Condensed Matter Physics in QCD

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Motivation

One of the challenges for a QCD theorist is to find techniques to compute quantities from first principles with controlled errors.

A condensed matter theorist faces very similar challenges.

Both these fields involve many body quantum mechanics of strongly correlated fermions.

Neutron Star: A QCD "Diode"

- At high quark densities matter is in the Color-Flavor-Locked phase

  Equal number of u, d and s quarks.

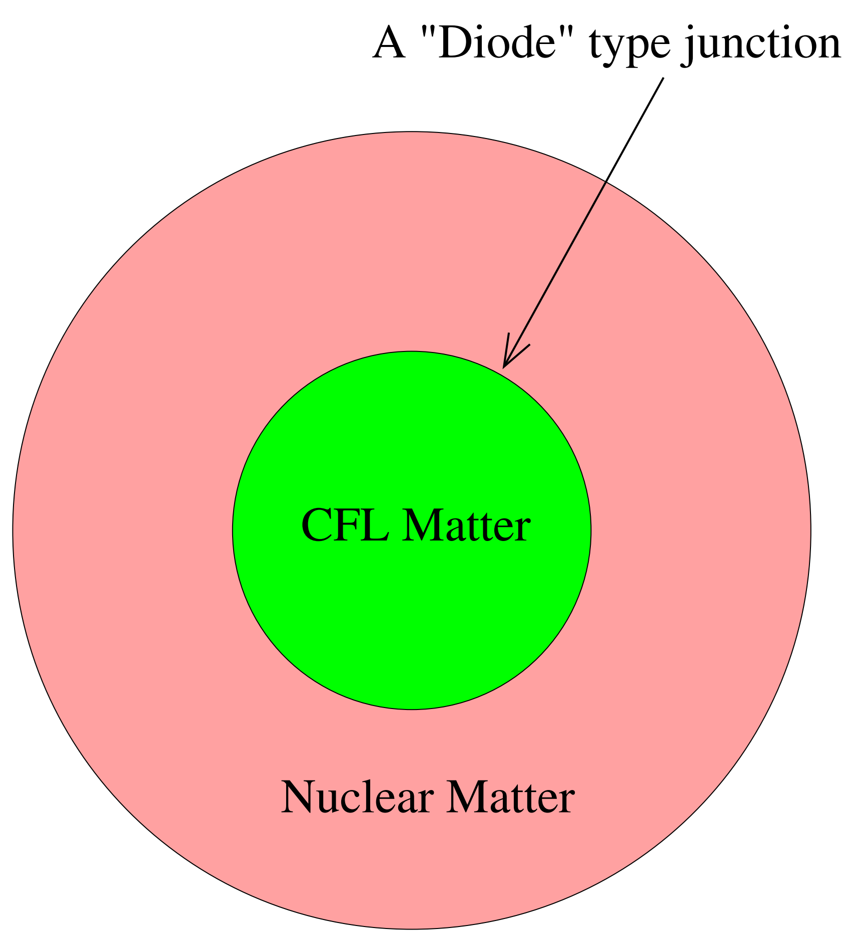
  (electrically neutral without electrons)

  Charged pseudo-scalar mesons are the lightest

  with masses $m \sim \sqrt{m_s m_u}\Delta/\mu$.

