Some observations on interpolating gauges and non-covariant gauges

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plan of talk

1. Introduction to the problem of Non-covariant gauges

2. Nature and subject of observations: Interpolating gauges and Prescription by Hand

3. Interpolating gauges: Boundary conditions important!

4. Delicate nature of the non-covariant gauge path-integral

5. Results from residual gauge transformation in BRS space

- Absurd results via RIGT; strong need for careful definition

- Corrected RIGT WT-identities are special cases of a careful BRST treatment

- Analogy with anomalies

6. Non-trivial constraints on Renormalization implied by a prescription

7. Conclusions

introduction

- Non-covariant gauges [Axial, temporal, light-cone, Coulomb, planar etc] have been widely used in standard model calculations as well as in formal arguments in gauge theories.

- Interpolating gauges have been used to enable one to define these from covariant ones.

- As compared to the covariant gauges, these gauges have however not been fully developed.

- Path-integrals for gauges with non-covariant gauges are ill-defined on account of residual gauge-invariance.

In axial gauges, it arises as the problem of spurious singularities in the propagator:

$$D_{\mu\nu}(k) = -\frac{1}{k^2+i\epsilon}\{g_{\mu\nu} - \frac{k_\mu\eta_\nu + k_\nu\eta_\mu}{\eta\cdot k} + k_\mu k_\nu\frac{\eta^2}{(\eta\cdot k)^2}\} \tag{1}$$

In Coulomb gauges, it arises as the problem of ill-defined energy-integrals as the propagator for the time-like component is :

$$D^c_{00}(k) = \frac{1}{|\mathbf{k}|^2} \sim k_0^0$$

is poorly damped as $k_0 \to \infty$ compared to the Lorentz gauges.

approaches to cure the problem (books by Bassetto et al, Leibbrandt)

- Impose by hand a prescription (CPV,L-M,...)

- use interpolating gauges [Burnel,Doust & Taylor,....]

- Obtain the definition using a field-transformation in BRS space: FF-BRST [SDJ,Misra:JMP**41**(2000),IJMP**A15** (2000)]

examples:

1. CPV: Replace

$$\frac{1}{\eta.k} \to \frac{1}{2}\left(\frac{1}{\eta.k+i\varepsilon}+\frac{1}{\eta.k-i\varepsilon}\right)$$

2. Define the Coulomb gauge by the Doust gauge:

$$F[A,\theta] = \theta\partial_0 A_0 - \frac{1}{\theta}\partial_i A_i$$

3. Use a FF-BRST to obtain an exact relation for the axial Green's function

$$\begin{aligned} << O[\phi] >>|_F = << O[\phi] >>|_L \\ +i\int_0^1 d\kappa \int D\phi exp\{iS^M_{eff}[\phi,\kappa] + \varepsilon\int d^4x(\frac{1}{2}A^2 - \overline{c}c)\} \\ \times \sum\{\delta_{1i}\widetilde{[\phi]} + \kappa\delta_{2i}\widetilde{[\phi]}\}(-i\Theta')\frac{\delta^L O}{\delta\phi_i} \end{aligned}$$

The subject of new observations

Several questions were evoked from comparison of the previous literature with the work I had done (in part with Aalok Misra and B.P.Mandal) that uses a FF-BRST transformation to *construct* the path-integral for various non-covariant gauges by correlating it with the well-defined path-integrals in Lorentz gauges such as to preserve the expectation values of gauge-invariant observables.

[PRD(95), JMP(2000), IJMPA(2000), IJMPA(2001)].

- Is it legitimate to impose a prescription by hand? Could it create difficulties? Would the symmetries not be affected by this ad hoc superimposition?

- Do interpolating gauges really interpolate gauge-invariantly? Going from a gauge to a neighboring gauge involves a ill-defined "singular" gauge-transformation. Does the proof of gauge-invariance hold in the face of the ill-defined nature of the gauge-transformation?

It will be seen that the definition of non-covariant gauges is not "a simple matter of fixing the spurious singularities in some way" but there are serious mathematical complications at the root of it. We will bring these out in this talk.

