2

Review of Stress, Linear Strain and Elastic Stress-Strain Relations

2.1 Introduction

In metal forming and machining processes, the work piece is subjected to external forces in order to achieve a certain desired shape. Under the action of these forces, the work piece undergoes *displacements* and deformation and develops internal forces. A measure of deformation is defined as *strain*. The intensity of internal forces is called as *stress*. The displacements, strains and stresses in a deformable body are interlinked. Additionally, they all depend on the geometry and material of the work piece, external forces and supports. Therefore, to estimate the external forces required for achieving the desired shape, one needs to determine the displacements, strains and stresses in the work piece. This involves solving the following set of governing equations : (i) strain-displacement relations, (ii) stress-strain relations and (iii) equations of motion.

In this chapter, we develop the governing equations for the case of *small deformation of linearly elastic materials*. While developing these equations, we disregard the molecular structure of the material and assume the body to be a *continuum*. This enables us to define the displacements, strains and stresses at every point of the body.

We begin our discussion on governing equations with the concept of stress at a point. Then, we carry out the analysis of stress at a point to develop the ideas of stress invariants, principal stresses, maximum shear stress, octahedral stresses and the hydrostatic and deviatoric parts of stress. These ideas will be used in the next chapter to develop the theory of plasticity. Next, we discuss the conditions which the principle of balance of linear momentum places on the derivatives of the stress components. These conditions lead to *equations of motion*. The concept of *linear strain*, which is a measure of small deformation, is discussed next. For the linear strain, the *strain-displacement relations* are linear. The linear strain measure is not directly useful in the analysis of plastic deformation, but it does provide a qualitative understanding of the deformation in solid bodies. We can draw upon it to develop a measure for large deformation which is to be used in the theory of plasticity. The analysis of linear strain at a point, similar to the analysis of stress at a point, is also carried out to develop the ideas of strain invariants, principal strains, maximum shear, volumetric strain and the hydrostatic and deviatoric parts of strain. Finally, the *stress-strain relations for small deformation of linearly elastic materials* are developed. Even though these relations are not directly useful for analyzing plastic behavior, their development provides a methodology of expressing qualitative material behavior into quantitative form. This will be useful for developing the plastic stress-strain relations in the next chapter.

Since the stress and strain at a point are *tensor* quantities, a simple definition of tensors involving transformation of components with respect to two Cartesian coordinate systems is provided. Essential elements of tensor algebra and calculus needed to develop the governing equations are discussed. For more elaborate definitions of tensor and for more details of tensor algebra and calculus, the reader is advised to refer to other books. There are quite a few well-written books on these topics like those by Jaunzemis [1], Malvern [2], Fung [3], Sokolnikoff [4] *etc*.

Both tensor and vector quantities are denoted by bold-face letters. Whether the quantity is a tensor or a vector can be understood from the context. Some tensor quantities, like the displacement gradient tensor, involve the use of symbol like the capital Greek letter delta. Most tensors used in the book are of second order. However, for brevity, the adjective "second order" is dropped. Thus, the word tensor without any qualifier means second order tensor. Higher order tensors are referred by their order. For example, the tensor relating stress and strain tensors in the stress-strain relations is of fourth order and is referred as such. The governing equations and some intermediate equations are expressed in *tensor notation*. This is done to emphasize the fact that these equations have a form which is independent of the coordinate system. However, while doing calculations, one needs a form of these equations which depends on the coordinate system being used. *Index notation* and the associated *summation convention* are useful for writing the component form of these equations in a condensed fashion. Since the reader is not expected to be familiar with the index notation and summation convention, both are discussed at length right in the beginning. Sometimes, for calculation purpose, an *array notation* is useful for writing the component form of these equations. This involves knowledge of matrix algebra. It is expected that the reader will have sufficient background in the matrix algebra and the associated array notation. Wherever possible, the equations are expressed in all the three notations: tensor, index and array notations. The calculations are carried out either in index notation or in array notation depending on the convenience of the situation.

The organization of this chapter is as follows. In Section 2.2, we introduce the index notation and summation convention. The idea of stress at point is developed in Section 2.3. Further, the analysis of stress at a point is also carried out. Equations of motion involving the derivatives of stress components are also presented in this section. The concept of linear strain tensor and associated strain-displacement relations are developed in Section 2.4. Additionally, analysis of the linear strain tensor and compatibility conditions for the strain components are also discussed in Section 2.4. Section 2.5 is devoted to the development of stress-strain relations for small deformation of linearly elastic materials. Finally, the whole chapter is summarized in Section 2.6. Worked out examples are provided at the end of Sections 2.2–2.5 to elaborate the concepts discussed in that section.

2.2 Index Notation and Summation Convention

In modeling of manufacturing processes, we encounter physical quantities in the form of scalars, vectors and tensors. (In this book, a tensor means the tensor of order two unless stated otherwise). Definition of a tensor is provided in Section 2.3. In 3-dimensional space, a vector has 3 components and tensor has 9 components. The *index notation* can be employed to represent these components as well as expressions and equations involving scalars, vectors and tensors. In the index notation, the coordinate axes (*x,y,z*) are labeled as (1,2,3). Thus, to represent a velocity vector (v_x, v_y, v_z), we use the notation v_i, where it is implied that the index *i* takes the values 1, 2 and 3 in a 3-dimensional space. In a 2-dimensional space, it will take the values 1 and 2. Similarly, the notation I_{ij} with the indices i and j is used to represent the following 9 components of an inertia tensor: $I_{xx}, I_{xy}, I_{xz}, I_{yx}, I_{yy}, I_{yz}, I_{zx}, I_{zy}, I_{zz}$.

Einstein's *summation convention* is employed for writing the sum of various terms in a condensed form. In this convention, if an index occurs twice in a term, then the term represents the sum of all the terms involving all possible values of the index. For example, $a_i b_i$ means $a_1 b_1 + a_2 b_2 + a_3 b_3$ in a 3-dimensional space. Similarly, I_{ii} means $I_{11} + I_{22} + I_{33}$. The repeated index is called *dummy index*, while the non-repeated index is called *free index*. Thus, in the term $c_{ij} b_j$, i is a free index and j is a dummy index. Any symbol can be used for a dummy index. Therefore, the expression $c_{ij} b_j$ can also be written as $c_{ik} b_k$. When there are two dummy indices, it means the sum over both. Thus in 3-dimensions, it will contain 9 terms. As an example, the term $p_{ij} q_{ij}$ means $p_{11}q_{11} + p_{12}q_{12} + p_{13}q_{13} + p_{21}q_{21} + p_{22}q_{22} + p_{23}q_{23} + p_{31}q_{31} + p_{32}q_{32} + p_{33}q_{33}$. If an index is repeated more than twice, then it is an *invalid* expression. An expression or equation containing no free index represents a scalar expression or scalar equation. Similarly, an expression or equation containing one free index denotes a vector expression or equation. An expression or equation containing two free indices represents a tensor expression or equation. As an example, the term I_{ii} represents a scalar, the term $c_{ij} b_j$ containing the free index i represents a vector while the term $p_{ij}\, q_{jk}$ containing the free indices i and k represents a tensor. Similarly, the equation

$$a_i b_i = d \,, \tag{2.1}$$

represents a scalar equation. Further, the equations

$$\sigma_{ij} n_j = t_i, \quad \text{(free index } i \text{, dummy index } j\text{)}, \tag{2.2}$$

$$p_{ij} = q_{ik} r_{kj}, \quad \text{(free indices } i \text{ and } j\text{, dummy index } k\text{)} \tag{2.3}$$

denote vector and tensor equations respectively. In an equation, all the terms should have the same number of free indices. Further, the notation for free indices should be the same in all the terms. Thus, the equations

$$I_{ii} = a_j, \quad \text{(no free index on left side)} \tag{2.4}$$

and

$$p_{ij} = q_{kl}, \text{(the two free indices have different notation on two sides)} \tag{2.5}$$

are *invalid* expressions.

***Example* 2.1:** Expand the following expression:

$$t_i = \sigma_{ij} n_j. \tag{2.6}$$

***Solution*:** This is a vector equation as there is only one free index, namely i, on each side of the equation. Dummy index j on the left side indicates that it is a sum of three terms. Expanding this sum, the equation becomes :

$$t_i = \sigma_{i1} n_1 + \sigma_{i2} n_2 + \sigma_{i3} n_3. \tag{2.7}$$

Now, since i is a free index and takes the values 1, 2 and 3, the above vector equation actually represents the following 3 scalar equations:

$$\begin{aligned} t_1 &= \sigma_{11} n_1 + \sigma_{12} n_2 + \sigma_{13} n_3, \\ t_2 &= \sigma_{21} n_1 + \sigma_{22} n_2 + \sigma_{23} n_3, \\ t_3 &= \sigma_{31} n_1 + \sigma_{32} n_2 + \sigma_{33} n_3. \end{aligned} \tag{2.8}$$

***Example* 2.2:** Write in index notation the following expression:

$$\begin{aligned} \sigma_n = {} & \sigma_{11} n_1^2 + \sigma_{22} n_2^2 + \sigma_{33} n_3^2 + (\sigma_{12} + \sigma_{21}) n_1 n_2 + (\sigma_{23} + \sigma_{32}) n_2 n_3 \\ & + (\sigma_{31} + \sigma_{13}) n_3 n_1. \end{aligned} \tag{2.9}$$

***Solution*:** Note that there are 9 terms. Therefore, the index notation must involve two dummy indices. In order to write the above equation in terms of the dummy indices, we rearrange the right side as follows:

$$\sigma_n = (\sigma_{11}\, n_1\, n_1 + \sigma_{12}\, n_1\, n_2 + \sigma_{13}\, n_1\, n_3) + (\sigma_{21}\, n_2\, n_1 + \sigma_{22}\, n_2\, n_2 + \sigma_{23}\, n_2\, n_3) + (\sigma_{31}\, n_3\, n_1 + \sigma_{32}\, n_3\, n_2 + \sigma_{33}\, n_3\, n_3). \tag{2.10}$$

Note that in each parenthesis, there is a sum over the second index of σ and the index of second n. This sum can be expressed using a dummy index which we denote by j. Then, the above expression becomes:

$$\sigma_n = \sigma_{1j} n_1 n_j + \sigma_{2j} n_2 n_j + \sigma_{3j} n_3 n_j . \tag{2.11}$$

Now, there is a sum over the first index of σ and the index of first n. We express this sum using another dummy index which we denote by i. Thus, the final expression in terms of the index notation can be written as:

$$\sigma_n = \sigma_{ij} n_i n_j . \tag{2.12}$$

Note that, as stated earlier, the symbols for the dummy indices can be different than i and j.