CFL quark matter is a super¤uid and a transparent insulator

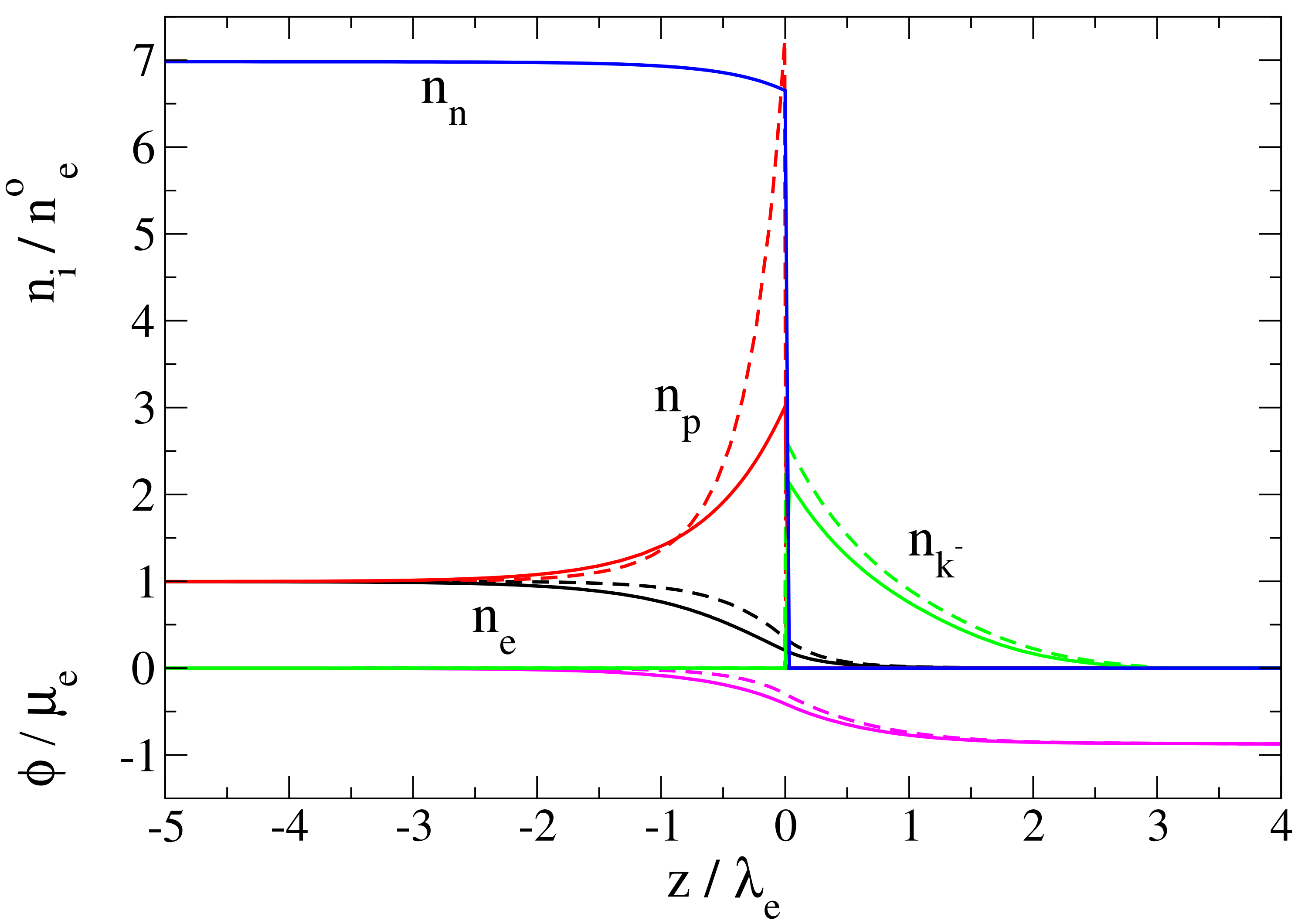
- CFL matter may be found at the core of neutron stars



A simple profile of a nuetron star

- A typical pro£le of the interface

[M. Alford, K. Rajagopal, S. Reddy, F. Wilczek, '01]



Positive charges (protons) accumulate on the nuclear matter side.

Negative charges (Kaons) accumulate on the CFL side.

Ideas of Condensed Matter Physics in QCD

- Attraction between quarks and anti-quarks

  leads to chiral symmetry breaking

  (we will see that this is similar to anti-ferromagnetism)

