Topological Field Patterns in the Yang-Mills Theory

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2. Characterisation of Monopoles : New framework

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't Hooft-Polyakov monopole. $SO(3)$, $a = 1, 2, 3$.

$$\mathcal{L} = -\frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} + \frac{1}{2}(D_\mu \Phi)^a (D_\mu \Phi)^a - \frac{\lambda}{4}(\Phi^2 - F^2)$$

$$G^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu - \epsilon^{abc} W^b_\mu W^c_\nu$$

$$(D_\mu \Phi)^a = \partial_\mu \Phi^a - \epsilon^{abc} W^b_\mu \Phi^c$$

Energy density

$$\begin{aligned} H = \frac{1}{2}[E^a_i E^a_i + B^a_i B^a_i + \Pi^a \Pi^a] \\ + \frac{1}{2}(D_i \Phi)^a (D_i \Phi)^a + V(\Phi) \end{aligned}$$

vacuum configuration Degenerate

- $G^a_{\mu\nu} = 0$
- $D_\mu \Phi^a = 0$ H, any sols. GI
- $V(\Phi) = 0$

Finite energy sols. $(2),(3)$

$r \to \infty$ $\Phi(\theta, \phi)$

$$\partial_\theta \Phi^a \neq 0, \quad \partial_\phi \Phi^a \neq 0.$$

$$\Phi^a \Phi^a \to F$$

$$S_2^{phys} \to S_2^{int}$$

boundary conditions

Monopole asymp. behavior
. no gauge filed

$U(1) \subset SO(3)$ mag. charge

$$\epsilon_{\mu\nu\lambda\sigma} \partial_\nu F_{\lambda\sigma} = K_\mu$$

't Hooft EM tensor

$$F_{\mu\nu} = \hat{\Phi}^a G^a_{\mu\nu} + \frac{1}{2}\epsilon^{abc}\hat{\Phi}^a D_\mu \hat{\Phi}^b D_\nu \hat{\Phi}^c$$

$$= \partial_\mu(\hat{\Phi}^a W^a_\nu) - \partial_\nu(\hat{\Phi}^a W^a_\mu) + \frac{1}{2}\epsilon^{abc}\hat{\Phi}^a \partial_\mu \hat{\Phi}^b \partial_\nu \hat{\Phi}^c$$
$$= M_{\mu\nu} + H_{\mu\nu}$$

$$K_\mu = \frac{1}{2}\epsilon_{\mu\nu\lambda\sigma}\epsilon^{abc}\partial_\nu\hat{\Phi}^a\partial_\lambda\hat{\Phi}^b\partial_\sigma\hat{\Phi}^c$$

$\partial_\mu K_\mu = 0$ topological

$$\begin{aligned} B_i &= \frac{1}{2}\epsilon_{ijk}F_{jk} \\ &= \epsilon_{ijk}\partial_i a_k + k_i. \end{aligned}$$

$$\begin{aligned} a_i &= \hat{\Phi}^a W_i^a \\ k_i &= \frac{1}{2}\epsilon_{ijk}\epsilon^{abc}\hat{\Phi}^a\partial_j\hat{\Phi}^b\partial_k\hat{\Phi}^c \\ \partial_i B_i &= \partial_i k_i = \delta^3(x - x_0) \end{aligned}$$

't Hooft sol spherical symm

$$\begin{aligned} \Phi^a &= \frac{r^a}{r^2}H(Fr) \\ W_i^a &= -\epsilon_{aij}\frac{r^j}{r^2}(1 - K(Fr)), \quad W_0^a = 0 \\ G_{ij}^a &= \epsilon_{ijk}\frac{r^a r^k}{r^4} \\ F_{ij} &= \hat{\Phi}^a G_{ij}^a = \frac{1}{r^3}\epsilon_{ijk}r^k, \quad B_i = \frac{1}{r^3}r_i \end{aligned}$$

$\Phi^a = \frac{r^a}{r^2} H(Fr)$

$\widehat{\Phi}^a = 0$ center

$\frac{1}{4\pi} \int d\sigma^i k_i = Q$ only $\widehat{\Phi}^a$, NO gauge field

- Topological aspects using the center

- Characterisation of monopoles using gauge potential alone is possible

New framework

$$S_{ij}(x) = B_i^a B_j^a$$

3×3, Real, Symmetric tensor

Eigenvalue Eqns.

$$S_{ij}\xi_j^A(x) = \lambda^A(x)\xi_i^A(x)$$

$\xi_i^A \xi_i^B = \delta^{AB}$ orthonormal frame

$S_{ij},\ \xi_i^A$ gauge invariant

W_i^a, B_i^a 9 dof

. 3 gauge degrees, 6 GI degrees

$\lambda^A(x)$ 3 degrees

$\xi_i^A(x)$ 3×3 Orthogonal, 3 degrees

λ^A, ξ_i^A give complete GI description of W_i^A, B_i^a

- By appropriate local gauge transformation, 3×3 matrix, B_i^a can be made symmetric. Its eigenvectors ξ_i^A can be used.