Two symbols often used to simplify and shorten expressions in index notation are *Kronecker's delta* and *permutation symbol*. The Kronecker's delta is defined by

$$\begin{aligned} \delta_{ij} &= 1 \quad \text{if} \quad i = j, \\ &= 0 \quad \text{if} \quad i \neq j. \end{aligned} \tag{2.13}$$

The permutation symbol is defined by

$$\begin{aligned} \in_{ijk} &= \;\; 0 \quad \text{if two or more indices are equal,} \\ &= +1 \quad \text{if } \left(i, j, k\right) \text{ are even permutations of } \left(1,2,3\right), \\ &= -1 \quad \text{if } \left(i, j, k\right) \text{ are odd permutations of } \left(1,2,3\right). \end{aligned} \tag{2.14}$$

The δ and $\in$ satisfy the following identities:

$$a_i \delta_{ij} = a_j , \quad A_{ij} \delta_{jk} = A_{ik} , \quad \delta_{ij} \delta_{jk} = \delta_{ik} , \tag{2.15}$$

$$\in_{ijk} \in_{pqr} = \delta_{ip} \left(\delta_{jq} \delta_{kr} - \delta_{jr} \delta_{kq} \right) + \delta_{iq} \left(\delta_{jr} \delta_{kp} - \delta_{jp} \delta_{kr} \right) + \delta_{ir} \left(\delta_{jp} \delta_{kq} - \delta_{jq} \delta_{kp} \right) . \tag{2.16}$$

***Example* 2.3:** Expand the following expressions:

(a) $c = a_i b_j \delta_{ij}$. (2.17)

(b) $\boldsymbol{d} = \in_{ijk} \hat{\boldsymbol{i}}_i a_j b_k$. (2.18)

***Solution*:** (a) This is a scalar equation involving two dummy indices i and j. Thus, it involves a sum of 9 terms. First expanding the sum over i, we get the following three terms on the left side of Eq. (2.17)

$$c = a_1 b_j \delta_{1j} + a_2 b_j \delta_{2j} + a_3 b_j \delta_{3j}. \tag{2.19}$$

Now, while expanding the sum over j in each of the three terms, we use Eq. (2.13) to substitute the values of δ. Since the value of δ is zero when its two indices are different, we get only one non-zero term in each expansion over j. Thus, the final expanded expression becomes

$$c = a_1 b_1 + a_2 b_2 + a_3 b_3. \tag{2.20}$$

Note that the expression on the right side of Eq. (2.20) is the expansion of $a_i b_i$ Thus, we get an identity

$$a_i b_j \delta_{ij} = a_i b_i. \tag{2.21}$$

(b) This is a vector equation involving 3 dummy indices. Therefore, it is a sum of 27 terms. However, the value of the permutation symbol $\in$ is zero when two of its indices are equal. Therefore, 21 terms are zero. The expansion with the remaining 6 non-zero terms is

$$\begin{aligned} \boldsymbol{d} &= \in_{123} \hat{\boldsymbol{i}}_{\mathbf{1}} a_2 b_3 + \in_{132} \hat{\boldsymbol{i}}_{\mathbf{1}} a_3 b_2 + \in_{231} \hat{\boldsymbol{i}}_{\mathbf{2}} a_3 b_1 + \in_{213} \hat{\boldsymbol{i}}_{\mathbf{2}} a_1 b_3 \\ &+ \in_{312} \hat{\boldsymbol{i}}_{\mathbf{3}} a_1 b_2 + \in_{321} \hat{\boldsymbol{i}}_{\mathbf{3}} a_2 b_1. \end{aligned} \tag{2.22}$$

Now, we use Eq. (2.14) to substitute the values of the permutation symbol. Then, we get:

$$\boldsymbol{d} = \hat{\boldsymbol{i}}_{\mathbf{1}}(a_2 b_3 - a_3 b_2) + \hat{\boldsymbol{i}}_{\mathbf{2}}(a_3 b_1 - a_1 b_3) + \hat{\boldsymbol{i}}_{\mathbf{3}}(a_1 b_2 - a_2 b_1). \tag{2.23}$$

Note that the expression on the right side is the cross product of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. Thus, we can write

$$\boldsymbol{a} \times \boldsymbol{b} = \in_{ijk} \hat{\boldsymbol{i}}_{\boldsymbol{i}} a_j b_k. \tag{2.24}$$

***Example* 2.4:** Determinant of a matrix $[A]$ is defined by

$$\det[A] = \frac{1}{6} \in_{lmn} \in_{xyz} A_{lx} A_{my} A_{nz}. \tag{2.25}$$

There are following constraints on the components of $[A]$.

(i) The matrix $[A]$ is symmetric, that is, its non-diagonal components satisfy the relation:

$$A_{ij} = A_{ji}. \tag{2.26}$$

(ii) Further, the sum of the diagonal components is zero.

$$A_{kk} = 0\,. \tag{2.27}$$

Using the above constraints, show that the expression for the determinant (Eq. 2.25) reduces to

$$\det[A] = \frac{1}{3} A_{lm} A_{mn} A_{nl}\,. \tag{2.28}$$

***Solution*:** Using the identity (2.16), the determinant of $[A]$ (Eq. 2.25) can be expressed in terms of δ :

$$\begin{aligned}\det[A] = \frac{1}{6}[&\delta_{lx}(\delta_{my}\delta_{nz} - \delta_{mz}\delta_{ny}) + \delta_{ly}(\delta_{mz}\delta_{nx} - \delta_{mx}\delta_{nz}) \\ &+ \delta_{lz}(\delta_{mx}\delta_{ny} - \delta_{my}\delta_{nx})]A_{lx}A_{my}A_{nz}.\end{aligned} \tag{2.29}$$

The above expression can be modified using the identity (2.15) in each of the 6 terms

$$\begin{aligned}\det[A] = \frac{1}{6}(&A_{ll}A_{mm}A_{nn} - A_{ll}A_{mn}A_{nm} + A_{ln}A_{ml}A_{nm} - A_{lm}A_{ml}A_{nn} \\ &+ A_{lm}A_{mn}A_{nl} - A_{ln}A_{mm}A_{nl}).\end{aligned} \tag{2.30}$$

Further modification in the 2nd, 4th and 6th terms arises because of the symmetry of $[A]$ (Eq. 2.26).

$$\begin{aligned}\det[A] = \frac{1}{6}(&A_{ll}A_{mm}A_{nn} - A_{ll}A_{mn}^2 + A_{ln}A_{ml}A_{nm} - A_{lm}^2A_{nn} \\ &+ A_{lm}A_{mn}A_{nl} - A_{ln}^2A_{mm}).\end{aligned} \tag{2.31}$$

Next, we use the constraint on the diagonal terms (Eq. 2.27) to simplify the above equation. Note that the index *k* in Eq. (2.27) is a dummy index, and thus, can be replaced by any other index. Therefore, 1st, 2nd, 4th and 6th terms become zero. Then, Eq. (2.31) becomes:

$$\det[A] = \frac{1}{6}(A_{ln}A_{ml}A_{nm} + A_{lm}A_{mn}A_{nl}) \ . \tag{2.32}$$

Next, we modify the 1st term using the symmetry of [*A*]:

$$\det[A] = \frac{1}{6}(A_{nl}A_{lm}A_{mn} + A_{lm}A_{mn}A_{nl}) \ . \tag{2.33}$$

Finally, shuffling the order in the 1st term, we find that both the terms are identical. Combining the two terms, we get the desired expression:

$$\det[A] = \frac{1}{6}(A_{lm}A_{mn}A_{nl} + A_{lm}A_{mn}A_{nl}) = \frac{1}{3}A_{lm}A_{mn}A_{nl} \ . \tag{2.34}$$

***Example* 2.5:** Express the derivative of A_{ij} with respect to A_{pq} in index notation.

***Solution*:** Note that the derivate of A_{ij} with respect to A_{pq} is 1 only if both the indices p and q are exactly equal to i and j. If any one index is different, then the derivative is zero. For example, choose $i = 2$ and $j = 3$. Then, if both $p = 2$ and $q = 3$, then the derivative of A_{23} with respect to A_{23} is one. However, the derivative of A_{23} with respect to A_{p3} for $p = 1,3$ or with respect to A_{2q} for $q = 1, 2$ is zero. Thus, we get

$$\frac{\partial A_{ij}}{\partial A_{pq}} = \delta_{ip}\delta_{jq}. \tag{2.35}$$

The first partial derivative of a component with respect to x_j is indicated by a *comma* followed by j. For example, $u_{i,j}$ means $\partial u_i / \partial x_j$, which in turn represents 9 expressions, because both i and j vary from 1 to 3.

***Example* 2.6:** Expand the following expression:

$$\sigma_{ij,j} = 0. \tag{2.36}$$

***Solution*:** This is a vector equation as there is one free index, namely i. Dummy index j represents a sum over three terms. Further, the comma before j indicates differentiation with respect to x_j. Hence, after expanding the sum over j, the above vector equation takes the form:

$$\frac{\partial \sigma_{i1}}{\partial x_1} + \frac{\partial \sigma_{i2}}{\partial x_2} + \frac{\partial \sigma_{i3}}{\partial x_3} = 0. \tag{2.37}$$

Since i is a free index and takes the values 1, 2 and 3, the above vector equation represents the following 3 scalar equations:

$$\begin{aligned} &\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} = 0, \\ &\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} = 0, \\ &\frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} = 0. \end{aligned} \tag{2.38}$$

2.3 Stress

As stated in the introduction, the stresses in a body vary from point to point. In this section, we first discuss the concept of stress at a point. Then, we carry out the analysis of stress at a point to develop the ideas of stress invariants, principal stresses, maximum shear stress, octahedral stresses and the hydrostatic and deviatoric parts of stress. Finally, we discuss the equations of motion which involve the derivatives of stress components. These equations arise as a consequence of the balance of linear momentum.

2.3.1 Stress at a Point

In this subsection, we first define the stress vector at a point. Then, the ideas of stress tensor and its relation with stress vector are developed. Definition of a tensor (or a second order tensor to be precise) is provided involving the transformation of components with a change in Cartesian coordinate system.

2.3.1.1 Stress Vector

Stress is a measure of the intensity of internal forces generated in a body. In general, stresses in a body vary from point to point. To understand the concept of stress at a point, consider a body subjected to external forces and supported in a suitable fashion, as shown in Figure 2.1. Note that, as soon as the forces are applied, the body gets deformed and sometimes displaced if the supports do not restrain the rigid body motion of the body. Thus, Figure 2.1 shows the deformed configuration. In fact, throughout this section, the configuration considered will be the deformed configuration. First, we define the stress vector (on a plane) at point P of the body. For this, pass a plane (called as cutting plane) through point P having a unit normal $\hat{\boldsymbol{n}}$. On each half of the body, there are distributed internal forces acting on the cutting plane and exerted by the other half. On the left half, consider a small area ΔA around point P of the cutting plane. Let $\boldsymbol{\Delta F}$ be the resultant of the distributed internal forces (acting on ΔA) exerted by the right half. Then, the *stress vector* (or traction) at point P (on the plane with normal $\hat{\boldsymbol{n}}$) is defined as