- Attraction between quarks

  leads to color superconductivity

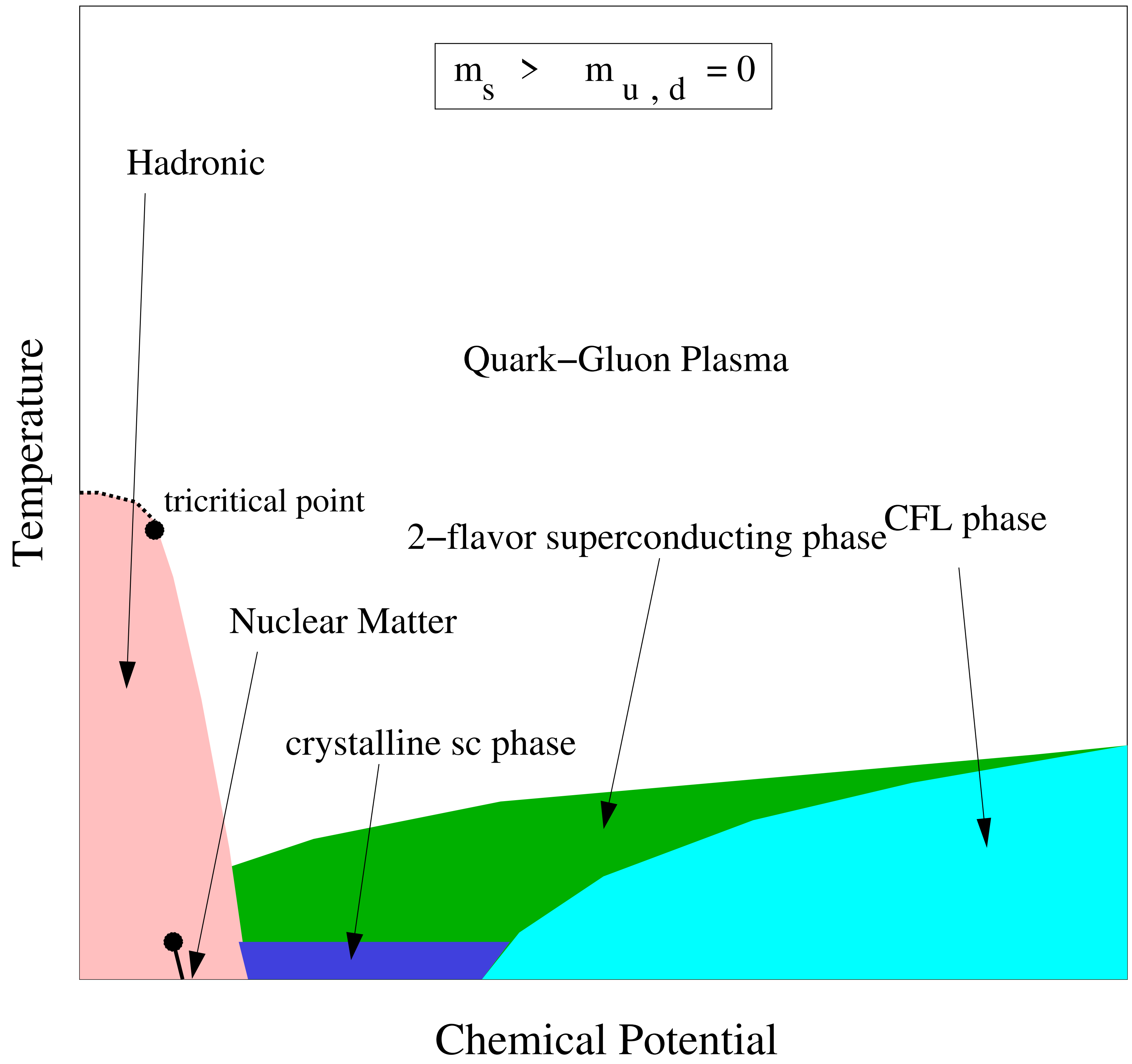
- Attraction between nucleons

  leads to nuclear superfluidity

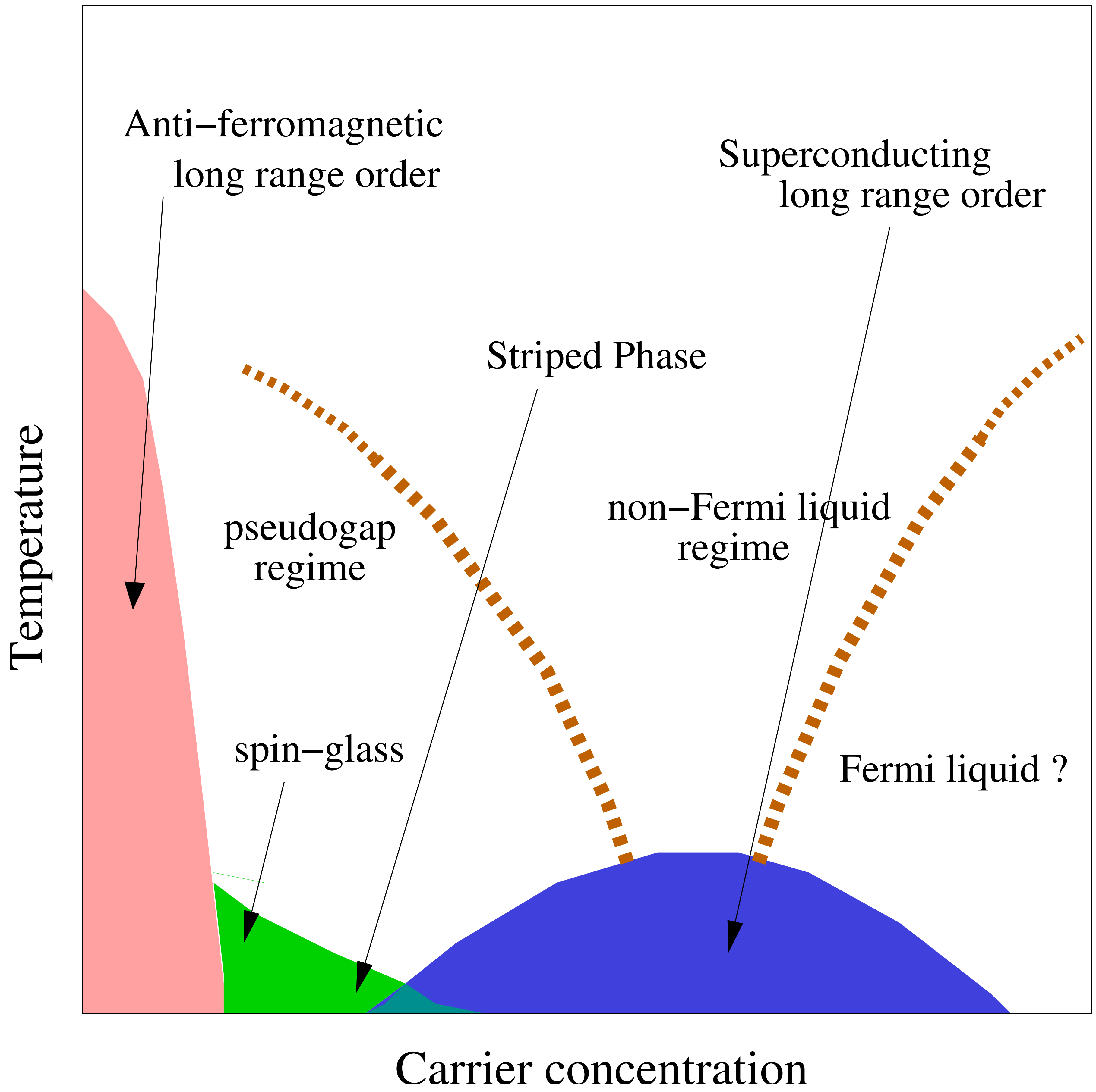
- Spontaneous breaking of translations and rotations can arise in QCD

  Example: crystalline color superconductivity

The QCD Phase Diagram



Phase Diagram of High T_c Materials



QCD vs. High T_c Superconductors

- **Similarities**

1 Both describe systems of strongly correlated fermions

2 Chiral Symmetry Breaking is analogous to Anti-Ferromagnetism

Both have long range bosonic excitations (Mesons vs. Magnons)

Fermionic excitations (nucleons vs. electrons) are gapped.

