomes

$$P^-_{fb} = P^-_0 + P^-_{hf}$$

where

$$\begin{aligned} P^-_0 &= \int dx^- a^2 \sum_{\mathbf{x}} [m^2 \eta^\dagger(\mathbf{x}) \frac{1}{i\partial^+} \eta(\mathbf{x}) \\ &\quad - \frac{1}{a^2} \sum_r \eta^\dagger(\mathbf{x}) \sum_r \frac{1}{i\partial^+} [\eta(\mathbf{x} + a\hat{\mathbf{r}}) - 2\eta(\mathbf{x}) + \eta(\mathbf{x} - a\hat{\mathbf{r}})] \\ p^-_{hf} &= \int dx^- a^2 \sum_{\mathbf{x}} \frac{1}{a^2} \eta^\dagger(\mathbf{x}) \sum_r (am\hat{\sigma}_r) \\ &\quad \frac{1}{i\partial^+} [\eta(\mathbf{x} + a\hat{\mathbf{r}}) - 2\eta(\mathbf{x}) + \eta(\mathbf{x} - a\hat{\mathbf{r}})]. \end{aligned}$$

$\hat{\sigma}_1 = \sigma_2$ and $\hat{\sigma}_2 = -\sigma_1$. Sign of the linear mass term changes if we switch the forward and backward derivatives. Considering the Fourier transform in transverse space we get the helicity nonflip term,

$$P^-_0 = \int dx^- \int \frac{d^2k}{(2\pi)^2} \phi^\dagger_{\mathbf{k}}(x^-) \frac{1}{i\partial^+} \phi_{\mathbf{k}}(x^-) \Big[m^2 + \sum_r k_r^2 \left(\frac{Sin\ k_r a/2}{k_r a/2} \right)^2 \Big]$$

=> No doublers.

Fixed Boundary Condition

For each transverse direction, we choose $2n+1$ lattice points ranging from $-n$ to $+n$ where fermions are allowed to hop. To implement fixed boundary condition we add two more points at the two ends and demand that the fermion remains fixed at these lattice points.

=> $k_p = \frac{\pi}{(2n+2)} p$ with $p = 1, 2, 3, ..., 2n+1$.

Periodic Boundary Condition

Again, for each transverse direction, we choose $2n + 1$ lattice points. We identify the $(2n + 2)^{th}$ lattice point with the first lattice point. $=> k_p = \pm\frac{2\pi}{(2n+1)}p$, $p = 0, 1, 2, ..., n$.

At finite volume, with fixed boundary condition, we do get a nodeless wave function. All the excited states have nodes characteristic of sine waves. $<< FIGURE1 >>$

With periodic boundary condition, for any n, we get a zero transverse momentum fermion with a flat wave function.

Now, considering the helicity flip term, we have the **full** eigenvalue equation

$$M^2 = m^2 + \frac{4}{a^2}\sum_r Sin^2\frac{k_r a}{2} \pm \frac{4m}{a}\sqrt{\sum_r Sin^4\frac{k_r a}{2}}.$$

$=>$ No doubling for physical particles $(m < \frac{1}{a})$.

With fixed boundary condition the lowest eigenstate has non vanishing transverse momentum in finite volume. In the absence of helicity flip term positive and negative helicity fermions are degenerate. The helicity flip term lifts the degeneracy. $<< FIGURE2 >>$

For the periodic boundary condition, the lowest state has exactly zero transverse momentum and we get two degenerate fermions for all n.

HAMILTONIAN WITH SYMMETRIC DERIVATIVE

The free fermionic Hamiltonian with symmetric lattice derivative

$$\begin{aligned} P^-_{sd}(\mathbf{x}) &= \int dx^- a^2 \sum_{\mathbf{x}} \Big\{ m^2 \eta^\dagger(\mathbf{x}) \frac{1}{i\partial^+} \eta(\mathbf{x}) \\ &+ \frac{1}{4a^2} \sum_r [\eta^\dagger(\mathbf{x}+a\hat{\mathbf{r}}) - \eta^\dagger(\mathbf{x}-a\hat{\mathbf{r}})] \frac{1}{i\partial^+} [\eta(\mathbf{x}+a\hat{\mathbf{r}}) - \eta(\mathbf{x}-a\hat{\mathbf{r}})] \Big\}. \end{aligned}$$