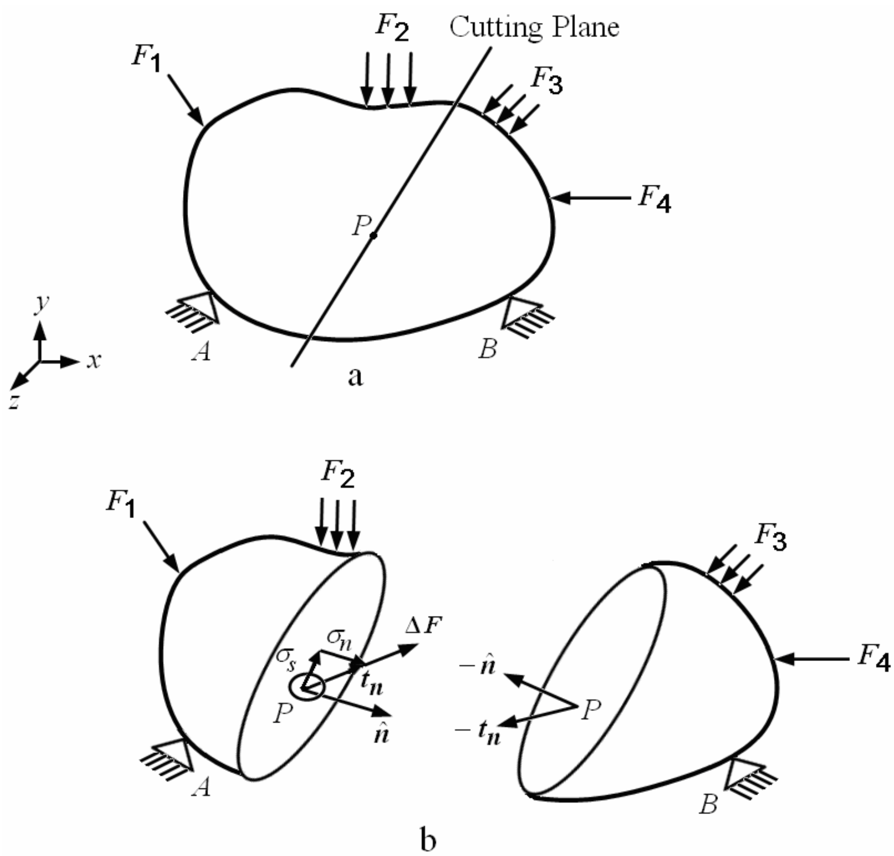


Figure 2.1. Stress vector at a point on a plane a. Cutting plane passing through point *P* of the deformed configuration; b. Stress vector $\boldsymbol{t}_n$, normal stress component σ_n and shear stress component σ_s acting at point *P* on the cutting plane

$$\boldsymbol{t}_n = \lim_{\Delta A \to 0} \frac{\boldsymbol{\Delta F}}{\Delta A}. \tag{2.39}$$

The component of $\boldsymbol{t}_n$ normal to the plane is called as the *normal stress* component. It is denoted by σ_n and given by

$$\sigma_n = (\boldsymbol{t}_n)_i\, n_i. \tag{2.40}$$

The component of $\boldsymbol{t_n}$ along the plane is called as the *shear stress* component. It is denoted by σ_s and given by

$$|\sigma_s| = \left[|\boldsymbol{t_n}|^2 - (\sigma_n)^2 \right]^{1/2}. \tag{2.41}$$

Note that, on the right half, the normal to the cutting plane will be $\boldsymbol{-\hat{n}}$ and the stress vector at *P* will be $-\boldsymbol{t}_n$ as per the Newton's third law.

2.3.1.2 State of Stress at a Point, Stress Tensor

One can pass an infinite number of planes through point *P* to obtain infinite number of stress vectors at point *P*. The set of stress vectors acting on every plane passing through a point describes the *state of stress* at that point.

It can be shown that a stress vector on any arbitrary plane can be uniquely represented in terms of the stress vectors on *three* mutually orthogonal planes. To show this, we consider *x, y* and *z* planes as the three planes, having normal vectors along the three Cartesian directions *x, y* and *z* respectively. Let the stress vectors on *x, y* and *z* planes be denoted by $\boldsymbol{t_x}$, $\boldsymbol{t_y}$ and $\boldsymbol{t_z}$ respectively. Further, we denote their components along *x, y* and *z* directions as follows:

$$\boldsymbol{t_x} = \sigma_{xx}\hat{\boldsymbol{i}} + \sigma_{xy}\hat{\boldsymbol{j}} + \sigma_{xz}\hat{\boldsymbol{k}}\,, \tag{2.42}$$

$$\boldsymbol{t_y} = \sigma_{yx}\hat{\boldsymbol{i}} + \sigma_{yy}\hat{\boldsymbol{j}} + \sigma_{yz}\hat{\boldsymbol{k}}\,, \tag{2.43}$$

$$\boldsymbol{t_z} = \sigma_{zx}\hat{\boldsymbol{i}} + \sigma_{zy}\hat{\boldsymbol{j}} + \sigma_{zz}\hat{\boldsymbol{k}}\,, \tag{2.44}$$

where ($\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{k}}$) are the unit vectors along (x, y, z) axes. The stress vectors and their components are shown in Figure 2.2. To derive the above result, we consider a small element at point *P* whose shape is that of a tetrahedron. The three sides of the tetrahedron are chosen perpendicular to *x, y* and *z* axes and the slant face is chosen normal to vector $\hat{\boldsymbol{n}}$. Then, equilibrium of the tetrahedron in the limit as its size goes to zero leads to the following result [1-5]:

$$\boldsymbol{t_n} = \boldsymbol{t_x} n_x + \boldsymbol{t_y} n_y + \boldsymbol{t_z} n_z\,, \tag{2.45}$$

where n_x, n_y and n_z are the components of the normal vector $\hat{\boldsymbol{n}}$. This result is true for every stress vector at point *P* no matter what the orientation of the normal vector $\hat{\boldsymbol{n}}$ is. Further, this result remains valid even if the body forces are not zero or the body is accelerating.

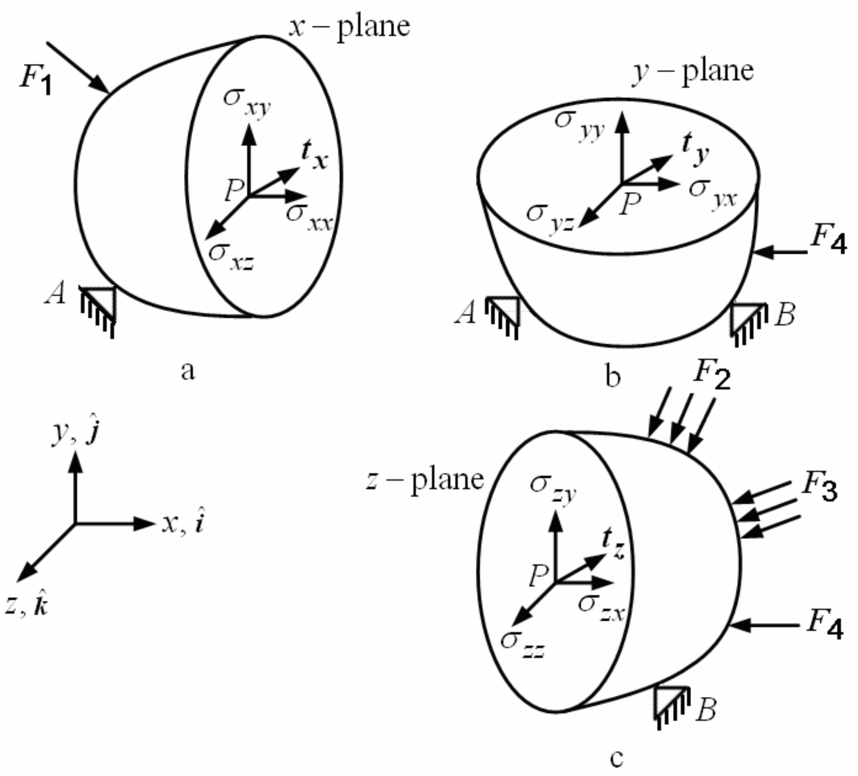


Figure 2.2. Stress vectors and their components on *x*, *y* and *z* planes a. Stress vector and its components on *x* plane; b. Stress vector and its components on *y* plane; c. Stress vector and its components on *z* plane

Let the components of the stress vector $\boldsymbol{t_n}$ be

$$\boldsymbol{t_n} = (t_n)_x\,\hat{\boldsymbol{i}} + (t_n)_y\,\hat{\boldsymbol{j}} + (t_n)_z\,\hat{\boldsymbol{k}}\ . \tag{2.46}$$

Substituting Eqs. (2.42-2.44) and (2.46), we get the component form of Eq. (2.45) as follows:

$$\begin{Bmatrix} (t_n)_x \\ (t_n)_y \\ (t_n)_z \end{Bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{yx} & \sigma_{zx} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{zy} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix}. \tag{2.47}$$

In array notation, this can be written as

$$\{t_n\} = [\sigma]^{\mathrm{T}}\,\{n\}\,, \tag{2.48}$$

where the matrix $[\sigma]$ is

$$[\sigma]=\begin{bmatrix}\sigma_{xx} & \sigma_{xy} & \sigma_{xz}\\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz}\\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz}\end{bmatrix}. \tag{2.49}$$

In index notation, it can be expressed as

$$(t_n)_i=\sigma_{ij}^{\mathrm{T}}\,n_j. \tag{2.50}$$

Equation (2.47) or (2.48) or (2.50) is called as the Cauchy's relation. Equations (2.45) and (2.47) indicate that the stress at a point can be completely described by means of just three stress vectors $\boldsymbol{t_x}, \boldsymbol{t_y}$ and $\boldsymbol{t_z}$ acting on mutually orthogonal planes or by their nine components: $\sigma_{xx}, \sigma_{xy}, \sigma_{xz}, \sigma_{yx}, \sigma_{yy}, \sigma_{yz}, \sigma_{zx}, \sigma_{zy}$ and σ_{zz}. Thus, the stress at a point is conceptually different than a *scalar* which has only one component or a *vector* which has three components (in three dimensions). In the next paragraph, we shall discuss a characteristic of the stress at a point which will indicate that it is a *tensor* (*of order two*).

2.3.1.3 Transformation Relations

Note that we can represent the stress vector $\boldsymbol{t_n}$ (at a point) as a combination of the stress vectors on any three mutually orthogonal planes. These planes can be x', y' and z' (Figure 2.3) instead of *x, y* and *z*. Then, following the earlier procedure, the stress vector $\boldsymbol{t_n}$ in the component form can be written as

$$\begin{Bmatrix}(t_n)_{x'}\\ (t_n)_{y'}\\ (t_n)_{z'}\end{Bmatrix}=\begin{bmatrix}\sigma_{x'x'} & \sigma_{y'x'} & \sigma_{z'x'}\\ \sigma_{x'y'} & \sigma_{y'y'} & \sigma_{z'y'}\\ \sigma_{x'z'} & \sigma_{y'z'} & \sigma_{z'z'}\end{bmatrix}\begin{Bmatrix}n_{x'}\\ n_{y'}\\ n_{z'}\end{Bmatrix}, \tag{2.51}$$

or

$$\{t_n'\}=[\sigma']^{\mathrm{T}}\{n'\}. \tag{2.52}$$

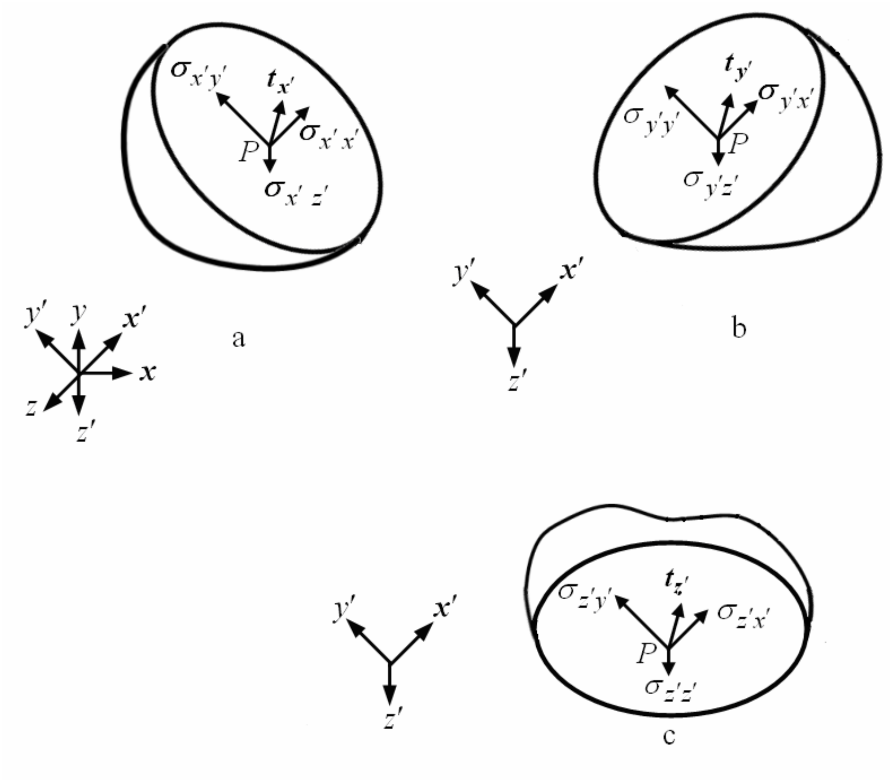


Figure 2.3. Stress vectors and their components on x', y' and z' planes. (Forces acting on the body and supports not shown) a. Stress vector and its components on x' plane; b. Stress vector and its components on y' plane; c. Stress vector and its components on z' plane