3 Finite chemical potential (or doping) is necessary to induce superconductivity.

4 BCS type pairing occurs at large chemical potential (doping).

5 Present computational techniques are insufficient at intermediate chemical potentials.

- **Differences**

1 Experimental results are more readily available in high T_c.

2 Gauge field dynamics in QCD is more complex.

3 Attraction between fermions is natural in QCD

leads to s-wave pairing instead of d-wave pairing

4 Crystal structure and inhomogeneities are natural in high T_c materials.

5 High T_c superconducting materials are two dimensional.

Models of Superconductivity

- Superconductivity arises due to attraction between
  spin-up and spin-down electrons
- The Hubbard Model

$$
\begin{aligned}
H &= \sum_{<xy>} -t(c^\dagger_{x,\uparrow} c_{y,\uparrow} + c^\dagger_{y,\uparrow} c_{x,\uparrow}) + -t(c^\dagger_{x,\downarrow} c_{y,\downarrow} + c^\dagger_{y,\downarrow} c_{x,\downarrow}) \\
&\quad + \sum_x U(n_{x,\uparrow} - 1/2)(n_{x,\downarrow} - 1/2) + \mu(n_{x,\uparrow} + n_{x,\downarrow})
\end{aligned}
$$

- $U > 0$ we get the repulsive Hubbard model
  may be a model for High T_c superconductivity
- $U < 0$ we get the attractive Hubbard model
  toy model for (s-wave) superconductivity
  forms an interesting model for Neutron Matter!

Predictions from Universality

- Pair Correlations:

$$P_x^+ = c^\dagger_{x,\uparrow} c^\dagger_{x,\downarrow}, \qquad P_x^- = c_{x,\downarrow} c_{x,\uparrow}$$

$$\langle P_x^+ P_y^- \rangle \propto \begin{cases} \mathrm{e}^{-|x-y|/\xi} & T > T_c \\ 1 \;/\; x^{\eta(T)} & T < T_c \end{cases}$$

- Predictions of Kosterlitz and Thouless

$$0 \leq \eta(T) \leq 0.25, \quad \eta(T_c) = 0.25 \quad \text{(up to log corrections)}$$

$$\chi \;=\; \frac{1}{L^2} \int d^2x \; d^2y \; \langle P_x^+ P_y^- \rangle \;=\; \begin{cases} \text{Const.} & T > T_c \\ L^{2-\eta(T)} & T < T_c \end{cases}$$

First principles calculations have not yet confirmed many universal predictions of superconductivity!

AN INVESTIGATION OF THE 2D ATTRACTIVE HUBBARD MODEL

R. Lacaze,A. Morel,B. Petersson, and J. Schröper

Published in Eur. Phys. J. **B2**, 509 (1998).

Abstract

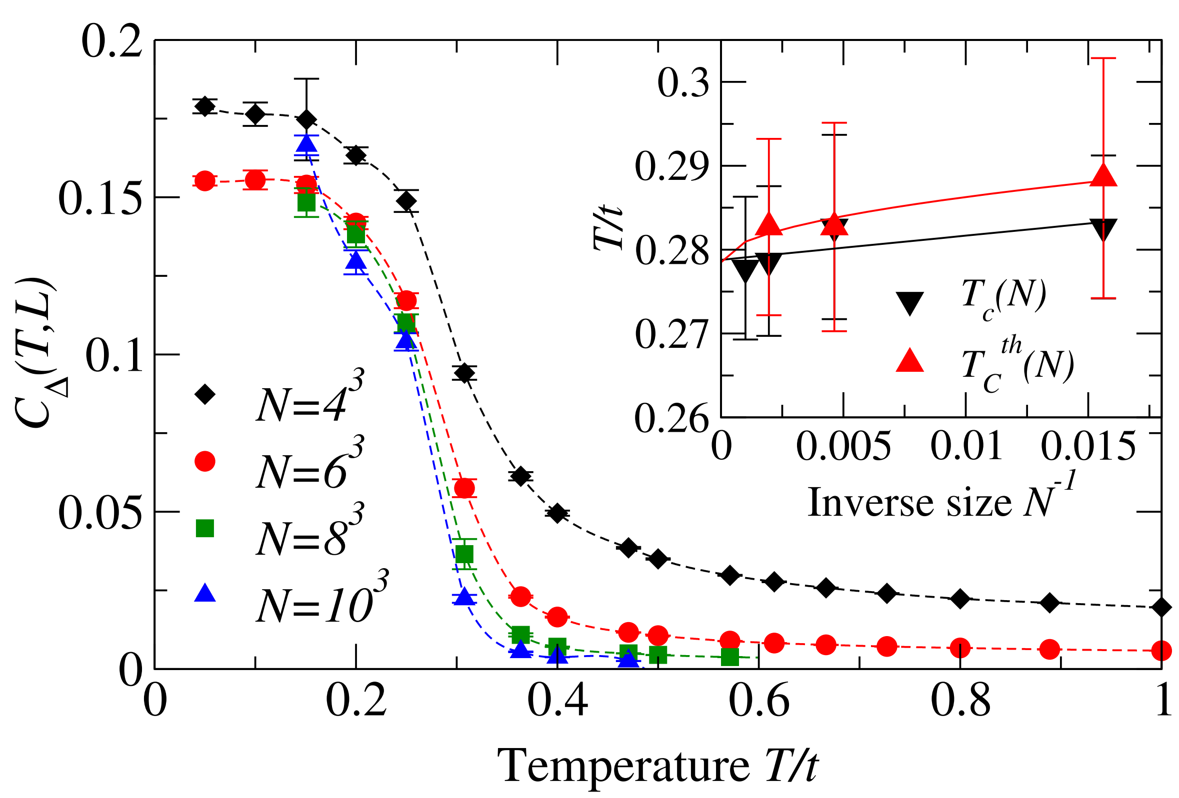
We present an investigation of the 2D attractive Hubbard model, ... We use both hybrid Monte Carlo simulations and existing hopping parameter expansions to explore the low temperature domain. The increase of the static S-wave pair correlation with decreasing temperature is analyzed in terms of an expected Kosterlitz-Thouless superconducting transition. The evidence for this transition is weak: If it exists, its temperature is very low and depends weakly on the band filling near half filling.....