It can be rewritten as

$$\begin{aligned} P^-_{sd} &= \int dx^- a^2 \sum_{\mathbf{x}=even} [m^2 \eta^\dagger(\mathbf{x}) \frac{1}{i\partial^+} \eta(\mathbf{x}) \\ &\quad - \frac{1}{4a^2} a^2 [\eta^\dagger(\mathbf{x}) \frac{1}{i\partial^+} \sum_r [\eta(\mathbf{x}+2a\hat{\mathbf{r}})] + \eta(\mathbf{x}-2a\hat{\mathbf{r}}) - 2\eta(\mathbf{x})]] \\ &+ \int dx^- a^2 \sum_{\mathbf{x}=odd} [m^2 \eta^\dagger(\mathbf{x}) \frac{1}{i\partial^+} \eta(\mathbf{x}) \\ &\quad - \frac{1}{4a^2} a^2 [\eta^\dagger(\mathbf{x}) \frac{1}{i\partial^+} \sum_r [\eta(\mathbf{x}+2a\hat{\mathbf{r}})] + \eta(\mathbf{x}-2a\hat{\mathbf{r}}) - 2\eta(\mathbf{x})]]. \end{aligned}$$

Hamiltonian is divided into even and odd sublattices each with lattice constant $2a$. => a momentum component in each sublattice is bounded by $\frac{\pi}{2a}$ in magnitude.

The free particle dispersion relation

$$k^-_{\mathbf{k}} = \frac{1}{k^+}[m^2 + \frac{1}{a^2} \sum_r \sin^2 \; k_r a].$$

Since $k_r a \leq \frac{\pi}{2}$, doublers can not arise from $k_r a = \pi$ but we can get zero momentum fermions from each sublattices.

Boundary Conditions: Two sequences for two sublattices
Fixed:=> $k_p = \frac{p\pi}{(2n+4)a}$, with $p = 1, 2, 3, ..., n+1$
and $k_p = \frac{p\pi}{(2n+2)a}$, with $p = 1, 2, 3, ..., n$.
Periodic:=> $k_p = \pm\frac{2\pi p}{(2n+2)a}$, $p = 0, 1, 2, ..., \frac{n+1}{2}$
and $k_p = \pm\frac{\pi}{na}p$, $p = 0, 1, 2, ..., \frac{n-1}{2}$.
Numerical results show that lowest four degenerate states are really lowest momentum states. $<< FIGURE3 >>$ $<< FIGURE4 >>$

Removal of doublers:

a) Staggered fermion on the light front transverse lattice (Kogut-Susskind formulation)
In analogy with the Euclidean staggered formualtion, define the spin diagonalisation transformation

$$\eta(x_1, x_2) \rightarrow (\hat{\sigma}^1)^{x_1}(\hat{\sigma}^2)^{x_2}\chi(x_1, x_2).$$

After spin diagonalisation, the full Hamiltonian in the free field limit becomes

$$\begin{aligned} P^-_{sf} &= \int dx^- a^2 \sum_{\mathbf{x}} \Big\{ m^2 \chi^\dagger(\mathbf{x}) \frac{1}{i\partial^+} \chi(\mathbf{x}) \\ &+ \frac{1}{4a^2} \sum_r [\chi^\dagger(\mathbf{x}+a\hat{\mathbf{r}}) - \chi^\dagger(\mathbf{x}-a\hat{\mathbf{r}})] \frac{1}{i\partial^+} [\chi(\mathbf{x}+a\hat{\mathbf{r}}) - \chi(\mathbf{x}-a\hat{\mathbf{r}})] \\ &- \frac{1}{2a} m \chi^\dagger(\mathbf{x}) \frac{1}{i\partial^+} \sum_r \phi(\mathbf{x}, r) [\chi(\mathbf{x}+a\hat{\mathbf{r}}) - \chi(\mathbf{x}-a\hat{\mathbf{r}})] \\ &- \frac{1}{2a} m \sum_r [\chi^\dagger(\mathbf{x}+a\hat{\mathbf{r}}) - \chi^\dagger(\mathbf{x}-a\hat{\mathbf{r}})] \phi(\mathbf{x}, r) \frac{1}{i\partial^+} \chi(\mathbf{x}) \Big\}. \end{aligned}$$

$\phi(\mathbf{x}, r) = 1$ for $r = 1$ and $\phi(\mathbf{x}, r) = (-1)^{x_1}$ for $r = 2$.