An interpolating gauge

One obvious way that has been in literature that can correlate two family of gauges is the *interpolating gauge*: A gauge function $F[A, \alpha]$ with a variable parameter(s) α such that for different parameter values, one has different gauges: Example: Doust gauge:

$$F[A, \theta] = \theta \partial_0 A_0 - \frac{1}{\theta} \partial_i A_i$$

For θ=1, we have the Feynman gauge, and for $\theta \to 0$, we have the Coulomb gauge limit. Then the generating functional for this gauge

$$W[J] = \int D\phi \, \text{exp}\{iS_0[A] - \frac{i}{2} F[A, \theta]^2 + S_{gh}[A, c, \overline{c}] + i \int d^4 x J A\}$$

This doesn't define the generating functional even for the Feynman gauge. We need the ϵ-term. An ϵ-term for the Feynman gauge would be

$$-i\varepsilon \int d^4 x \{\frac{A^2}{2} - \overline{c} c\}$$

and modify $W[J]$ as

$$W[J] = \int D\phi \, \text{exp}\{iS_0[A] - \frac{i}{2}F[A,\theta]^2 + S_{gh}[A,c,\overline{c}] \\ + \varepsilon \int d^4x\{\frac{A^2}{2} - \overline{c}c\} + i\int d^4x JA\}$$

as was done by Doust. This certainly would define the Green's function for all θ ; $0 \leq \theta \leq 1$.

But Does it interpolate Gauge-invariantly?

One would expect that if there is no unusual reason for it, a simple procedure such as this should work.

There is a nontrivial problem in proving gauge-invariance. It stems from the ***singular nature of the gauge transformations involved.*** [SDJ:EPJ*direct* C(2001)]

Consider the transformation that leads to $F[A,\theta] \rightarrow F[A,\theta+\delta\theta]$. The transformation is (The Abelian case)

$$A_\mu = A'_\mu + \partial_\mu M_0^{-1} \Delta F'; M_0 \equiv [\theta^2 \partial_0^2 - \nabla^2];$$
$$\Delta F = \delta\theta \Delta_\mu A'^\mu$$

Now M_0 is a singular operator and this at least calls for care in handling it. It is necessary that we do the calculation self-consistently. we start form

$$W[J] = \int D\phi \,\mathrm{exp}\{iS_0[A] - \frac{i}{2}F[A,\theta]^2 + S_{gh}[A,c,\overline{c}]$$
$$+\varepsilon \int d^4x\{\frac{A^2}{2} - \overline{c}c\} + i\int d^4xJA\}$$

keeping the ϵ-term in tact and perform some (regularized form of) field transformation on the path-integral and do not drop any apparently small terms. Calculate the new propagator. One finds that the boundary condition for the propagator for the A_0 and A_l (spatial longitudinal) components changes from C (causal) to $\frac{1}{2}(C+A)$ for $\delta\theta >> \varepsilon$.

Consider, for example,the change in εA^2term. It is $\varepsilon \int d^4x A^\mu \frac{1}{[\theta^2\partial_0^2 - \nabla^2 + i\varepsilon]} \Delta_\mu A^\mu \delta\theta$. In the momentum space, in the domain $\theta^2 k_0^2 - |\mathbf{k}|^2 \sim 0$, this term is not of $O(\varepsilon\delta\theta)$,but of $O(\delta\theta)$. A careful calculation shows that this sort of terms dominate the ϵ-term as $\epsilon \to 0$ and alters the nature of boundary conditions for A_0 and A_l.

Results on the interpolating gauges:

[SDJ:EPJ*directC*(2001) & hep-th/0201011]

[1] There is no choice of a *local* quadratic ϵ-term that will keep the gauge-invariance of a vev of a GI operator for $\theta \in [0,1]$. At best it can be kept in an infinitesimal interval in [0,1].

[2]Given a Green's function, say a 4-point Green's function in 1-loop approximation at slightly off-shell momenta, is not a differentiable function of θ at some $\theta \in (0,1)$. This value of θ is where the threshold of 2 unphysical degrees of freedom opens up.

[3] Consequently, if the gauge variation of such a Green's function across such a $\theta = \theta_0$ is not amenable to Taylor expansion: i.e. G[θ_0+δ]-G[θ_0-δ]$\neq$O[δ].

[4] On account of such possibilities, the gauge variation WT-identity for G has an unusual behavior at such a value of θ: the *ϵ-term contributes to the WT-identity* in such a manner that the difference of the gauge-variation and the ϵ-term contribution is O[δ].

Delicate nature of Path-integrals and WT involved in Non-covariant gauges

- *IMPOSING PRESCRIPTION BY HAND IS A MATHEMATICALLY DANGEROUS OPERATION*

Path-integrals are ill-defined owing to a residual gauge invariance. Equivalently, the propagators are ill-defined over a 3-dimensional subspace $\eta.k = 0$. A prescription chooses its value by hand in this subspace; and moreover it is a "large" value. Thus, going from one prescription to another amounts to replacing, in this subspace, a large quantity by another large quantity.

It is not obvious that this procedure of defining the pole this way is not going to interfere with the BRS identities.