- Lattice sizes at most 32×32.

QUANTUM MONTE CARLO STUDY OF THE 3D ATTRACTIVE HUBBARD MODEL

Alain Sewer,Xenophon Zotos,Hans Beck

cond-mat/0204053



- Note the lattice sizes:

$$10 \times 10 \sim 32 \times 32$$

Challenges for QCD shared by CMP

- Find computational techniques for strongly interacting fermions
- The typical partition function

$$
\begin{aligned}
Z &= \int d\phi \, d\Psi d\overline{\Psi} \; \exp\left(-S[\phi] + \overline{\Psi} M[\phi] \Psi\right) \\
  &= \int d\phi \; \exp\left(-S[\phi]\right) \; \mathrm{Det}[M(\phi)]
\end{aligned}
$$

At zero density (or half-filling) $\mathrm{Det}[M(\phi)]$ is real and positive

- Conventional (local) algorithms become inefficient

$$
N_{Flops} = 2.8 \left(\frac{\#\mathrm{Confs}}{1000}\right) \left(\frac{0.6}{M_\pi/M_\rho}\right)^6 \left(\frac{L}{3fm}\right)^5 \left(\frac{1/a}{(2GeV)}\right)^7
$$

Physical $M_\pi/M_\rho \sim 0.18$.

The current algorithms breakdown close to the chiral limit.

- At non-zero density $\mathrm{Det}[M]$ can become negative!

This leads to "Sign Problems"

The Sign Problem

- Consider the partition function

$$Z_f = \sum_{[n]} \mathrm{Sign}[n]\; W[n]$$

where $W[n] > 0$ and $\mathrm{Sign}[n] = \pm 1$. Ignoring the sign we get

$$Z_b = \sum_{[n]} \; W[n]$$

- Observables

$$\langle O \rangle = \frac{1}{Z_f} \sum_{[n]} O[n]\; \mathrm{Sign}[n]\; W[n]$$

- Brute force Monte-Carlo

$$\langle O \rangle = \frac{\sum_{[n]} O[n]\; \mathrm{Sign}[n]\; W[n]}{\sum_{[n]} \mathrm{Sign}[n]\; W[n]} \;=\; \frac{\langle O\; \mathrm{Sign}\rangle_b}{\langle \mathrm{Sign}\rangle_b}$$

- Due to severe cancellations in large volumes

$$\langle \mathrm{Sign}\rangle_b = \frac{Z_f}{Z_b} \sim \exp(-\Delta F V) \to 0$$

One needs exponentially large statistics to evaluate observables!
This is called the sign problem.

Cluster Algorithms and Sign Problems

- Cluster Algorithms: Find new ways to rewrite the partition function

$$Z = Tr[\exp(-H/T)] = \sum_{[\phi]} \det[M(\phi)] = \sum_{[\mathcal{C}]} W[\mathcal{C}]$$

ϕ local spin field but $\det[M(\phi)]$ is non-local

$\mathcal{C}$ configuration of non-local clusters but $W[\mathcal{C}]$ is local

- Usefulness of cluster variables have been demonstrated in many models:

q-state Potts Model, Swendsen and Wang, 1987

$O(N)$ classical spin Models, Wolff, 1988

$SU(2)$ quantum spin models, Evertz, Lana and Marcu , 1993

- Cluster variables lead to a new approach to solve sign problems

$O(3)$ Classical spin Model with θ vacuum, Bietenholz, et.al., 1995

certain four fermi-models, S.C and Wiese, 1999

Potts models in complex magnetic fields, Alford, et.al., 2000

strong coupling $U(1)$ gauge theories, S.C, 2001

strong coupling $U(N)$ gauge theories S.C and Adams, 2002

- Cluster algorithms lead to very efficient algorithms

  they are difficult to design.

  cluster variables encode a lot of physics of the model.