The two linear mass terms cancel with each other in the free theory, but since they are present in the interacting theory we keep them for investigation.
Since the Hamiltonian is now spin diagonal we put single component field at each transverse site. Thus we have thinned the degrees of freedom by half. Interpret the remaining two species as two flavors.
Define $<< FIGURE 5 >>$
$u_1 = \frac{1}{\sqrt{2}}(\eta_1 + \eta_{1'}), \quad u_2 = \frac{1}{\sqrt{2}}(\eta_2 + \eta_{2'}),$
$\tilde{d}_1 = \frac{1}{\sqrt{2}}(\eta_1 - \eta_{1'}), \quad \tilde{d}_2 = \frac{1}{\sqrt{2}}(\eta_2 - \eta_{2'})$
and $d = \hat{\sigma}^1 \tilde{d}$. The picture becomes clear when written in terms of block derivatives:

$$\begin{aligned} P^-_{sf} = \quad & \int dx^- a^2 \sum_x \{ m^2 f^\dagger \frac{1}{i\partial^+} f \\ & + \frac{1}{4}[\hat{\nabla}_r f^\dagger \frac{1}{i\partial^+} \hat{\nabla}_r f + a^2 \hat{\nabla}^2_r f^\dagger \frac{1}{i\partial^+} \hat{\nabla}^2_r f \\ & + \frac{ia}{2}(\hat{\nabla}_r f^\dagger \frac{1}{i\partial^+} \sigma^r T^r \hat{\nabla}^2_r f + \hat{\nabla}^2_r f^\dagger \frac{1}{i\partial^+} \sigma^r T^r \hat{\nabla}_r f)] \\ & - \frac{1}{2} m (f^\dagger \frac{1}{i\partial^+} \hat{\sigma}^r \hat{\nabla}_r f + \frac{a}{2} f^\dagger \frac{1}{i\partial^+} \sigma^3 T^r \hat{\nabla}^2_r f + h.c) \} \end{aligned}$$

where, the flavor isospin doublet

$$f = \begin{bmatrix} u \\ d \end{bmatrix},$$

$\hat{\nabla}_r$ and $\hat{\nabla}^2_r$ are respectively first and second order block derivatives, $T^1 = -i\sigma^2, \;\; T^2 = -i\sigma^1$ matrices in the flavor space.
NO flavor mixing in free theory.

b) Wilson fermions

To remove doublers, add a term to the Lagrangian density

$$\delta\mathcal{L}(\mathbf{x}) = \frac{\kappa}{a}\bar{\psi}(\mathbf{x})[\psi(\mathbf{x}+a\hat{\mathbf{r}}) - 2\psi(\mathbf{x}) + \psi(\mathbf{x}-a\hat{\mathbf{r}})],$$

and the Hamiltonian with Wilson term reads

$$\begin{aligned} P_w^- &= \int dx^- a^2 \sum_{\mathbf{x}} \Big[\mu^2 \eta^\dagger(\mathbf{x}) \frac{1}{i\partial^+}\eta(\mathbf{x}) \\ &+ \frac{1}{2a}[\eta^\dagger(\mathbf{x}+a\hat{\mathbf{r}}) - \eta^\dagger(\mathbf{x}-a\hat{\mathbf{r}})]\frac{1}{i\partial^+}\frac{1}{2a}[\eta(\mathbf{x}+a\hat{\mathbf{r}}) - \eta(\mathbf{x}-a\hat{\mathbf{r}})] \\ &+ \frac{\kappa^2}{a^2}[\eta^\dagger(\mathbf{x}+a\hat{\mathbf{r}}) - 2\eta^\dagger(\mathbf{x}) + \eta^\dagger(\mathbf{x}-a\hat{\mathbf{r}})] \\ &\qquad \frac{1}{i\partial^+}[\eta(\mathbf{x}+a\hat{\mathbf{r}}) - 2\eta(\mathbf{x}) + \eta(\mathbf{x}-a\hat{\mathbf{r}})] \\ &- 2\frac{\mu\kappa}{a}\eta^\dagger(\mathbf{x})\frac{1}{i\partial^+}[\eta(\mathbf{x}+a\hat{\mathbf{r}}) - 2\eta(\mathbf{x}) + \eta(\mathbf{x}-a\hat{\mathbf{r}})]\Big], \end{aligned}$$

Where $\mu^2 = (m + 4\frac{\kappa}{a})^2$.