- *THE NON-COVARIANT PATH-INTEGRALS ARE DANGEROUS OBJECTS!*

The results to follow are best illustrated with the help of the simple integral which, in many ways, mimics the path-integral in non-covariant gauges. Let

$$I[J_1, J_2] = \int\int_{-\infty}^{\infty} dx_1 dx_2 \exp\{-ax_1^2 + J_1 x_1 + J_2 x_2\}$$

$I[0,0]$ is ill-defined because it has an infinite "volume factor" $\int dx_2$; and so is $I[J_1, J_2]$ .This is very similar to the path-integral for noncovariant gauges:

$$W[J] = \int D\phi \exp\{iS_0[A] - \frac{i}{2}F[A]^2 + S_{gh}[A, c, \overline{c}] \\ + i\int d^4x JA\}$$

which has a residual gauge-invariance in the BRS space and is consequently ill-defined. Its analogue for $I[0,0]$ is the invariance under $x_2 \to x_2 + a$. Imagine performing an infinitesimal translation of this kind and equate the change to zero:

$$\delta I = 0 = J_2 \int\int_{-\infty}^{\infty} dx_1 dx_2 \exp\{-ax_1^2 + J_1 x_1 + J_2 x_2\}$$

Assuming that the procedure is well-defined, (as one normally does in the derivation of WT identities), this would lead one to an erroneous conclusion:

$$J_2 = 0!$$

Evidently, to avoid such pitfalls, it is essential to make the procedure well-defined to find the correct conclusions out of it. The very fact that we need to make the procedure well-defined before a correct conclusion is possible indicates that the procedure is highly sensitive to the way the path-integral is made well-defined.

THE EPSILON TERM CAN & HAS TO CONTRIBUTE!

Thus, let

$$I[J_1, J_2; \varepsilon] = \int \int_{-\infty}^{\infty} dx_1 dx_2$$

$$\bullet \exp\{-ax_1^2 + J_1x_1 + J_2x_2 - \frac{1}{2}\varepsilon x_2^2\}$$

Then the above procedure of deriving a WT-identity will then lead to a correct equation:

$$\delta I = 0 = \int \int_{-\infty}^{\infty} dx_1 dx_2 [J_2 - \varepsilon x_2]$$
$$\bullet \exp\{-ax_1^2 + J_1 x_1 + J_2 x_2 - \frac{1}{2}\varepsilon x_2^2\}$$

which implies

$$0 = J_2 I[J_1, J_2; \varepsilon] - \varepsilon \int \int_{-\infty}^{\infty} dx_1 dx_2 x_2$$
$$\bullet \exp\{-ax_1^2 + J_1 x_1 + J_2 x_2\}$$

i.e.

$$J_2 = \frac{-\varepsilon \int \int_{-\infty}^{\infty} dx_1 dx_2 x_2 \exp\{-ax_1^2 + J_1 x_1 + J_2 x_2\}}{I[J_1, J_2; \varepsilon]}$$

which undoubtedly correct. But this then implies that the last term which comes from the symmetry variation of the "ϵ-term" is crucial for its validity! Thus, the ϵ-term must contribute to this "WT-IDENTITY" .

Results on Non-covariant gauges

We would like to present several general results on non-covariant gauges. These happen to be more easily perceptible in a very general formalism we will present. Before we do so, we would like to define the setting.

"TYPE R-GAUGES"

Definition: A gauge with a gauge functional F^α is called a *"Type-R gauge"* if the following set of three conditions are satisfied:

(i)There exists a set of infinitesimal gauge transformations characterized by gauge parameters $\sigma\theta^\alpha(x)$, with σ a constant infinitesimal:

$$\delta A^\alpha_\mu(x) = \sigma D^{\alpha\beta}_\mu \theta^\beta(x); \delta\psi = ig\sigma T^\alpha \psi \theta^\alpha; \tag{2}$$

$$\delta\psi^\dagger = -ig\sigma\psi^\dagger T^\alpha \theta^\alpha \tag{3}$$

such that the gauge-fixing term S_{gf}, is left invariant under them.

(ii) The ghost action, S_{gh} is invariant under the above gauge transformations when combined with " local vector" transformations on c and $\overline{c}$; i.e. under

$$\delta A_\mu^\alpha(x) = \sigma D^{\alpha\beta}\theta^\beta(x); \delta c^\alpha(x) = -\sigma g f^{\alpha\beta\gamma} c^\beta(x)\theta^\gamma(x);$$
$$\delta\overline{c}^\alpha(x) = -\sigma g f^{\alpha\beta\gamma}\overline{c}^\beta(x)\theta^\gamma(x)$$
$$\delta\psi = ig\sigma T^\alpha\psi\theta^\alpha; \delta\psi^\dagger = -ig\sigma\psi^\dagger T^\alpha\theta^\alpha \quad (4)$$

In other words, the Faddeev-Popov effective term in the net action viz. $\{-ilndetM\}$ is invariant under the gauge transformations in Not a special case of BRS in form!