Obviously, the components of the matrices $[\sigma]$ and $[\sigma']$ must be related as the stress vector $\boldsymbol{t_n}$ (at point *P*) has a unique magnitude and direction. To get this relation, we consider equilibrium of three tetrahedra (at point *P*) whose three faces are perpendicular to *x, y* and *z* directions. The fourth face is normal to x' direction for the first tetrahedron, normal to y' direction for the second tetrahedron and normal to z' direction for the third tetrahedron. Three equilibrium equations for each of the three tetrahedra lead to the following result:

$$\begin{bmatrix} \sigma_{x'x'} & \sigma_{x'y'} & \sigma_{x'z'} \\ \sigma_{y'x'} & \sigma_{y'y'} & \sigma_{y'z'} \\ \sigma_{z'x'} & \sigma_{z'y'} & \sigma_{z'z'} \end{bmatrix} = \begin{bmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}. \tag{2.53}$$

Here, if ($\hat{\boldsymbol{i}}', \hat{\boldsymbol{j}}', \hat{\boldsymbol{k}}'$) are the unit vectors along (x', y', z') axes, then (ℓ_1, m_1, n_1) denote the direction cosines of $\hat{\boldsymbol{i}}'$ with respect to (x, y, z) axes. Similarly, (ℓ_2, m_2, n_2) denote the direction cosines of $\hat{\boldsymbol{j}}'$ with respect to (x, y, z) axes and (ℓ_3, m_3, n_3) denote the direction cosines of $\hat{\boldsymbol{k}}'$ with respect to (x, y, z) axes. Define the matrix $[Q]$ as

$$[Q] = \begin{bmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{bmatrix}. \tag{2.54}$$

Then, the relation (2.53) can be written as

$$[\sigma'] = [Q][\sigma][Q]^{\mathrm{T}}, \tag{2.55}$$

or, in index notation, it can be expressed as

$$\sigma'_{ij} = Q_{ik}\sigma_{kl}Q^{\mathrm{T}}_{lj}. \tag{2.56}$$

The result of Eq. (2.53) or (2.55) or (2.56) remains valid even if the body forces are not zero or the body is accelerating.

Any quantity whose components with respect to two Cartesian coordinate systems transform according to the relation (2.53) or (2.55) or (2.56) is called as a *tensor* (*or tensor of second order*). Thus, the stress at a point is a tensor, called as *stress tensor*. We denote it by the symbol $\boldsymbol{\sigma}$. It is related to the stress vector on plane with normal $\hat{\boldsymbol{n}}$ by the relation (2.47) or (2.48) or (2.50). In tensor notation, this relation can be written as

$$\boldsymbol{t_n} = \boldsymbol{\sigma}^{\mathrm{T}}\hat{\boldsymbol{n}}. \tag{2.57}$$

The relation (2.53) or (2.55) or (2.56) is called as the *tensor transformation relation*. The stress tensor $\boldsymbol{\sigma}$ is called the Cauchy stress tensor. In the next chapter, we shall discuss other types of stress tensors.

Thus, there is a difference between a tensor and its matrix. A tensor represents a physical quantity which has an existence independent of the coordinate system being used. On the other hand, matrix of a tensor contains its components with respect to some coordinate system. If the coordinate system is changed, the components change giving a different matrix. Matrices with respect to two different coordinate systems are related by the tensor transformation relation.

Let (a_x, a_y, a_z) be the components of a vector $\boldsymbol{a}$ with respect to the coordinate system (x, y, z). Further, denote the components of $\boldsymbol{a}$ with respect to the coordinate system (x', y', z') as (a'_x, a'_y, a'_z). Then these two sets of components are related by the following transformation law:

$$\begin{Bmatrix} a'_x \\ a'_y \\ a'_z \end{Bmatrix} = \begin{bmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{bmatrix} \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix}, \tag{2.58}$$

or

$$\{a\}' = [Q]\{a\}, \tag{2.59}$$

or, in index notation

$$a' = Q_{ij}\, a_j \,. \tag{2.60}$$

The relation (2.58) or (2.59) or (2.60) is called as the *vector transformation relation*. The matrix $[Q]$, which appears in vector and tensor transformation relations, is called as the *transformation matrix*. It can be easily verified that $[Q]$ is an *orthogonal matrix*, that is, it satisfies the relation

$$Q_{ik} Q_{kj}^{\mathrm{T}} = Q_{ik}^{\mathrm{T}} Q_{kj} = \delta_{ij} \,. \tag{2.61}$$

There are two types of orthogonal matrices. The first type represents the rotation of the coordinate axes and its determinant is +1. The second type represents the reflection of the coordinate axes and its determinant is -1. It can be shown that the matrix $[Q]^{\mathrm{T}}$ represents the rotation of the (x, y, z) coordinate axes to (x', y', z') axes and therefore it is called as the *rotation matrix*. Its determinant is +1.

2.3.1.4 Stress Components

A tensor component is always represented by two subscript indices. In the case of a component of the stress tensor, the meaning of the indices is as follows. The first index describes the direction of the normal to the plane on which the stress component acts while the second index represents the direction of the stress component itself. Thus, σ_{xy} indicates a stress component acting in y-direction on x-plane. When both the indices are same, it means the stress component is along the normal to the plane on which it acts. It is called as the normal stress component. Thus, σ_{xx}, σ_{yy} and σ_{zz} are the *normal stress* components. When the two indices are different, it means the direction of the component is within the plane. Such a component is called as the *shear stress* component. Thus, σ_{ij} where

$i \neq j$ are the shear stress components. We adopt the following *sign convention* for the stress components. We first define positive and negative planes. A plane i is considered positive if the outward normal to it points in the positive i direction, otherwise it is considered as negative. A stress component is considered positive if it acts in positive direction on positive plane or in negative direction on negative plane. Otherwise, it is considered as negative. Figure 2.4 illustrates positive and negative normal and shear stress components.

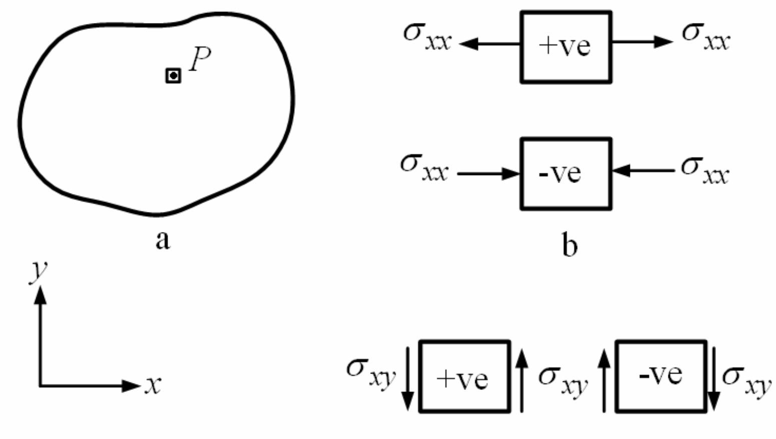


Figure 2.4. Sign convention for normal and shear stress components a. Small element at point 'P' in the deformed configuration. Forces on the body and supports are not shown; b. Positive and negative 'σ_{xx}'; c. Positive and negative 'σ_{xy}'

2.3.1.5 Symmetry of Stress Tensor

By considering the moment equilibrium of a small element (of parallelepiped shape) at point P in the limit as the size of the element tends to zero, it can be shown that [2]

$$\sigma_{ij} = \sigma_{ji} . \tag{2.62}$$

Thus, the stress tensor is *symmetric*. Now, the Cauchy relation (Eq. 2.48 or 2.50) may be written as:

$$\{t_n\} = [\sigma]\{n\} , \tag{2.63}$$

or

$$(t_n)_i = \sigma_{ij} n_j . \tag{2.64}$$

In tensor notation, it can be expressed as

$$\boldsymbol{t}_n = \boldsymbol{\sigma}\hat{\boldsymbol{n}} . \tag{2.65}$$

The result of Eq. (2.62) is valid even if the body forces are not zero or the body is accelerating.

***Example* 2.7:** Components of the stress tensor $\boldsymbol{\sigma}$ at point P of the beam of Figure 2.5, with respect to (x, y, z) coordinate system, are given as:

$$[\sigma] = \begin{bmatrix} 35 & -25 & 0 \\ -25 & -15 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ (MPa)}. \tag{2.66}$$

(a) Find the stress vector $\boldsymbol{t}_n$ on the plane whose normal is given by

$$\hat{\boldsymbol{n}} = (1/\sqrt{3})(\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}) . \tag{2.67}$$

Find the normal (σ_n) and shear (σ_s) components of $\boldsymbol{t}_n$.