Using the Fourier transform in the transverse space, we get,

$$\begin{aligned} &\int dx^- \int \frac{d^2k}{(2\pi)^2)} \phi_{\mathbf{k}}^\dagger(x^-) \frac{1}{i\partial^+} \phi_{\mathbf{k}}(x^-) \Big[\mu^2 + \sum_{\hat{\mathbf{r}}} k_r^2 \Big(\frac{\sin k_r a}{k_r a}\Big)^2 \\ &+ 2a\mu\kappa \sum_{\hat{\mathbf{r}}} k_r^2 \Big(\frac{\sin k_r a/2}{k_r a/2}\Big)^2 + a^2\kappa^2 \sum_{\hat{\mathbf{r}}} k_r^4 \Big(\frac{\sin k_r a/2}{k_r a/2}\Big)^4\Big]. \end{aligned}$$

Note that, as anticipated, Wilson term removes the doublers.

Since the Wilson term connects even and odd lattices, the extra fermions that appear at zero transverse momentum are removed. For large n, we get the expected spectra but, numerical results suggest that the finite volume effect is larger for small κ.

Doubling and symmetry on the light front transverse lattice:

In lattice gauge theory in the Euclidean or equal time formalism, there has to be explicit chiral symmetry breaking in the kinetic part of the action or Hamiltonian. Translated to the light front formalism, this would then require helicity flip in the kinetic part since chirality *is* helicity even for a massive fermion in front form. A careful observation of all the above methods that get rid of fermion doublers on the light front transverse lattice reveals that this is indeed true. In particular, we draw attention to the even-odd helicity flip transformation

$$\eta(x_1, x_2) \rightarrow (\hat{\sigma}_1)^{x_1} (\hat{\sigma}_2)^{x_2} \eta(x_1, x_2)$$

Invariance => doublers
Non-invariance => No doublers.

SUMMARY AND CONCLUSIONS

• Hamiltonian with forward and backward derivatives is free from fermion doubling. The helicity flip term proportional to the fermion mass in the full light front QCD becomes an irrelevant term in free field limit. With periodic boundary condition one can get the helicity up and helicity down fermions to be degenerate for any transverse lattice size n. With fixed boundary condition, there is splitting between the two states at any n but the splitting vanishes in the large volume limit.

• Hamiltonian with symmetric derivative results in four fermion species as it has only next to nearest neighbor interactions and even and odd lattice points are decoupled.

• Two ways of removing the doublers are discussed:

(a) staggered fermion: irrelevant flavor mixing only in interacing theory,

(b) Wilson fermion: generates extra terms, small values of Wilson parameter are sensitive to small volume effects.

• We have tried to understand the origin of doublers in terms of symmetries of the transverse lattice Hamiltonian and identified the even-odd helicity flip symmetry to be relevant for doubling.