(iii) Under IRGT, the boundary conditions on the path-integral variables are unaltered as $t \to \pm\infty$ for all $\mathbf{x}$.

We shall denote such a set as in the infinitesimal residual gauge transformations (IRGT), for the gauge F.

Examples of type -R gauges:

(1) The Coulomb family:$F = \nabla . A; \theta(x) = \theta(t)$; with either of the following satisfied:

(i) $\theta(t)$ and $\frac{d\theta}{dt}$ both $\rightarrow 0$ as $t \rightarrow \pm\infty$, OR

(ii) $\frac{d\theta}{dt} \rightarrow 0$ and $\theta(t) \rightarrow$ finite as $t \rightarrow \pm\infty$; if the boundary conditions on the fields in the original path-integral are : $\phi(\mathbf{x}, t) \rightarrow 0$ as $t \rightarrow \pm\infty$

(2) An axial gauge: $F = A_3; \theta(x) = \theta(x_0, x_1, x_2)$; with either of the following satisfied:

(i) $\theta(x)$ and $\frac{\partial\theta}{\partial x^\mu}; \mu = 0, 1, 2$ all $\rightarrow 0$ as $t \rightarrow \pm\infty$, OR

(ii) $\frac{\partial\theta}{\partial x^\mu}; \mu = 0, 1, 2 \rightarrow 0$ and $\theta(x) \rightarrow$ finite as $t \rightarrow \pm\infty$; if the boundary conditions on the fields in the original path-integral are : $\phi(\mathbf{x}, t) \rightarrow 0$ as $t \rightarrow \pm\infty$

Mathematical dangers inherent in the path-integrals

Theorem I: The Minkowski space Lagrangian path-integral of (i) for "type-R gauges" leads to physically unacceptable results from the IRGT WT-identities (derived along standard lines):

$$0 = << \int d^4x\{J^{\alpha\mu}(x) D_\mu^{\alpha\beta}\theta^\beta(x)\}$$

$$+ \int d^4x\{K^\dagger(x) igT^\alpha\theta^\alpha(x)\psi(x) - ig\psi(x)^\dagger T^\alpha\theta^\alpha(x)K(x) >> \quad (5)$$

- The propagator for the scalar field vanishes $\Delta(\mathbf{y}, t; \mathbf{z}, t') = 0$ for $\mathbf{z} \neq \mathbf{y}$ for any $t' \neq t$; evidently a conclusion incompatible with the corresponding one for the Lorentz gauges.

- we could differentiate IRGT WT-identities with respect to $J_0^\alpha(\mathbf{z}, t')$ and set all sources to zero. We are then lead to
$$\frac{d}{dt}\delta(t - t') = 0! \quad (6)$$
which is even mathematically unsound.

- We could also differentiate with respect to $J_\nu^\delta(w)$ and $J_\sigma^\rho(v)$ and then set all sources to zero. This gives vanishing of the gauge boson propagator $\Delta_{\mu\nu}(v,w)$ for $v \neq w$.

Comments on theorem I:

The fact that the generating functional is ill-defined for Type-R gauges is well-known. Nonetheless, it can lead to diverse absurd results by mathematical manipulations *along standard lines* has been exhibited above. The result above brings out the inevitable role that any corrective measure for the path-integral is likely to play. This is in fact born out by the results below!

corrective measure: a general approach

Normally, even in Lorentz gauges, the path-integral is understood to contain a suitable ϵ-term to make the Green's functions well-defined; nevertheless it is generally ignored in the formal manipulations and in the derivation of WT-identities (and this is appropriate for the family of the Lorentz gauges in the context of unbroken gauge theory: EPJC(2001). Suppose we modify the path-integral as

$$W[J,K,\overline{K};\xi,\overline{\xi}] = \int D\phi exp\{iS_{eff}[A,c,\overline{c},\psi] + \varepsilon O[\phi] \quad (7)$$
$$+source - terms\} \quad (8)$$

to include *a symmetry breaking* ϵ-term $-i\varepsilon O[\phi]$.

- this ϵ-term must, in particular, break invariance under IRGT.

- Need not limit ϵ-term to have dimension two,or be local.

- Various prescriptions, say the L-M for the LCG and the CPV for axial gauges etc, can be understood as special cases

- *The exact BRST is broken* .