(b) Find the components of $\boldsymbol{\sigma}$ with respect to the rotated coordinate system (x', y', z'). The unit vectors $(\hat{\boldsymbol{i}}', \hat{\boldsymbol{j}}', \hat{\boldsymbol{k}}')$ along the (x', y', z') axes are given as:

$$\begin{aligned} \hat{\boldsymbol{i}}' &= 0.6\hat{\boldsymbol{i}} + 0.8\hat{\boldsymbol{k}}, \\ \hat{\boldsymbol{j}}' &= \hat{\boldsymbol{j}}, \\ \hat{\boldsymbol{k}}' &= -0.8\hat{\boldsymbol{i}} + 0.6\hat{\boldsymbol{k}} . \end{aligned} \tag{2.68}$$

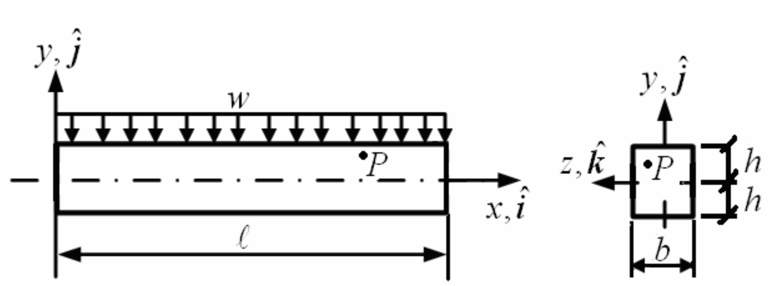


Figure 2.5. A cantilever beam subjected to uniformly distributed load on top surface

***Solution*:** (a) We use the Cauchy's relation in array form to evaluate the stress vector $\boldsymbol{t}_n$. As per Eq. (2.46), we denote its components with respect to

(x, y, z) coordinate system by $(t_n)_x$, $(t_n)_y$ and $(t_n)_z$. Further, the given components of the unit normal vector $\hat{\boldsymbol{n}}$ are

$$n_x = n_y = n_z = 1/\sqrt{3}\,. \tag{2.69}$$

Writing the components of $\boldsymbol{t_n}$ and $\hat{\boldsymbol{n}}$ in the array form and using Eq. (2.47), we get

$$\begin{Bmatrix} (t_n)_x \\ (t_n)_y \\ (t_n)_z \end{Bmatrix} = \begin{bmatrix} 35 & -25 & 0 \\ -25 & -15 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \dfrac{1}{\sqrt{3}} \\ \dfrac{1}{\sqrt{3}} \\ \dfrac{1}{\sqrt{3}} \end{Bmatrix} = \begin{Bmatrix} \dfrac{10}{\sqrt{3}} \\ -\dfrac{40}{\sqrt{3}} \\ 0 \end{Bmatrix}. \tag{2.70}$$

Thus, the stress vector is:

$$\boldsymbol{t_n} = \frac{10}{\sqrt{3}}\hat{\boldsymbol{i}} - \frac{40}{\sqrt{3}}\hat{\boldsymbol{j}} \text{ (MPa)}. \tag{2.71}$$

Then, using Eq. (2.40), we get the normal component of the stress vector:

$$\sigma_n = (t_n)_i n_i = \left(\frac{10}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right) + \left(-\frac{40}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right) + (0)\left(\frac{1}{\sqrt{3}}\right) = -10 \text{ (MPa)}. \tag{2.72}$$

Further, using Eq. (2.41), we get the magnitude of the shear component of the stress vector:

$$|\sigma_s| = \left[|\boldsymbol{t_n}|^2 - (\sigma_n)^2\right]^{1/2} = \left[\left(\frac{10}{\sqrt{3}}\right)^2 + \left(\frac{-40}{\sqrt{3}}\right)^2 - (-10)^2\right]^{1/2} = \frac{10\sqrt{14}}{\sqrt{3}} \text{ (MPa)}. \tag{2.73}$$

(b) To find the components of $\boldsymbol{\sigma}$ with respect to (x', y', z') coordinate system, we first evaluate the transformation matrix $[Q]$. We get the direction cosines of the unit vectors ($\hat{\boldsymbol{i}}', \hat{\boldsymbol{j}}', \hat{\boldsymbol{k}}'$) from Eq. (2.68). Substituting them in Eq. (2.54), we get the following expression for $[Q]$:

$$[Q] = \begin{bmatrix} 0.6 & 0 & 0.8 \\ 0 & 1 & 0 \\ -0.8 & 0 & 0.6 \end{bmatrix}. \tag{2.74}$$

Using the tensor transformation relation (Eq. 2.55), we get the following matrix of the components of the stress tensor with respect to (x', y', z') coordinate system:

$$[\sigma]' = [Q][\sigma][Q]^{\mathrm{T}} = \begin{bmatrix} 0.6 & 0 & 0.8 \\ 0 & 1 & 0 \\ -0.8 & 0 & 0.6 \end{bmatrix} \begin{bmatrix} 35 & -25 & 0 \\ -25 & -15 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.6 & 0 & -0.8 \\ 0 & 1 & 0 \\ 0.8 & 0 & 0.6 \end{bmatrix},$$
$$= \begin{bmatrix} 12.6 & -15 & -16.8 \\ -15 & -15 & 20 \\ -16.8 & 20 & 22.4 \end{bmatrix} (\text{MPa}). \tag{2.75}$$

Equation (2.75) shows that the matrix of $\boldsymbol{\sigma}$ is symmetric with respect to the coordinate system (x', y', z') as well.

Note that the stress components σ_{xz}, σ_{yz} and σ_{zz} are zero at point P of the beam (Eq. 2.66). Such a state is called as the *state of plane stress* (*at a point*) *in* $x-y$ *plane*. When these stress components are zero at every point of the body and if, additionally, the remaining stress components σ_{xx}, σ_{yy} and σ_{xy} are independent of z coordinate, it is called as the *state of plane stress* (*in a body*) *in* $x-y$ *plane*. It can be shown that the state of stress in the beam of Figure 2.5 is of this type.

2.3.2 Analysis of Stress at a Point

As stated earlier, in this subsection, we carry out the analysis of stress at a point to discuss the concepts of principal stresses and principal directions, principal invariants, maximum shear stress, octahedral stresses and the hydrostatic and deviatoric parts of stress.

2.3.2.1 Principal Stresses, Principal Planes and Principal Directions

There exist at least 3 mutually perpendicular planes (in the deformed configuration) such that there are no shear stress components on them *i.e.*, the stress vector is normal to these planes. These planes are called as the *principal planes* and normals to these planes are called as the *principal directions* (of stress). The values of the normal stress components are called as the *principal stresses*. We denote the principal stresses as σ_1, σ_2 and σ_3. The unit vectors along the principal directions are normally denoted as $\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$ and $\hat{\boldsymbol{e}}_3$. We arrange the principal stresses as

$$\sigma_1 \geq \sigma_2 \geq \sigma_3. \tag{2.76}$$

The senses of the unit vectors are so chosen that they always form a right-sided system. Thus,

$$\hat{\boldsymbol{e}}_1 . \hat{\boldsymbol{e}}_2 \times \hat{\boldsymbol{e}}_3 = +1. \tag{2.77}$$

Since the stress vector on a principal plane i (*i.e.*, the plane perpendicular to the principal direction i) has only the normal component equal to σ_i, the components of the stress tensor, with respect to the principal directions as the coordinate system, become:

$$[\sigma]^p = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}. \tag{2.78}$$

It can be easily verified that, at a point, *maximum* value of the *normal stress* component (σ_n) with respect to the orientation of the normal vector $\hat{\boldsymbol{n}}$ is σ_1. Further, the *minimum* value is σ_3.

It can be shown that the principal stresses are the *eigen values* or *principal values* and the unit vectors along the principal directions are the normalized *eigen vectors* of the stress tensor [2-4]. Before we write the equation which the eigen values and eigenvectors of a tensor satisfy, we define a *unit tensor*. It is denoted by the symbol $\mathbf{1}$. A unit tensor is defined as a tensor whose components with respect to every coordinate system are given by the following array:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.79}$$

Thus, in index notation, the components of the unit tensor are represented as δ_{ij}. If λ is an eigen value of the stress tensor $\boldsymbol{\sigma}$ and if $\boldsymbol{x}$ is the corresponding eigenvector, λ and $\boldsymbol{x}$ satisfy the following equation:

$$\left([\sigma] - \lambda [1] \right) \{x\} = \{0\}. \tag{2.80}$$

Here, the arrays $[\sigma]$, $[1]$ and $\{x\}$ contain the components of respectively $\boldsymbol{\sigma}$, $\mathbf{1}$ and $\boldsymbol{x}$ with respect to the given coordinate system (x, y, z). For $\boldsymbol{x}$ to be an eigen vector of $\boldsymbol{\sigma}$, Eq. (2.80) should have a non-trivial solution. For this to happen, the determinant of the coefficient matrix $\left([\sigma] - \lambda [1] \right)$ must be zero. This condition leads to the following cubic equation in λ:

$$\lambda^3 - I_\sigma \lambda^2 - II_\sigma \lambda - III_\sigma = 0, \tag{2.81}$$

where

$$I_\sigma = \sigma_{ii}\,, \tag{2.82}$$

$$II_\sigma = \frac{1}{2}(\sigma_{ij}\sigma_{ij} - \sigma_{ii}\sigma_{jj})\,, \tag{2.83}$$

$$III_\sigma = \in_{ijk}\ \sigma_{1i}\sigma_{2j}\sigma_{3k}\,. \tag{2.84}$$

Thus, the principal stresses σ_i are found as the roots of the above equation. Once σ_i are determined, The unit vectors $\hat{\boldsymbol{e}}_{\boldsymbol{i}}$ along the principal directions are found from the following equation:

$$\left([\sigma] - \sigma_i[1]\right)\{e_i\} = \{0\}\,, \quad \text{(no sum over } i\text{)}. \tag{2.85}$$

where the array $\{e_i\}$ contains the components of $\hat{\boldsymbol{e}}_{\boldsymbol{i}}$ with respect to the given coordinate system (x, y, z).

2.3.2.2 Principal Invariants

Trace of tensor $\boldsymbol{\sigma}$ (denoted by $tr\boldsymbol{\sigma}$) is a scalar function of $\boldsymbol{\sigma}$ which is defined as

$$tr\boldsymbol{\sigma} = \sigma_{ii}\,. \tag{2.86}$$

Thus, using Eq. (2.82), we get

$$I_\sigma = tr\boldsymbol{\sigma}\,. \tag{2.87}$$

Note that, in Eq. (2.86), the scalar function $tr\boldsymbol{\sigma}$ is evaluated using the components of $\boldsymbol{\sigma}$ with respect to the given coordinate system (x, y, z). Let σ'_{ij} be the components of $\boldsymbol{\sigma}$ in a rotated coordinate system (x', y', z'). If we use the rotated coordinate system to evaluate the scalar function $tr\boldsymbol{\sigma}$, then it would be

$$tr\boldsymbol{\sigma} = \sigma'_{ii}\,. \tag{2.88}$$

Using the tensor transformation relation (2.56), and the orthogonality of $[Q]$ (Eq. (2.61)), it can be shown that

$$\sigma'_{ii} = \sigma_{ii}\,. \tag{2.89}$$

Thus, Eqs. (2.86), (2.88) and (2.89) show that the value of $tr\boldsymbol{\sigma}$ is independent of the coordinate system. A scalar function of a tensor whose value is independent of the coordinate system is called as an *invariant* of the tensor. Thus, I_σ is an invariant of the tensor $\boldsymbol{\sigma}$. Similarly, it can be shown that II_σ and III_σ are also the invariants of the tensor $\boldsymbol{\sigma}$. Using the definition of the trace and the symmetry of $\boldsymbol{\sigma}$, it can be shown that

$$II_\sigma = \frac{1}{2}\left\{tr(\boldsymbol{\sigma}^2) - (tr\boldsymbol{\sigma})^2\right\}. \tag{2.90}$$

Further, it can be shown that

$$III_\sigma = \det \boldsymbol{\sigma}, \tag{2.91}$$

where $\det \boldsymbol{\sigma}$ means the *determinant* of the matrix of $\boldsymbol{\sigma}$ in any coordinate system.

Every other invariant of $\boldsymbol{\sigma}$ can be expressed in terms of these three invariants [1]. Therefore, I_σ, II_σ and III_σ are called as the *principal invariants* of the tensor $\boldsymbol{\sigma}$.

