1. (a) • $\sigma_{xx} = \sigma_0, \quad \sigma_{yy} = \nu\sigma_0, \quad \sigma_{ij} = 0$ for other i and j. (Before yielding)

$$\therefore [\sigma] = \begin{bmatrix} \sigma_0 & 0 & 0 \\ 0 & \nu\sigma_0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The principal stresses are: $\sigma_1 = \sigma_0, \quad \sigma_2 = \nu\sigma_0, \quad \sigma_3 = 0$

• Mises initial yield criterion:

$$(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2 - 2\sigma_Y^2 = 0$$

$$\therefore (\sigma_0 - \nu\sigma_0)^2 + (\nu\sigma_0 - 0)^2 + (0-\sigma_0)^2 - 2\sigma_Y^2 = 0$$

$$\therefore \sigma_0^2(1-2\nu+\nu^2) + \nu^2\sigma_0^2 + \sigma_0^2 - 2\sigma_Y^2 = 0$$

$$\therefore \sigma_0^2(\cancel{2} - \cancel{2}\nu + \cancel{2}\nu^2) - \cancel{2}\sigma_Y^2 = 0$$

$$\therefore \sigma_0 = \frac{\sigma_Y}{\sqrt{1-\nu+\nu^2}}$$

(b) $$[\sigma] = \begin{bmatrix} \sigma_{xx} & 0 & 0 \\ 0 & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ (After yielding)

$$\Rightarrow \tfrac{1}{3}\,\mathrm{tr}\,[\sigma] = \tfrac{1}{3}(\sigma_{xx}+\sigma_{yy})$$

$$[\sigma]' = [\sigma] - \left(\tfrac{1}{3}\,\mathrm{tr}\,[\sigma]\right)[1]$$

$$= \begin{bmatrix} \sigma_{xx} - \frac{1}{3}(\sigma_{xx}+\sigma_{yy}) & 0 & 0 \\ 0 & \sigma_{yy} - \frac{1}{3}(\sigma_{xx}+\sigma_{yy}) & 0 \\ 0 & 0 & 0 - \frac{1}{3}(\sigma_{xx}+\sigma_{yy}) \end{bmatrix} = \begin{bmatrix} \frac{2\sigma_{xx}-\sigma_{yy}}{3} & 0 & 0 \\ 0 & \frac{2\sigma_{yy}-\sigma_{xx}}{3} & 0 \\ 0 & 0 & -\frac{\sigma_{xx}+\sigma_{yy}}{3} \end{bmatrix}$$

(c) Plastic part of Prandtl-Reuss relations:

$$\dot\epsilon^p_{kk} = 0, \quad \dot\epsilon^{p\prime}_{ij} = \frac{3}{2}\frac{\dot\epsilon^p_{ep}}{\sigma_Y}\sigma'_{ij} \quad \Rightarrow \quad \dot\epsilon^p_{ij} = \frac{3}{2}\frac{\dot\epsilon^p_{ep}}{\sigma_Y}\sigma'_{ij}$$

(i) yy-component: $\dot\epsilon^p_{yy} = \frac{3}{2}\frac{\dot\epsilon^p_{ep}}{\sigma_Y}\sigma'_{yy} \quad \Rightarrow \quad \dot\epsilon^p_{ep} = \frac{2\sigma_Y}{3}\frac{\dot\epsilon^p_{yy}}{\sigma'_{yy}}$

Using (1), $\sigma'_{yy} = \frac{2\sigma_{yy}-\sigma_{xx}}{3} = \frac{1}{3}\left[\cancel{2}\left(\frac{2A-3}{\cancel{4}A_2}\right)-1\right]\sigma_{xx} = \frac{1}{3}\left[\frac{\cancel{2A}-3-\cancel{2A}}{2A}\right]\sigma_{xx} = -\frac{1}{2A}\sigma_{xx}$

Using (2) and (3), $\dot\epsilon^p_{yy} = -\frac{1}{E}(\dot\sigma_{yy} - \nu\dot\sigma_{xx}) = -\frac{1}{E}\left[\left(\frac{1}{2}+A\right)-\nu\right]\dot\sigma_{xx} = -\frac{1}{E}\left[\frac{1-2\nu}{2}+A\right]\dot\sigma_{xx}$

Then

$$\dot\epsilon^p_{ep} = \frac{2\sigma_Y}{3}\frac{\dot\epsilon^p_{yy}}{\sigma'_{yy}} = \frac{2\sigma_Y}{3}\,\frac{\frac{1}{E}\left[\frac{1-2\nu}{2}+A\right]\dot\sigma_{xx}}{\frac{1}{2A}\sigma_{xx}} = \frac{4\sigma_Y}{3E}\left[\left(\frac{1-2\nu}{2}\right)A + A^2\right]\frac{\dot\sigma_{xx}}{\sigma_{xx}}$$

(ii) xx – component : $\dot{\epsilon}^{p}_{xx} = \frac{3}{2\sigma_Y} \dot{\epsilon}^{p}_{eq} \sigma'_{xx}$