We then we have the following result, (which we can well suspect from the theorem I):

BRST WT-Identity and the ϵ-term

Theorem II: The BRST WT-identity can receive a nontrivial contribution from the ϵ-term for type-R gauges even as $\epsilon \to 0$.

- Even though IRGT do not appear a spacial case of BRS, The IRGT WT-identities are derivable from the BRS WT-identities *with the ϵ-term included.*

- The IRGT WT-identity ***must*** receive contributions as $\epsilon \to 0$

- These ϵ-term contributions (which have to be finite) protect the IRGT WT-identity from leading to absurd conclusions.

- Even tree WT-identities receive contributions from the ϵ-term.

ADDITIONAL CONSTRAINTS ON ACCOUNT OF THE ϵ-TERM

Theorem III: The local quadratic ϵ-term (e.g. one below) implies additional constraints on Green's functions that are derivable from the IRGT WT-identities for the type-R gauges.

Example:

$$O = -i\varepsilon \int d^4x \{ \frac{A^2}{2} - \overline{c}c \}$$

We have, under IRGT

$$\varepsilon(\Delta O)^\alpha = \ \varepsilon \partial . A^\alpha$$

Now we shall illustrate the proof for the Coulomb gauge. The ϵ-term then leads to a term in the IRGT WT-identity that depends on

$$-i \int d^3\mathbf{x} \varepsilon (\Delta O)^\alpha = i\varepsilon \int d^3\mathbf{x} \frac{\partial}{\partial t} A^\alpha_\nu(\mathbf{x}, t) \qquad (9)$$

Now,

$$<< i\varepsilon\int d^3\mathbf{x}\frac{\partial}{\partial t}A^{\alpha}_{\nu}(\mathbf{x},t) >>= \varepsilon\int d^3\mathbf{x}\frac{\partial}{\partial t}\frac{\delta W}{\delta J^{\alpha}_{\nu}(\mathbf{x},t)} \tag{10}$$

Suppose we now differentiate the IRGT WT-identities with respect to $J^{\alpha}_{\sigma}(\mathbf{y},t')$ and set all sources to zero, we find (α not summed):

$$-\frac{d}{dt}\delta(t-t')g_{\sigma 0}+\varepsilon\int d^3\mathbf{x}\frac{\partial}{\partial t}\frac{\delta^2 W}{\delta J^{\alpha}_{0}(\mathbf{x},t)\delta J^{\alpha}_{\sigma}(\mathbf{y},t')}\mid_{0=J=\dots}=0 \tag{11}$$

This is a "corrected" version of an earlier relation; and similar "corrected" versions will follow for others.

Evidently, (11) this implies a constraint on the unrenormalized propagator $\Delta_{\nu\sigma}(p)$ to all orders:

$$\varepsilon p_0 i\Delta_{0\sigma}(\mathbf{p}=0,p_0)=-g_{\sigma 0}p_0$$

Thus, for $p_0\neq 0$, we have the constraint on $\Delta_{\nu\sigma}(\mathbf{p}=0,p_0)$ that

$$\varepsilon\Delta_{\sigma}(\mathbf{p}=0,p_0)=ig_{\sigma 0}$$

- Such relations are contained in the careful BRS identity.

- A type-R gauge is specified by *two* things: (i) The BRST-invariant effective action S_{eff} ,(ii) The way of interpreting the singularities in propagators.The extra relations refer to the latter.

- Above constraint is trivially true in the lowest order,

- Such a constraint is satisfied *by the unrenormalized propagator to all orders* [*if the prescription term is fixed*]. There is no parallel for such a constraint in Lorentz gauges.

- The other "absurd" relations get corrected through such ϵ-term contributions.

- Such extra relations can conflict with renormalization.

Renormalization and extra relations

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- Extra relations that follow from imposition of a prescription can clash with a multiplicative renormalization procedure that preserves BRS.

- We may have to allow renormalization of the ϵ-term.

- Even then, they lead to additional constraints that may not be compatible with a multiplicative renormalization procedure.

conclusions

1. Boundary conditions for the A_0 and A_l required to maintain gauge invariance are "unstable" with respect to a small variation of θ in the Doust gauge. A simple ϵ-term won't work. (A similar conclusion is expected to hold for other interpolating gauges).

2. The associated path-integral is not differentiable with θ.

3. The non-covariant gauge path-integral is poorly defined, and sensitively depends on the prescription term.

4. The careful BRS identities contain in addition to the "formal" BRS identities, IRGT WT-identities that are specific to a prescription term.

5. The IRGT WT-identities imply additional constraints on Green's functions

6. These have been studied to an extent that it would appear miraculous if they were trivially satisfied.