- Cluster variables possess many interesting physical properties.

  size of clusters contribute to condensates

  2-pt correlation functions get contribution from a single cluster

  cluster topology can have interesting properties

  update algorithms decorrelate faster

A Kosterlitz Thouless Transition

S.C., J. Osborn, 01′

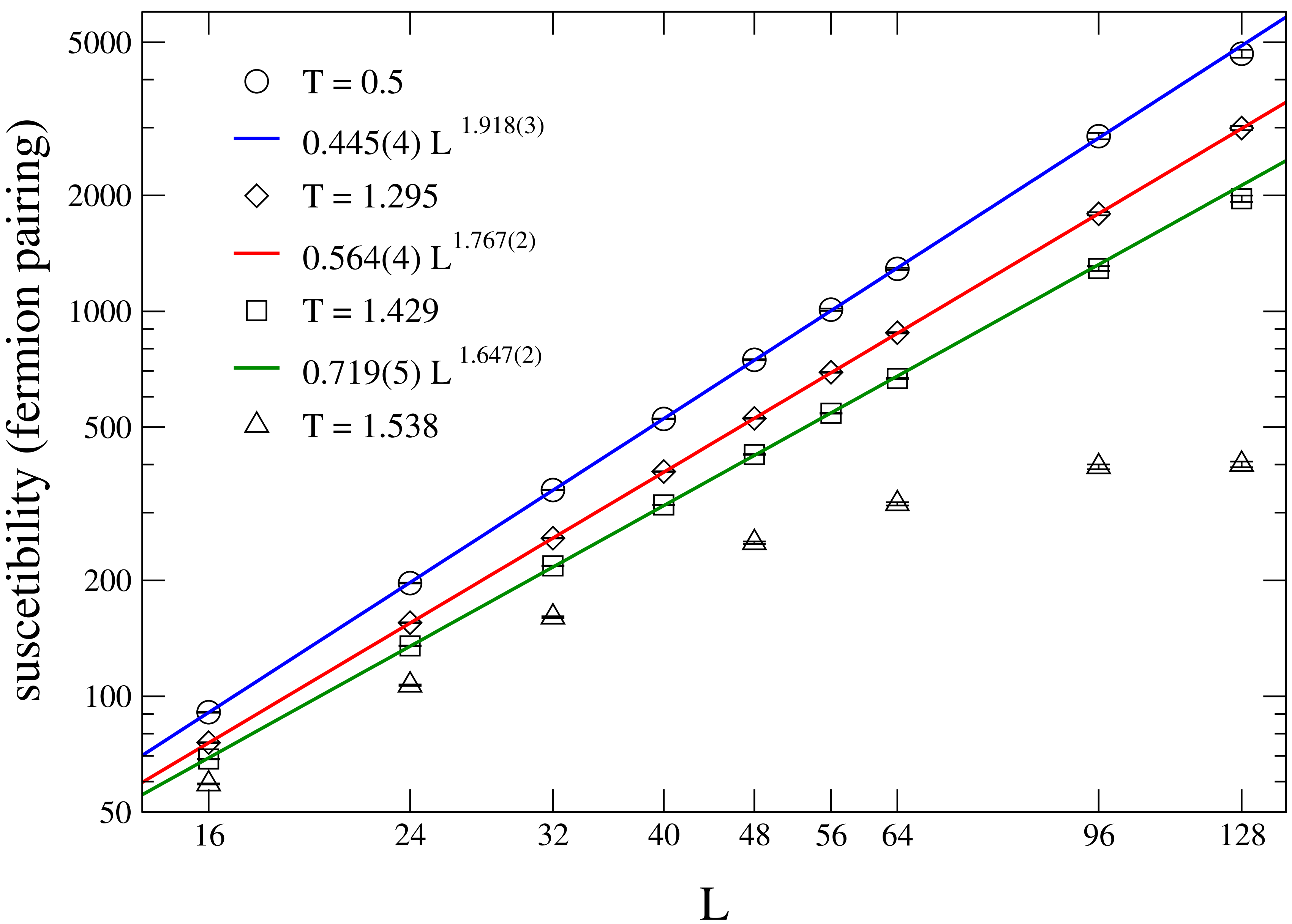
- Hamiltonian with the symmetries of the Hubbard Model

  but solvable with a meron-cluster algorithm

$$
\begin{aligned}
H &= \sum_{<xy>} \Big[\sum_{s=\uparrow,\downarrow} ({c_{x,s}}^{\dagger} c_{y,s} + {c_{y,s}}^{\dagger} c_{x,s})(1 - 3n_{xy} + n_{xy}^2) \\
&- \quad 4 \quad \left(n_{x,\uparrow} - \frac{1}{2}\right)\left(n_{x,\downarrow} - \frac{1}{2}\right)\left(n_{y,\uparrow} - \frac{1}{2}\right)\left(n_{y,\downarrow} - \frac{1}{2}\right) \\
&+ \quad 4 \quad [\vec{S}_x \cdot \vec{S}_y + \vec{J}_x \cdot \vec{J}_y - J_x^3 J_y^3]\Big] \\
&- \quad 4 \quad \sum_x \left(n_{x,\uparrow} - \frac{1}{2}\right)\left(n_{x,\downarrow} - \frac{1}{2}\right).
\end{aligned}
$$