2.3.2.3 Maximum Shear Stress

It can be shown that, at a point, *maximum* value of the *shear stress* component with respect to the orientation of the normal vector $\hat{\boldsymbol{n}}$ is [4]

$$|\sigma_s|_{\max} = \frac{(\sigma_1 - \sigma_3)}{2}. \tag{2.92}$$

Further, the normals to the planes on which $|\sigma_s|_{\max}$ acts are given by [4]

$$\hat{\boldsymbol{n}} = \pm\frac{1}{\sqrt{2}}(\hat{\boldsymbol{e}}_1 \pm \hat{\boldsymbol{e}}_3). \tag{2.93}$$

This result will be useful when we discuss the yield criteria later.

2.3.2.4 Octahedral Stresses

A plane whose normal is equally inclined to the three principal directions is called as *octahedral plane*. Let $\hat{\boldsymbol{n}}$ be the unit normal to an octahedral plane. Further, let n_i be its components with respect to the principal directions $\hat{\boldsymbol{e}}_i$. Since n_i are equal in magnitude and

$$n_i\, n_i = 1, \tag{2.94}$$

we get

$$n_i = \pm \frac{1}{\sqrt{3}}. \tag{2.95}$$

From Eq. (2.95), we get eight different normal vectors: $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$, $(1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3}), \ldots\ldots\ldots, (-1/\sqrt{3}, -1/\sqrt{3}, -1/\sqrt{3})$. Thus there are *eight* octahedral planes.

Let $\boldsymbol{t_n}$ be the stress vector on an octahedral plane. Further, let $(t_n)_i$ be its components with respect to the principal directions $\hat{\boldsymbol{e}}_{\boldsymbol{i}}$. Substituting the components of $\boldsymbol{\sigma}$ and $\hat{\boldsymbol{n}}$ with respect to the principal directions (expressions 2.78 and 2.95) in Eq. (2.64), we get the following expression for $(t_n)_i$:

$$(t_n)_i = \pm \frac{1}{\sqrt{3}} \sigma_i . \tag{2.96}$$

Substituting the expressions (2.95-2.96) for n_i and $(t_n)_i$ in Eq. (2.40), we get the following expression for the normal stress component (denoted by σ_{oct}) on the octahedral planes:

$$\sigma_{oct} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{I_\sigma}{3}. \tag{2.97}$$

Similarly, substituting the expressions (2.96-2.97) for $(t_n)_i$ and σ_{oct} in Eq. (2.41), we get the following expression for the magnitude of the shear stress component on the octahedral planes (denoted by $|\tau_{oct}|$):

$$|\tau_{oct}| = \left[\frac{1}{3}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) - \frac{1}{9}\left(\sigma_1 + \sigma_2 + \sigma_3\right)^2 \right]^{1/2} = \left[\frac{2}{9}\left(I_\sigma^2 + 3II_\sigma\right) \right]^{1/2}. \tag{2.98}$$

Note that when the stress tensor at a point has only the deviatoric part, then the octahedral planes are free of the normal stress component. The expression for the shear stress on the octahedral planes will be useful when we discuss the yield criteria in Chapter 3.

2.3.2.5 Decomposition into the Hydrostatic and Deviatoric Parts

Every tensor can be decomposed as a sum of a scalar multiple of a unit tensor **1** and a traceless tensor. Thus, for the stress tensor $\boldsymbol{\sigma}$, we can write

$$\boldsymbol{\sigma} = \left(\frac{1}{3} tr\boldsymbol{\sigma}\right)\mathbf{1} + \boldsymbol{\sigma}' , \qquad (tr\boldsymbol{\sigma}' = 0). \tag{2.99}$$

In index notation, this can be written as

$$\sigma_{ij} = \left(\frac{1}{3}\sigma_{kk}\right)\delta_{ij} + \sigma'_{ij} , \ (\sigma'_{kk} = 0) . \tag{2.100}$$

Note that, since $\boldsymbol{\sigma}$ is a symmetric tensor, $\boldsymbol{\sigma}'$ is also a symmetric tensor. The unit tensor $\mathbf{1}$ is of course symmetric. The stress vector corresponding to the first part is always normal to the plane and has the same magnitude on every plane, namely $(1/3)tr\boldsymbol{\sigma}$. Thus, this part of the stress tensor is similar to the state of stress in water at rest, except that whereas $(1/3)tr\boldsymbol{\sigma}$ may be tensile or compressive, the state of stress in water is always compressive. Therefore, this part of the stress tensor is called as *hydrostatic part* of $\boldsymbol{\sigma}$. The second part is called as the *deviatoric part* of $\boldsymbol{\sigma}$ and represents a pure shear state.

In *isotropic materials*, the deformation caused by the hydrostatic part consists of only a change in volume (or size) but no change in shape. On the other hand, the deformation caused by the deviatoric part consists of no change in volume but only the change in shape. We shall see in Chapter 3 that, in an isotropic ductile material, yielding is caused only by the deviatoric part of the stress tensor.

2.3.2.6 Principal Invariants of the Deviatoric Part

The principal invariants of $\boldsymbol{\sigma}'$ are denoted by J_1 , J_2 and J_3 . Like the principal invariants of $\boldsymbol{\sigma}$ (Eqs. 2.82-2.84, 2.87, 2.90, 2.91), they are defined as

$$J_1 = tr\boldsymbol{\sigma}' = \sigma'_{ii} , \tag{2.101}$$

$$J_2 = \frac{1}{2}\left\{tr(\boldsymbol{\sigma}'^2) - (tr\boldsymbol{\sigma}')^2\right\} = \frac{1}{2}(\sigma'_{ij}\sigma'_{ij} - \sigma'_{ii}\sigma'_{jj}) , \tag{2.102}$$

$$J_3 = \det\boldsymbol{\sigma}' = \in_{ijk} \sigma'_{1i}\sigma'_{2j}\sigma'_{3k} . \tag{2.103}$$

Since $tr\boldsymbol{\sigma}' = 0$ (Eq. 2.99), J_1 has the value zero. Further, J_2 also gets simplified. Thus,

$$J_1 = 0 , \tag{2.104}$$

$$J_2 = \frac{1}{2} tr\left(\boldsymbol{\sigma}'^2\right) = \frac{1}{2}\sigma'_{ij}\sigma'_{ij} . \tag{2.105}$$

The expressions for these invariants will be useful while discussing the yield criteria of isotropic materials in Chapter 3.

***Example* 2.8:** Components of the stress tensor $\boldsymbol{\sigma}$ at a point, with respect to the (x, y, z) coordinate system, are given as

$$[\sigma] = \begin{bmatrix} 18 & 24 & 0 \\ 24 & 32 & 0 \\ 0 & 0 & -20 \end{bmatrix} \text{ (MPa)}. \tag{2.106}$$

(a) Find the principal invariants of $\boldsymbol{\sigma}$.
(b) Find the principal stresses σ_i and the unit vectors $\hat{\boldsymbol{e}}_i$ along the principal directions. Arrange σ_i such that $\sigma_1 \geq \sigma_2 \geq \sigma_3$. Further, choose the senses of $\hat{\boldsymbol{e}}_i$ such that $\hat{\boldsymbol{e}}_1 \cdot \hat{\boldsymbol{e}}_2 \times \hat{\boldsymbol{e}}_3 = +1$.
(c) Find the maximum shear stress $|\sigma_s|_{max}$ and the normals to the planes on which $|\sigma_s|_{max}$ acts. Express the normals in terms of the unit vectors $(\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{k}})$.
(d) Find the octahedral normal (σ_{oct}) and shear (τ_{oct}) stresses .
(e) Find the hydrostatic and deviatoric parts of $\boldsymbol{\sigma}$.

***Solution*:** (a) Substituting the values of σ_{ij} from Eq. (2.106) and the values of permutation symbol $\in_{ijk}$ from Eq. (2.14), we get

$$I_\sigma = \sigma_{ii} = 18 + 32 - 20 = 30 \text{ (MPa)}; \tag{2.107}$$

$$\begin{aligned} II_\sigma &= \frac{1}{2}(\sigma_{ij}\sigma_{ij} - \sigma_{ii}\sigma_{jj}), \\ &= \frac{1}{2}\left[(18)^2 + 2(24)^2 + (32)^2 + (-20)^2 + 4(0)^2 \ - (30)(30)\right] = 1000\left(\text{MPa}\right)^2; \end{aligned} \tag{2.108}$$

$$\begin{aligned} III_\sigma &= \in_{\boldsymbol{ijk}} \sigma_{1\boldsymbol{i}}\sigma_{2\boldsymbol{j}}\sigma_{3\boldsymbol{k}}, \\ &= \sigma_{11}(\sigma_{22}\sigma_{33} - \sigma_{23}\sigma_{32}) + \sigma_{12}(\sigma_{23}\sigma_{31} - \sigma_{21}\sigma_{33}) + \sigma_{13}(\sigma_{21}\sigma_{32} - \sigma_{22}\sigma_{31}), \\ &= 18[32\times(-20) - 0\times 0] + 24[0\times 0 - 24\times(-20)] + 0[24\times 0 - 32\times 0], \\ &= 0 \ \left(\text{MPa}\right)^3. \end{aligned} \tag{2.109}$$

(b) Substituting the values of I_σ, II_σ and III_σ from part (a), the cubic equation for λ (Eq. 2.81) becomes:

$$\lambda^3 - 30\lambda^2 - 1000\lambda - 0 = 0. \tag{2.110}$$

The roots of this equation are: $\lambda = 0, -20, 50$. Arranging them in decreasing order, we get the following values of the principal stresses:

$$\sigma_1 = 50 \text{ MPa}, \ \sigma_2 = 0 \text{ MPa}, \ \sigma_3 = -20 \text{ MPa}. \tag{2.111}$$

To find the unit vectors $\hat{\boldsymbol{e}}_i$ along the principal directions, we use Eq. (2.85). Let the unit vector along the first principal direction be:

$$\hat{\boldsymbol{e}}_1 = e_{1x}\hat{\boldsymbol{i}} + e_{1y}\hat{\boldsymbol{j}} + e_{1z}\hat{\boldsymbol{k}}. \tag{2.112}$$

Then for $i = 1$, Eq. (2.85) becomes:

$$\begin{bmatrix} \sigma_{11} - \sigma_1 & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \sigma_1 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \sigma_1 \end{bmatrix} \begin{Bmatrix} e_{1x} \\ e_{1y} \\ e_{1z} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{2.113}$$

Substituting $\sigma_1 = 50$ and the values of σ_{ij} from Eq. (2.106) and expanding the above equation, we get

$$\begin{aligned} &(18-50)\boldsymbol{e}_{1\boldsymbol{x}} + 24\boldsymbol{e}_{1\boldsymbol{y}} + 0\boldsymbol{e}_{1\boldsymbol{z}} = 0, \\ &24\boldsymbol{e}_{1\boldsymbol{x}} + (32-50)\boldsymbol{e}_{1\boldsymbol{y}} + 0\boldsymbol{e}_{1\boldsymbol{z}} = 0, \\ &0\boldsymbol{e}_{1\boldsymbol{x}} + 0\boldsymbol{e}_{1\boldsymbol{y}} + (-20-50)\boldsymbol{e}_{1\boldsymbol{z}} = 0. \end{aligned} \tag{2.114}$$

From third equation, we obtain $e_{1z} = 0$. Note that the first two equations are linearly dependent. Each of them gives $e_{1y} = (4/3)e_{1x}$. Since $\hat{\boldsymbol{e}}_1$ is a unit vector, we have

$$e_{1x}^2 + e_{1y}^2 + e_{1z}^2 = 1. \tag{2.115}$$

Substituting $e_{1z} = 0$ and $e_{1y} = (4/3)e_{1x}$ in the above equation, we obtain $e_{1x} = \pm(3/5)$. Choosing the positive sign, we get the following expression for the unit vector along the first principal direction:

$$\hat{\boldsymbol{e}}_1 = \frac{3}{5}\hat{\boldsymbol{i}} + \frac{4}{5}\hat{\boldsymbol{j}}. \tag{2.116}$$

Similarly, we get the following expressions for the unit vectors along the other two principal directions:

$$\hat{\boldsymbol{e}}_2 = \frac{4}{5}\hat{\boldsymbol{i}} - \frac{3}{5}\hat{\boldsymbol{j}}, \tag{2.117}$$

$$\hat{\boldsymbol{e}}_3 = -\hat{\boldsymbol{k}}. \tag{2.118}$$

Note that whereas the sense of the second unit vector has been chosen to be arbitrary, that of the third one has been selected so as to satisfy the condition $\hat{\boldsymbol{e}}_1 \cdot \hat{\boldsymbol{e}}_2 \times \hat{\boldsymbol{e}}_3 = +1$.