- By local gauge transformation, columns of B_i^A can be made orthogonal. These columns serve as ξ_i^A

topological aspects singularities of ξ_i^a

No ref. to asymptotic behavior is needed
Asymptotic behavior of fields can be obtained

$$
\begin{aligned}
S^{ab}(x) &= B_i^a B_i^b \\
S^{ab}\xi_A^a &= \lambda^A \xi_A^a, \quad A = 1,2,3 \\
\xi_a^A \xi_a^B &= \delta^{AB}
\end{aligned}
$$

$\xi_A^a \quad \rightarrow \quad$ 3 sets of fields in place of the Higgs
. $\widehat{\Phi}^a$ of 't Hooft-Polyakov monopole.

$$
\begin{aligned}
Q^A &= \frac{1}{8\pi}\int d\sigma^i \epsilon_{ijk}\epsilon^{abc}\xi_a^A \partial_j \xi_b^A \partial_k \xi_c^A \\
&= \frac{1}{4\pi}\int d\sigma^i k_i^A
\end{aligned}
$$

$\partial_i k_i^A = 0$ $\qquad\qquad$ $k_i^A = (\nabla \times \quad)_i^A$

- 3×3, orthogonal matrices, ξ_a^A generate local gauge transformations.

- pure gauge fields can be constructed using ξ_a^A.

$$
\omega_i = \frac{1}{2}\epsilon^{ABC}\xi_a^B \partial_i \xi_a^C
$$

Then

$$\begin{aligned} k_i^A &= \frac{1}{2}\epsilon_{ijk}\epsilon^{abc}\xi_a^A\partial_j\xi_b^A\partial_k\xi_c^A \\ &= \frac{1}{2}\epsilon_{ijk}\epsilon^{ABC}\omega_j^B\omega_k^C \end{aligned}$$

$$\epsilon_{ijk}\left(\partial_j\omega_k^A - \partial_k\omega_j^A - \epsilon^{ABC}w_j^B\omega_k^C\right) = 0$$

$$k_i^A = \epsilon_{ijk}\partial_j\omega_k^A$$

- k_i^A is the curl of a vector field
- ω_i^A is non-Abelian pure gauge, constructed from ξ_a^A

't Hooft-Polyakov Monopole

for 't Hooft-Polyakov monopole, generic S^{ab} is

$$S^{ab} = \alpha(r)\delta^{ab} \ + \ \beta(r) r^a r^b$$

r is the distance from the center where all ξ_a^A are degenerate

$$\xi_a^1 = \frac{\widehat{r}^a}{r}, \quad \xi_a^2 = \widehat{\theta}^a, \quad \xi_a^3 = \widehat{\phi}^a$$

$\lambda^2 = \lambda^3 = \alpha(r)$ double degeneracy

$$k_i^1 = \frac{\widehat{r}_i}{r^2}, \quad k_i^2 \ = \ \frac{cos\theta}{r^2 sin\theta}\, \widehat{r}_i, \quad k_i^3 = 0$$

$$\frac{1}{4\pi}\int d\sigma^i k_i^1 \ = \ 1$$

$$\int d\sigma^i k_i^2 \ = \ 0$$

k_i^1 mag. field B of a Dirac monopole
$k_i^2 \neq 0$ radial flux from $-\infty \to +\infty$ along z-axis without a source.

Now 't Hooft-Polyakov solutions in terms of ω_i^A are

$$\omega_i^1 = -\frac{cos\theta}{rsin\theta}\widehat{\phi}_i, \quad \omega_i^2 = \frac{1}{r}\widehat{\phi}_i, \quad \omega_i^3 = -\frac{1}{r}\widehat{\theta}_i$$

ω_i^2, ω_i^3 flux without sources

ω_i^A is pure gauge everywhere
$B_i^A = 0$ everywhere

Orthogonal matrices ξ_a^A can be used for gauge transformations

Applying on Yang-Mills potential,

$$a_i^A = \xi_a^A W_i^a + \frac{1}{2}\epsilon^{ABC}\xi_a^B \partial_i \xi_a^C$$

$\xi_a^A W_i^a = \tilde{A}_i$ is 't Hooft's Abelian gauge field, now constructed from ξ_a^A
Then $a_i^A = \tilde{A}_i + \omega_i^A$

for 't Hooft-Polyakov monopole,

$$W_i^a = -\epsilon_{aij}\frac{r^j}{r^2}(1-K(r))$$

$$\tilde{A}_i^1 = 0,$$

$$\tilde{A}_i^2 = -\widehat{\phi}_i\frac{(1-K(r))}{r}, \quad \tilde{A}_i^3 = \widehat{\theta}_i\frac{(1-K(r))}{r}$$

at ∞, $K(r) \to 0$

$$a_i^1 = -\frac{cos\theta}{rsin\theta}\widehat{\phi}_i, \quad a_i^2 = \frac{K(r)}{r}\widehat{\phi}_i, \quad a_i^3 = -\frac{K(r)}{r}\widehat{\theta}_i$$

for Dirac monopole, we have

$$A_i^I = \frac{1}{r}\frac{(1-cos\theta)}{sin\theta}\ \widehat{\phi}_i$$

$$A_i^{II} = -\frac{1}{r}\frac{(1+cos\theta)}{sin\theta}\ \widehat{\phi}_i$$

$$\frac{A_i^I + A_i^{II}}{2} = a_i^1$$

for $A = 1$ we get Dirac potential, $A = 2, 3$ Potential without sources and vanishes asymptotically.

CONCLUSION

- Using orthonormal frames ξ_i^A or ξ_a^A and corresponding eigenvectors λ^A, gauge invariant/covariant description of monopole is obtained.

- Topological aspects are defined using the behavior of fields at the center/interior of the monopole

- We have re-derived the 't Hooft-Polyakov potential and winding no in this framework. We have constructed a pure gauge ω_i^A from ξ_a^A and showed that the Abelian field strength of ω_i^A captures all the topological features of the monopole.

- Using the orthonormal frames ξ_i^A, ξ_a^A all possible topological configurations of Yang-Mills theory can be handled.