Pair Susceptibility vs. Lattice size

$$\chi = \frac{1}{L^2} \int d^2x\, d^2y\, \langle P_x^+ P_y^- \rangle = \begin{cases} \text{Const.} & T > T_c \\ L^{2-\eta(T)} & T < T_c \end{cases}$$



A Z_2 Chiral Transition

S.C., J. Osborn, 00'

- Staggered fermion Hamiltonian

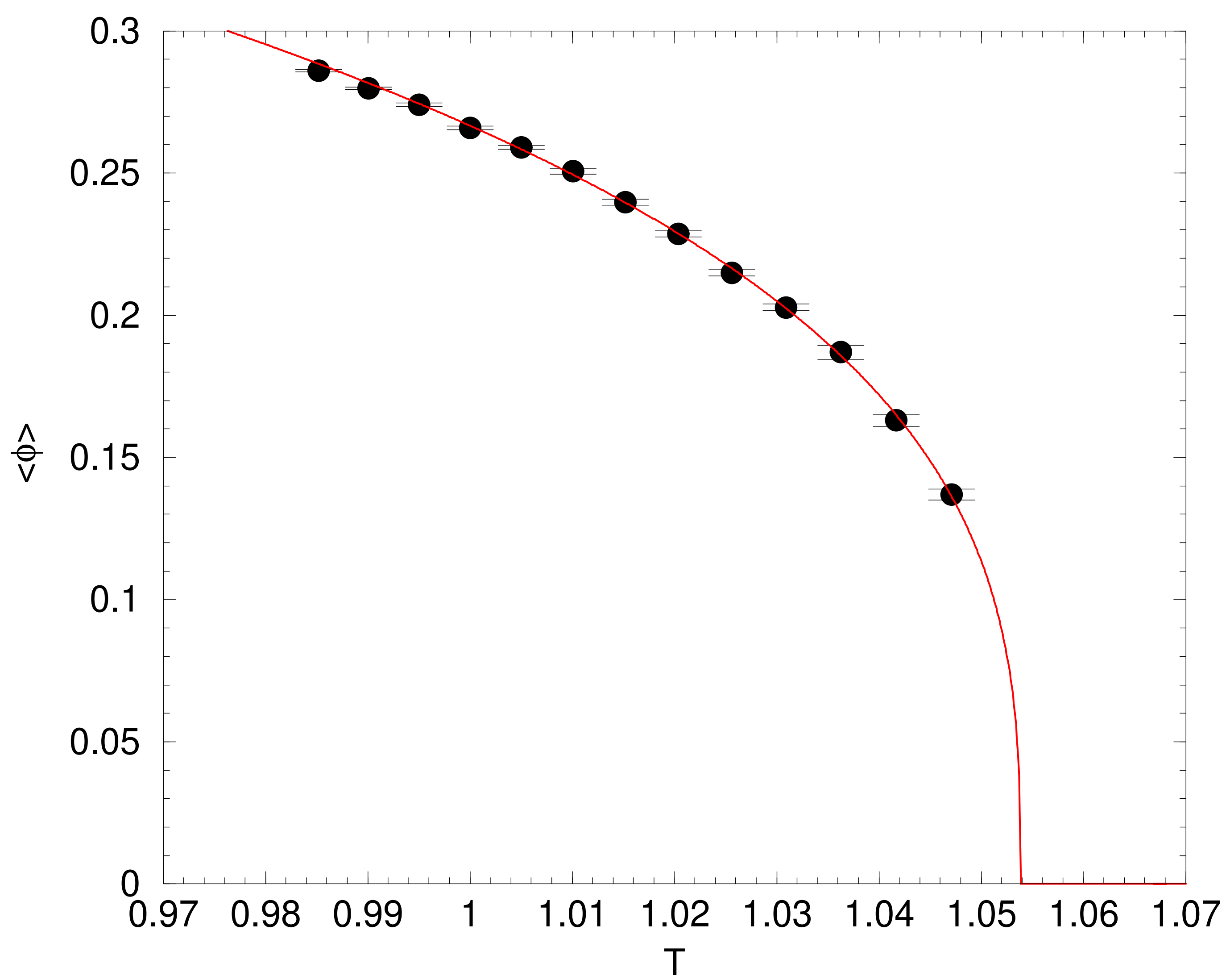
$$\begin{aligned} H &= \sum_{<xy>} \frac{-i\eta_{xy}}{2} \left(c_x^\dagger c_y - c_y^\dagger c_x \right) \\ &\quad + \left(n_x - \frac{1}{2} \right) \left(n_y - \frac{1}{2} \right). \end{aligned}$$

- The Hamiltonian is invariant under a chiral Z_2 transformation.
- The corresponding chiral condensate

$$\langle \phi \rangle = \left\langle \left(n_x - \frac{1}{2} \right) \right\rangle$$

Chiral Condensate vs. Temperature

$$\langle\phi\rangle \;=\; \begin{cases} 0 & T > T_c \\ (T_c - T)^{\beta} & T < T_c \end{cases} ,$$



β (3d Ising Model) $= 0.324(1)$

β (fit) $= 0.314(6)$

Strong Coupling QCD with Staggered Fermions

[Karsch and Mutter, 1989]