(c) Maximum shear stress is given by Eq. (2.92). Substituting the values of σ_1 and σ_3 from part (b) in this equation, we get

$$|\sigma_s|_{\max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{50-(-20)}{2} = 35 \text{ (MPa)}. \tag{2.119a}$$

The normals $\hat{\boldsymbol{n}}$ to the planes on which $|\sigma_s|_{max}$ acts are given by Eq. (2.93). Substituting the expressions for $\hat{\boldsymbol{e}}_1$ and $\hat{\boldsymbol{e}}_3$ from part (b) in this equation, we obtain the following expressions for $\hat{\boldsymbol{n}}$:

$$\hat{\boldsymbol{n}} = \pm\frac{1}{\sqrt{2}}(\hat{\boldsymbol{e}}_1 \pm \hat{\boldsymbol{e}}_3) = \pm\frac{1}{\sqrt{2}}\left(\frac{3}{5}\hat{\boldsymbol{i}} + \frac{4}{5}\hat{\boldsymbol{j}} \mp \hat{\boldsymbol{k}}\right). \tag{2.119b}$$

(d) Octahedral normal (σ_{oct}) and shear (τ_{oct}) stresses are calculated using Eqs. (2.97) and (2.98). Substituting the values of I_σ and II_σ from part (a), we get

$$\sigma_{oct} = \frac{I_\sigma}{3} = \frac{30}{3} = 10 \text{ (MPa)}, \tag{2.120a}$$

$$|\tau_{oct}| = \left[\frac{2}{9}(I_\sigma^2 + 3II_\sigma)\right]^{1/2} = \left\{\frac{2}{9}[(30)^2 + 3(1000)]\right\}^{1/2} = \frac{10\sqrt{78}}{3}\text{(MPa)}. \tag{2.120b}$$

(e) As per Eq. (2.100), components of the hydrostatic part are given by $[(1/3)\sigma_{kk}]\delta_{ij}$. Since $\sigma_{kk} = 30$ from part (a), the matrix of the hydrostatic part of $\boldsymbol{\sigma}$ becomes

$$\begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix} \text{(MPa)}. \tag{2.121}$$

Using $\sigma_{kk} = 30$ and Eq. (2.100), components of the deviatoric part can be expressed as:

$$\sigma'_{ij} = \sigma_{ij} - 10\delta_{ij}. \tag{2.122a}$$

Using the values of σ_{ij} from Eq. (2.106), we get the following expression for the matrix of the deviatoric part:

$$[\sigma'] = \begin{bmatrix} 18 & 24 & 0 \\ 24 & 32 & 0 \\ 0 & 0 & -20 \end{bmatrix} - \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix} = \begin{bmatrix} 8 & 24 & 0 \\ 24 & 22 & 0 \\ 0 & 0 & -30 \end{bmatrix} (\text{MPa}). \tag{2.122b}$$

In the state of stress given by Eq. (2.106), σ_{zz} is not zero. Therefore, it is not a state of plane stress (at a point) in $x-y$ plane. However, since the principal stress σ_2 is zero, it is a state of plane stress (at a point) in the plane perpendicular to $\hat{\boldsymbol{e}}_2$.

2.3.3 Equations of Motion

Let

$$\boldsymbol{a} = a_x\hat{\boldsymbol{i}} + a_y\hat{\boldsymbol{j}} + a_z\hat{\boldsymbol{k}} \tag{2.123}$$

be the *acceleration vector* at a point of the deformed configuration. The acceleration vector is related to the time derivatives of the displacement vector and velocity vector. But, that relation will be discussed later. Further, let

$$\boldsymbol{b} = b_x\hat{\boldsymbol{i}} + b_y\hat{\boldsymbol{j}} + b_z\hat{\boldsymbol{k}} \tag{2.124}$$

be the *body force vector per unit mass* acting on the body. We shall denote the density in the deformed configuration by the symbol ρ. Note that, in general, $\boldsymbol{u}$, $\boldsymbol{b}$ and ρ vary from point to point. Thus, they are functions of the coordinates (x, y, z).

Now, we apply the principle of balance of linear momentum in x, y and z directions to a small element (of parallelepiped shape) at a point of the deformed configuration. In the limit as the size of the element tends to zero, it leads to the following three *equations of motion*:

$$\begin{aligned}
\rho a_x &= \rho b_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}, \\
\rho a_y &= \rho b_y + \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z}, \\
\rho a_z &= \rho b_z + \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}.
\end{aligned} \tag{2.125}$$

When the acceleration vector is zero, we get the *equilibrium equations*.

As stated in the introduction, there are 3 sets of equations which govern the displacements, strains and stresses in a body. Equations (2.125) represent the *first set of governing equations*. The other two sets will be discussed in the remaining sections.

Divergence of the stress tensor $\boldsymbol{\sigma}$ is denoted by $\nabla \cdot \boldsymbol{\sigma}$. It is a vector and defined by

$$\begin{aligned}
\nabla \cdot \boldsymbol{\sigma} = & \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} \right) \hat{\boldsymbol{i}} + \left(\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} \right) \hat{\boldsymbol{j}} \\
& + \left(\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \right) \hat{\boldsymbol{k}}.
\end{aligned} \tag{2.126}$$

In index notation, the component i of $\nabla \cdot \boldsymbol{\sigma}$ can be written as

$$(\nabla \cdot \boldsymbol{\sigma})_i = \sigma_{ij,j}. \tag{2.127}$$

Using the definition of $\nabla \cdot \boldsymbol{\sigma}$, the equations of motion (Eq. 2.125) become

$$\rho \boldsymbol{a} = \rho \boldsymbol{b} + \nabla \cdot \boldsymbol{\sigma}^{\mathrm{T}}. \tag{2.128}$$

In index notation, they can be expressed as

$$\rho a_i = \rho b_i + \sigma_{ji,j}. \tag{2.129}$$

But, since $\boldsymbol{\sigma}$ is a symmetric tensor, the above equations can be written as

$$\rho \boldsymbol{a} = \rho \boldsymbol{b} + \nabla \cdot \boldsymbol{\sigma}, \tag{2.130}$$

or

$$\rho a_i = \rho b_i + \sigma_{ij,j}. \tag{2.131}$$

***Example* 2.9:** For the beam of Figure 2.5, expressions of the stress components with respect to (x, y, z) coordinate system are:

$$\begin{aligned} \sigma_{xx} &= \frac{3wb}{6I_{zz}}(\ell - x)^2 y, \\ \sigma_{yy} &= -\frac{wb}{6I_{zz}}\left(3h^2 y - y^3 + 2h^3\right), \\ \sigma_{xy} &= -\frac{3wb}{6I_{zz}}(l - x)\left(h^2 - y^2\right), \\ \sigma_{xz} &= \sigma_{yz} = \sigma_{zz} = 0. \end{aligned} \tag{2.132}$$

Here, w is the uniform stress acting on the top surface of the beam in negative y direction and b, ℓ and h are the geometric dimensions of the beam (Figure 2.5). Further, I_{zz} is the moment of inertia of the cross-section of the beam about z-axis. Assuming the body force vector $\boldsymbol{b}$ to be zero, verify that the above stress expressions satisfy the equations of motion (Eq. 2.125).

***Solution*:** Since the beam is in equilibrium, the acceleration vector is zero. Therefore,

$$a_x = a_y = a_z = 0. \tag{2.133}$$

Further, since the body force vector is given as zero,

$$b_x = b_y = b_z = 0. \tag{2.134}$$

Then, the equations of motion (Eq. 2.125) reduce to:

$$\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} &= 0, \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} &= 0, \\ \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} &= 0 \end{aligned} \tag{2.135}$$

They are called as the equilibrium equations since the acceleration vector is zero. Differentiating the expressions (Eq. 2.132) for σ_{ij} and substituting the derivatives in the first two equilibrium equations, we get

$$\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{yx}}{\partial y}+\frac{\partial \sigma_{zx}}{\partial z}=\frac{3wb}{6I_{zz}}[-2(\ell-x)y-(\ell-x)(-2y)]+0=0\,, \quad (2.136a)$$

$$\frac{\partial \sigma_{xy}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}+\frac{\partial \sigma_{zy}}{\partial z}=\frac{wb}{6I_{zz}}\left[(-3)(-1)\left(h^2-y^2\right)-\left(3h^2-3y^2\right)\right]+0=0. \quad (2.136b)$$

Because σ_{xz} , σ_{yz} and σ_{zz} are zero (Eq. 2.132), the third equilibrium equation is identically satisfied.

2.4 Deformation

While discussing stresses in a body, we considered only the deformed configuration. However, for describing the deformation of a body, one must consider both the initial (undeformed) and the deformed configurations of the body. Those are shown in Figure 2.6. However, the forces acting on the deformed configuration and the supports are not shown as they are not necessary to discuss the deformation. For the sake of clarity, overlapping of the undeformed and deformed configurations is avoided by assuming the translation of the body to be very large as shown in the figure. Deformation in a body varies from point to point. Deformation at a point has two aspects. When the body is deformed, a small line element P_0Q_0 at a point undergoes a change in its initial length (Figure 2.6). In general, this happens for the line elements in all directions at that point. Similarly, a pair of line elements P_0R_0 and P_0S_0 undergo a change in their initial angle (Figure 2.6). Again, generally, this happens for every pair of line elements at that point. Strain at a point is a measure of the deformation at that point. Thus, strain at a point consists of the following two infinite sets:

- A measure of change in linear dimension in every direction at that point
- A measure of change in angular dimension for every pair of directions at that point.

One can choose various measures to define the strain at a point. For example, one can choose either the change in length per unit length or the change in square length per unit square length or the logarithm of the ratio of new length to the initial length as measures of the change in linear dimension. Further, one can choose the change in angle, the sine function of the change in angle *etc*. as the measures of the change in angle. For specifying the measure of change in angle, usually, the initial angle is chosen to be $\pi/2$ radians.