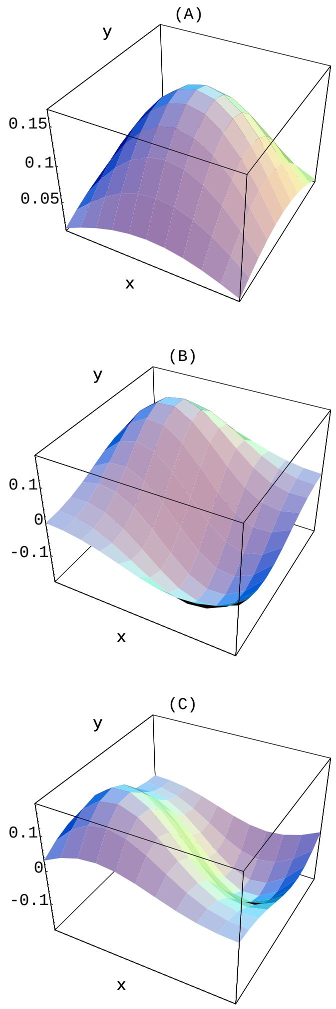

Figure 1: Eigenfunctions of first three states for P_0^-

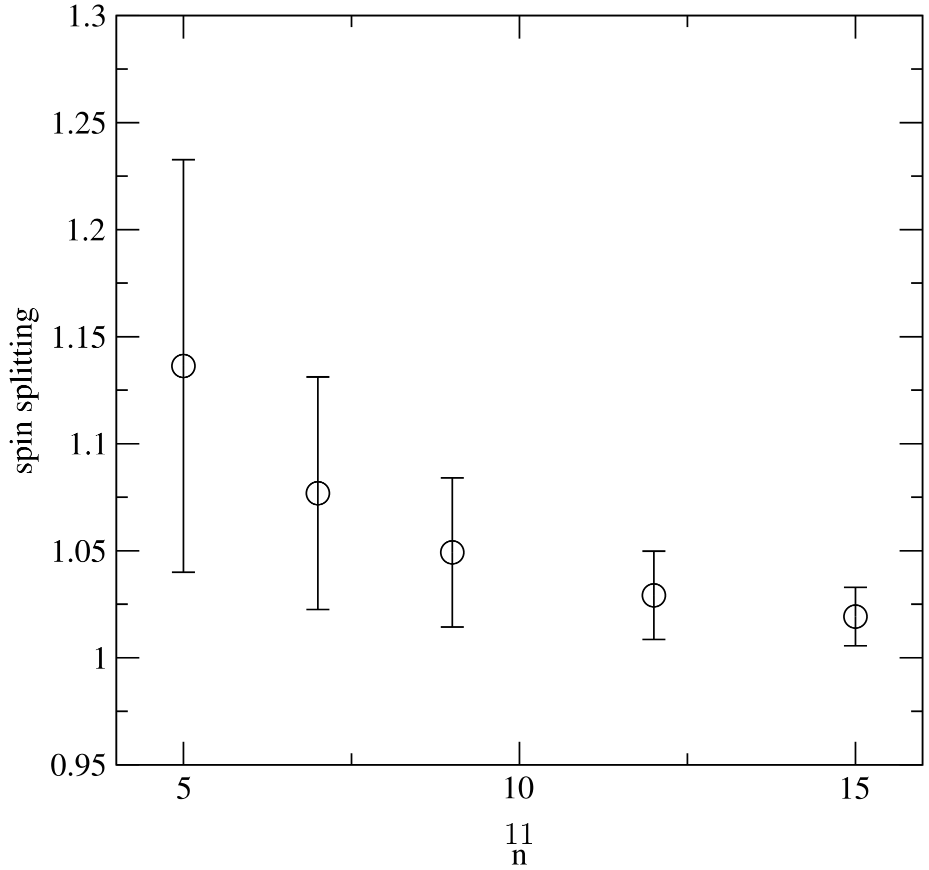

Figure 2: Spin splitting of the ground state caused by the spin dependent interaction as a function of n.