- Partition function in the chiral limit

$$Z = \int dU \; d\Psi d\overline{\Psi} \; \exp\left(\overline{\Psi} M[U] \Psi\right) = \int d\Psi d\overline{\Psi} \prod_{<xy>} F(x,y)$$

- $F(x,y)$ can be written in terms of Meson and Baryon fields

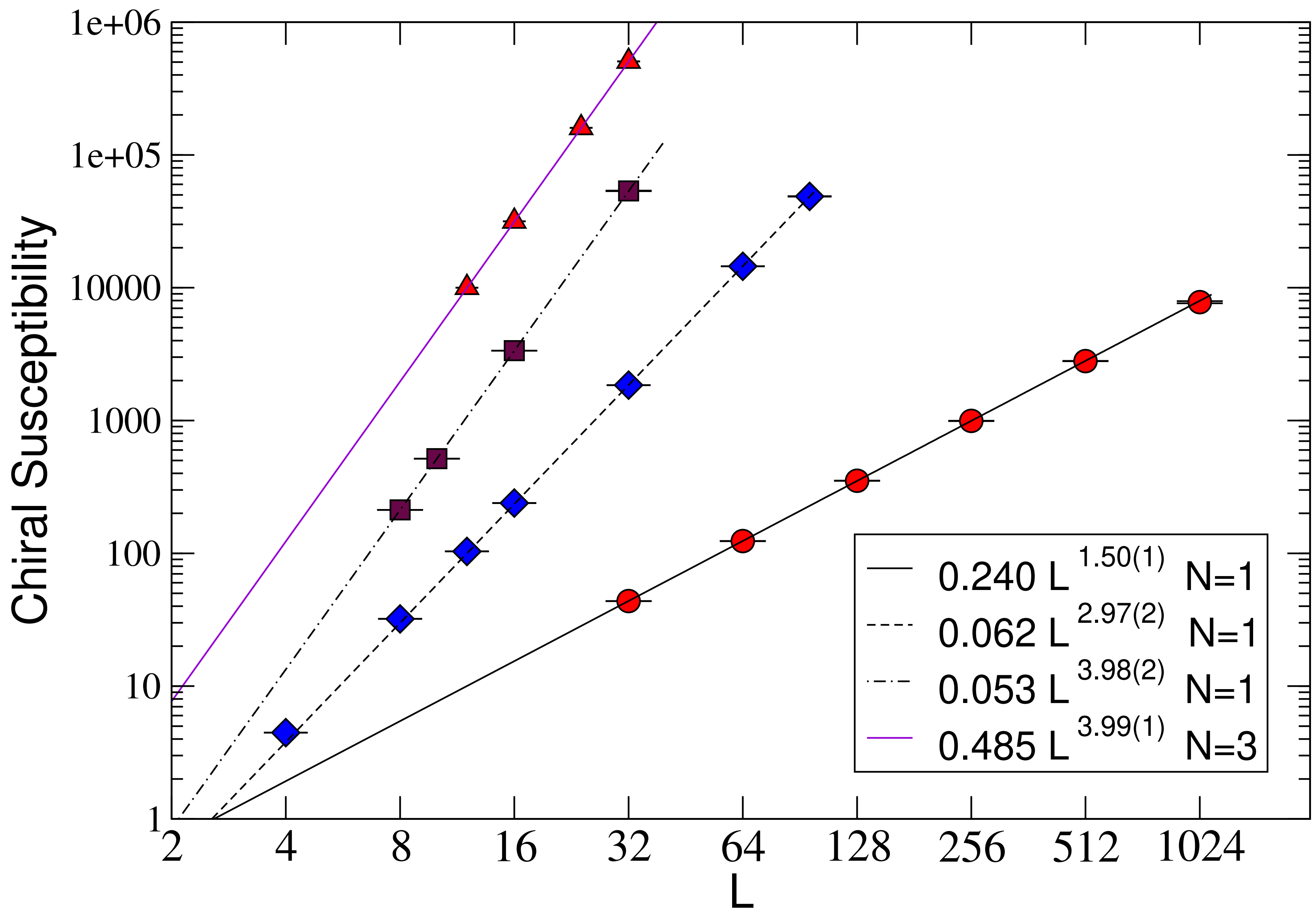
- Chiral Symmetry breaking must lead to

$$\chi = \frac{1}{L^d} \sum_{x,y} \left\langle \overline{\Psi}(x)\Psi(x) \;\; \overline{\Psi}(y)\Psi(y) \right\rangle \sim L^d$$

This observable gets contributions form a disconnected diagram hence it is impossible to calculate it with controlled errors with conventional algorithms.

Chiral Susceptibility vs. Lattice size

[S.C, D. Adams, '02]



Chiral Symmetry breaks in $d = 3, 4$ but not in $d = 2$
consistent with Mermin-Wagner theorem.

Conclusions

- Many ideas have recently emerged in QCD that owe their origin to condensed matter physics

  novel phases, scales, symmetries are continuing to be observed experimentally in CMP. Will they have analogies in QCD?

- Problems in performing calculations in QCD are common to CMP.

  Novel algorithms are necessary to make progress. Some new directions have emerged in the last few years.

- Strongly correlated fermionic many body theory is in its infancy.

  "Precise" numerical experiments on models can teach us a lot