Using (1), $\sigma'_{xx} = \frac{2\sigma_{xx} - \sigma_{yy}}{3} = \frac{1}{3}\left[2 - \frac{2A-3}{4A}\right]\sigma_{xx} = \frac{1}{3}\,\frac{8A - 2A + 3}{4A}\sigma_{xx} = \frac{6A+3}{12A}\sigma_{xx}$

$$= \frac{2A+1}{4A}\sigma_{xx}$$

Then,

$$\dot{\epsilon}^{p}_{xx} = \frac{3}{2\sigma_Y} \dot{\epsilon}^{p}_{eq} \sigma'_{xx}$$

$$= \frac{3}{2\sigma_Y}\frac{4\sigma_Y}{3E}\left[\left(\frac{1-2\nu}{2}\right)A + A^2\right]\frac{\dot{\sigma}_{xx}}{\sigma_{xx}}\left(\frac{2A+1}{4A}\right)\sigma_{xx}$$

$$= \frac{1}{2E}\left[\left(\frac{1-2\nu}{2}\right) + A\right](2A+1)\,\dot{\sigma}_{xx}$$

$$= \frac{1}{2E}\left[(1-2\nu)A + 2A^2 + \frac{1-2\nu}{2} + A\right]\dot{\sigma}_{xx}$$

$$= \frac{1}{2E}\left[\frac{1-2\nu}{2} + (1-2\nu+1)A + 2A^2\right]\dot{\sigma}_{xx}$$

$$= \frac{1}{2E}\left[\left(\frac{1-2\nu}{2}\right) + 2(1-\nu)A + 2A^2\right]\dot{\sigma}_{xx}$$

$$= \frac{1}{E}\left[\left(\frac{1-2\nu}{4}\right) + (1-\nu)A + A^2\right]\dot{\sigma}_{xx}$$

2. $v_x = \dfrac{V_0 l}{[(l-x)+(h_2/h_1)x]}$, $v_y = -\dfrac{V_0[1-(h_2/h_1)]l}{[(l-x)+(h_2/h_1)x]^2}\,y$, $v_z = 0$

(a) $$\dot\epsilon_{xx} = \frac{\partial v_x}{\partial x} = V_0 l\left\{-\frac{(-1+h_2/h_1)}{[(l-x)+(h_2/h_1)x]^2}\right\} = \frac{V_0 l(1-h_2/h_1)}{[(l-x)+(h_2/h_1)x]^2}$$

$$\dot\epsilon_{yy} = \frac{\partial v_y}{\partial y} = -\frac{V_0[1-(h_2/h_1)]l}{[(l-x)+(h_2/h_1)x]^2} = -\dot\epsilon_{xx}$$

$$\dot\epsilon_{zz} = \frac{\partial v_z}{\partial z} = 0$$

$$\dot\epsilon_{xy} = \frac{1}{2}\left(\cancel{\frac{\partial v_x}{\partial y}}^{\,0} + \frac{\partial v_y}{\partial x}\right) = \cancel{\tfrac{1}{2}}(-)V_0[1-(h_2/h_1)]l\,y\left\{-\cancel{2}\frac{(-1+h_2/h_1)}{[(l-x)+(h_2/h_1)x]^3}\right\}$$

$$= -\frac{V_0[1-(h_2/h_1)]^2 l y}{[(l-x)+(h_2/h_1)x]^3}$$

$$\dot\epsilon_{yz} = \frac{1}{2}\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right) = 0, \qquad \dot\epsilon_{zx} = \frac{1}{2}\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right) = 0$$

(b) $\dot\epsilon^p_{ij} = \dot\epsilon_{ij}$

(i) Along the path $y=0$, $\dot\epsilon^p_{xx} = \dfrac{V_0 l(1-h_2/h_1)}{[(l-x)+(h_2/h_1)x]^2} = -\dot\epsilon^p_{yy}$

$\dot\epsilon^p_{xy} = 0$ (for $y=0$) , $\dot\epsilon^p_{zz} = \dot\epsilon^p_{yz} = \dot\epsilon^p_{zx} = 0$

$$\therefore\ \dot\epsilon^p_{eq} = \sqrt{2/3\,(\dot\epsilon^{p\,2}_{xx} + \dot\epsilon^{p\,2}_{yy})} = \sqrt{2/3\,(\dot\epsilon^{p\,2}_{xx} + \dot\epsilon^{p\,2}_{xx})} = \frac{2}{\sqrt3}\dot\epsilon^p_{xx} = \frac{2}{\sqrt3}\frac{V_0 l(1-h_2/h_1)}{[(l-x)+(h_2/h_1)x]^2}$$

(ii) Along the path $y=0$, at $x=l$:

$$\epsilon^p_{eq} = \int_0^l \dot\epsilon^p_{eq}\,dt = \int_0^l \dot\epsilon^p_{eq}\frac{dx}{v_x} = \int_0^l \frac{2}{\sqrt3}\frac{V_0 l(1-h_2/h_1)}{[(l-x)+(h_2/h_1)x]^{2}}\frac{[(l-x)+(h_2/h_1)x]}{V_0 l}\,dx$$

$$= \frac{2}{\sqrt3}(1-h_2/h_1)\int_0^l \frac{dx}{[(l-x)+(h_2/h_1)x]}$$

Change of variable: $\xi = (l-x)+(h_2/h_1)x \Rightarrow d\xi = (-1+h_2/h_1)dx = -(1-h_2/h_1)dx$

Limits: $x=0 \Rightarrow \xi = l$, $x=l \Rightarrow \xi = (h_2/h_1)l$

$$\therefore\ \epsilon^p_{eq} = \frac{2}{\sqrt3}(1-h_2/h_1)\int_0^l \frac{dx}{[(l-x)+(h_2/h_1)x]} = \frac{2}{\sqrt3}(1-h_2/h_1)\int_l^{(h_2/h_1)l}\frac{1}{\xi}\,\frac{-d\xi}{(1-h_2/h_1)}$$

$$= \frac{2}{\sqrt3}\int_{(h_2/h_1)l}^{l}\frac{d\xi}{\xi}$$ (Limits interchanged ∴ − sign becomes +)

$$= \frac{2}{\sqrt3}\ln\xi\,\Big|_{(h_2/h_1)l}^{l}$$

$$= \frac{2}{\sqrt3}\ln\frac{\cancel{l}}{(h_2/h_1)\cancel{l}} = \frac{2}{\sqrt3}\ln\frac{h_1}{h_2}$$