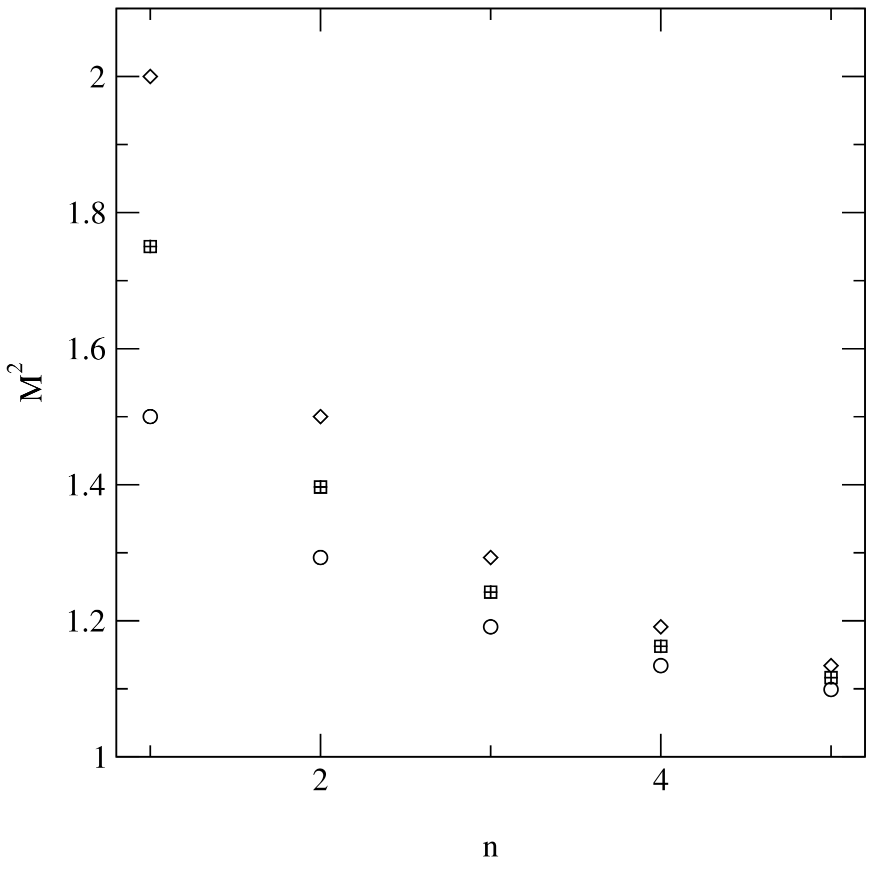

Figure 3: First four eigenvalues as a function of n(symmetric derivative).

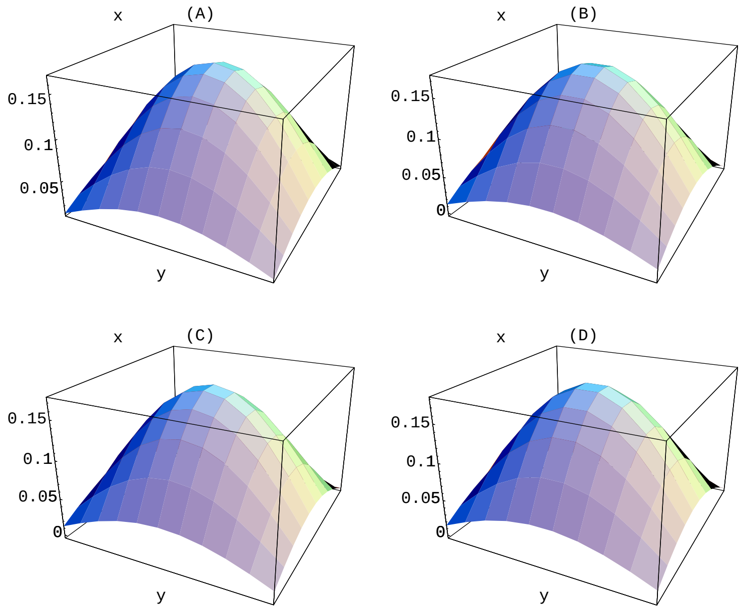

Figure 4: Eigenfunctions of first four (degenerate) states for the case of symmetric derivative.

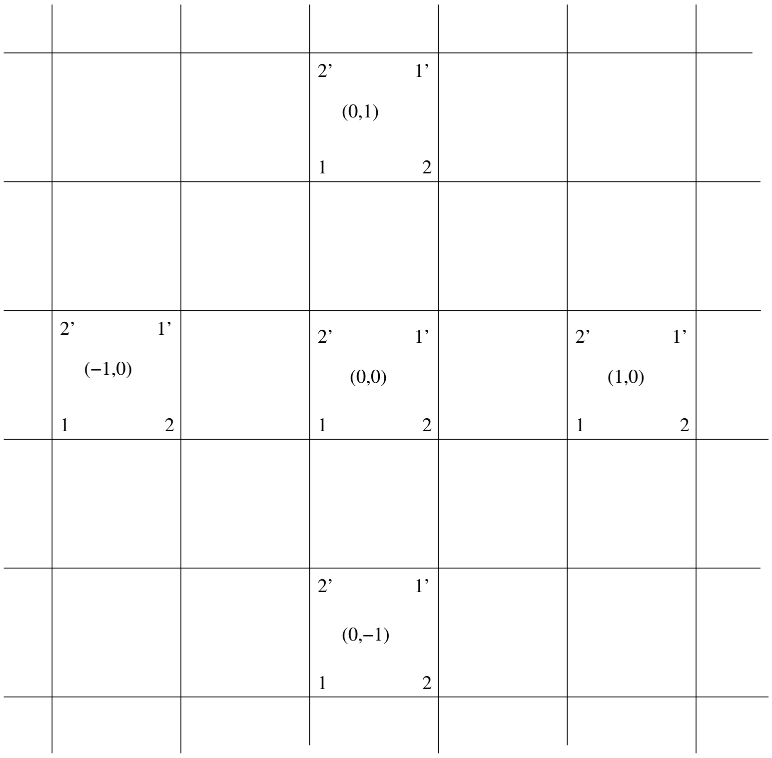

Figure 5: Staggered distribution