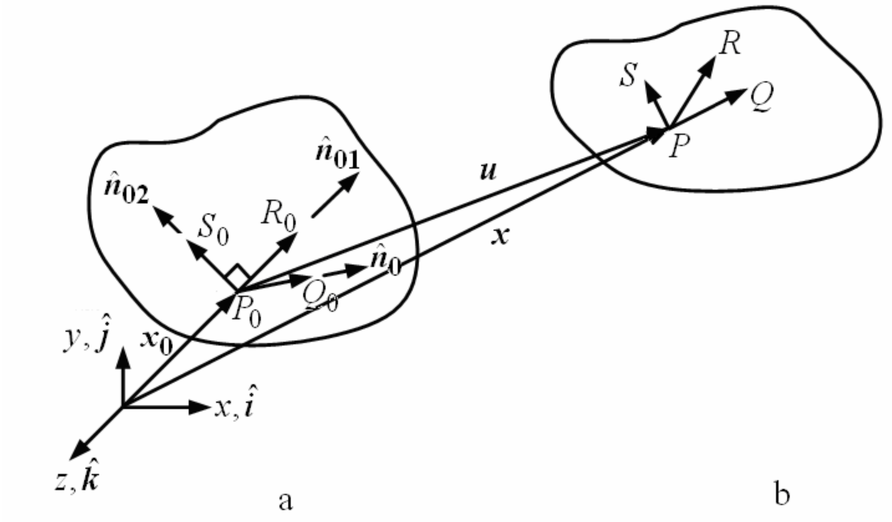


Figure 2.6. Deformation at a point. The length P_0Q_0 changes to PQ in the deformed configuration. The angle $S_0P_0R_0$ changes to SPR in the deformed configuration a. Undeformed configuration; b. Deformed configuration

Deformation at a point is related to the displacement of the neighborhood of that point. The neighborhood of a point is defined as a set of points in the close vicinity of that point. The displacement consists of three parts: (i) displacement due to translation of the neighborhood of that point, (ii) displacement due to rotation of the neighborhood of that point and (iii) displacement due to deformation of the neighborhood of that point. If we consider only the relative displacement of a point with respect to the center of its neighborhood, then it contains the displacement only due to rotation and deformation of the neighborhood. We start our discussion on linear strain tensor at a point with displacement gradient tensor which is a measure of the relative displacement.

2.4.1 Linear Strain Tensor

In this section, we first define the displacement gradient tensor at a point. Then, we decompose it into the symmetric and antisymmetric parts. It is shown that the symmetric part can completely describe the deformation at a point when the deformation is small. It is called as the linear strain tensor. The antisymmetric part represents the rotation when the rotation is small.

2.4.1.1 Displacement Gradient Tensor

Let

$$\boldsymbol{u} = u_x\hat{\boldsymbol{i}} + u_y\hat{\boldsymbol{j}} + u_z\hat{\boldsymbol{k}} \tag{2.137}$$

be the displacement vector at point P_0 whose position vector in the *initial* configuration is given by

$$\boldsymbol{x_0} = x_0\hat{\boldsymbol{i}} + y_0\hat{\boldsymbol{j}} + z_0\hat{\boldsymbol{k}} \tag{2.138}$$

(Figure 2.6). Consider the following array:

$$[\nabla_0 u] = \begin{bmatrix} \dfrac{\partial u_x}{\partial x_0} & \dfrac{\partial u_x}{\partial y_0} & \dfrac{\partial u_x}{\partial z_0} \\ \dfrac{\partial u_y}{\partial x_0} & \dfrac{\partial u_y}{\partial y_0} & \dfrac{\partial u_y}{\partial z_0} \\ \dfrac{\partial u_z}{\partial x_0} & \dfrac{\partial u_z}{\partial y_0} & \dfrac{\partial u_z}{\partial z_0} \end{bmatrix}. \tag{2.139}$$

The subscript zero is used with the symbol ∇ to emphasize the fact that the derivatives are to be taken with respect to the coordinates in the *initial* configuration. Consider a rotated coordinate system (x', y', z') with unit vectors ($\hat{\boldsymbol{i}}', \hat{\boldsymbol{j}}', \hat{\boldsymbol{k}}'$) along them (not shown in Figure 2.6). Further, let the components of the displacement vector $\boldsymbol{u}$ and the position vector $\boldsymbol{x_0}$ along the rotated coordinates be

$$\boldsymbol{u}' = u'_x\hat{\boldsymbol{i}}' + u'_y\hat{\boldsymbol{j}}' + u'_z\hat{\boldsymbol{k}}', \tag{2.140}$$

$$\boldsymbol{x_0'} = x'_0\hat{\boldsymbol{i}}' + y'_0\hat{\boldsymbol{j}}' + z'_0\hat{\boldsymbol{k}}'. \tag{2.141}$$

In (x', y', z') coordinate system, the array of the displacement derivatives can be written as

$$[\nabla_0 u]' = \begin{bmatrix} \dfrac{\partial u'_x}{\partial x'_0} & \dfrac{\partial u'_x}{\partial y'_0} & \dfrac{\partial u'_x}{\partial z'_0} \\ \dfrac{\partial u'_y}{\partial x'_0} & \dfrac{\partial u'_y}{\partial y'_0} & \dfrac{\partial u'_y}{\partial z'_0} \\ \dfrac{\partial u'_z}{\partial x'_0} & \dfrac{\partial u'_z}{\partial y'_0} & \dfrac{\partial u'_z}{\partial z'_0} \end{bmatrix}. \tag{2.142}$$

Using the vector transformation relation (Eq. 2.58) for the components of $\boldsymbol{u}$ and $\boldsymbol{x_0}$, and the chain rule for the derivatives, it can be shown that

$$[\nabla_0 u]' = [Q][\nabla_0 u][Q]^{\mathrm{T}}, \tag{2.143}$$

where the matrix $[Q]$ (Eq. 2.54) represents the transformation from (x, y, z) coordinate system to (x', y', z') system. Thus, the components of the array $[\nabla_0 u]$ are the components of a tensor. It is denoted by $\nabla_{\mathbf{0}} \boldsymbol{u}$ and is called as the *displacement gradient tensor* at the point.

2.4.1.2 Linear Strain Tensor

Every tensor can be decomposed as a sum of symmetric and antisymmetric parts. Thus,

$$\nabla_{\mathbf{0}} \boldsymbol{u} = \frac{1}{2}\left(\nabla_{\mathbf{0}} \boldsymbol{u} + (\nabla_{\mathbf{0}} \boldsymbol{u})^{\mathrm{T}}\right) + \frac{1}{2}\left(\nabla_{\mathbf{0}} \boldsymbol{u} - (\nabla_{\mathbf{0}} \boldsymbol{u})^{\mathrm{T}}\right). \tag{2.144}$$

Here, the first part is symmetric part of the tensor $\nabla_{\mathbf{0}} \boldsymbol{u}$ while the second part is the antisymmetric part. At a point, define tensor $\boldsymbol{\varepsilon}$ as the symmetric part of $\nabla_0 u$:

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\nabla_{\mathbf{0}} \boldsymbol{u} + (\nabla_{\mathbf{0}} \boldsymbol{u})^{\mathrm{T}}\right). \tag{2.145}$$

In matrix notation, this can be written as

$$[\varepsilon] = \frac{1}{2}\left([\nabla_0 u] + [\nabla_0 u]^{\mathrm{T}}\right) , \tag{2.146}$$

while in index notation, it can be expressed as

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) , \tag{2.147}$$

where it is understood that the comma indicates the derivatives with respect to the coordinates in the *initial* configuration.

Assume that the components of the tensor $\nabla_{\mathbf{0}} \boldsymbol{u}$ are small compared to 1 everywhere in the body. In many aerospace, civil and mechanical engineering applications, the components of $\nabla_{\mathbf{0}} \boldsymbol{u}$ are of the order of $10^{-4} - 10^{-6}$. Therefore, this assumption is not very restrictive. Let ε_n denote the *unit extension* along the direction $\hat{\boldsymbol{n}}_{\mathbf{0}}$ at point P_0 of the initial configuration (Figure 2.6), *i.e.*, the change in length per unit length at P_0 along the direction $\hat{\boldsymbol{n}}_{\mathbf{0}}$. Further, let γ_{n1n2} denote the *shear* associated with the directions $\hat{\boldsymbol{n}}_{\mathbf{01}}$ and $\hat{\boldsymbol{n}}_{\mathbf{02}}$ at point P_0 of the initial configuration (Figure 2.6), *i.e.*, the change in angle between the two perpendicular directions $\hat{\boldsymbol{n}}_{\mathbf{01}}$ and $\hat{\boldsymbol{n}}_{\mathbf{02}}$ at P_0. We denote the arrays of the components of $\hat{\boldsymbol{n}}_{\mathbf{0}}$, $\hat{\boldsymbol{n}}_{\mathbf{01}}$ and $\hat{\boldsymbol{n}}_{\mathbf{02}}$ with respect to (x, y, z) coordinates by $\{n_0\}$, $\{n_{01}\}$ and $\{n_{02}\}$. Then, under the above assumption, it can be shown that [5]

$$\varepsilon_n = \{n_0\}^{\mathrm{T}}[\varepsilon]\{n_0\}, \tag{2.148}$$

$$\gamma_{n1n2} = 2\{n_{01}\}^{\mathrm{T}}[\varepsilon]\{n_{02}\}. \tag{2.149}$$

Therefore, under the above assumption, if the tensor $\boldsymbol{\varepsilon}$ is given at a point, we can find the change in length per unit length along any direction at that point. Further, we can find the change in angle between any pair of perpendicular directions at that point. Thus, under the above assumption, the tensor $\boldsymbol{\varepsilon}$ can completely describe the deformation at a point and, therefore, can be used as a strain tensor. It is called as *linear or infinitesimal strain tensor.* Note that the assumption of the components of the tensor $\nabla_{\mathbf{0}}\boldsymbol{u}$ being small implies that the components of the tensor $\boldsymbol{\varepsilon}$ are also small. Therefore, this assumption is called as the *small deformation assumption*. Thus, $\boldsymbol{\varepsilon}$ can be used as a strain tensor, only when the deformation is small. The plastic deformation is often not small. Therefore, to describe the plastic deformation, we shall have to look for some other tensor. Such tensors are discussed in Chapter 3.

Note that, by definition (Eq. 2.145), the tensor $\boldsymbol{\varepsilon}$ is symmetric. Therefore, its components with respect to (x, y, z) coordinate system can be expressed as

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{zx} \\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{yz} & \varepsilon_{zz} \end{bmatrix}. \tag{2.150}$$

Substituting expressions (2.150) and (2.139) into Eq. (2.146), we get the following expressions for the strain components:

$$\begin{aligned} &\varepsilon_{xx} = \frac{\partial u_x}{\partial x_0},\ \varepsilon_{yy} = \frac{\partial u_y}{\partial y_0},\ \varepsilon_{zz} = \frac{\partial u_z}{\partial z_0}. \\ &\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y_0} + \frac{\partial u_y}{\partial x_0}\right), \\ &\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z_0} + \frac{\partial u_z}{\partial y_0}\right), \\ &\varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial u_z}{\partial x_0} + \frac{\partial u_x}{\partial z_0}\right). \end{aligned} \tag{2.151}$$

These are called as the *strain-displacement relations*. The tensor, array and index forms of these equations are given by expressions (2.145-2.147). Note that the strain-displacement relations are *linear* when the deformation is small. For plastic deformation, the strain-displacement relations may be non-linear. They are discussed in Chapter 3.

As stated in the introduction, there are 3 sets of equations which govern the displacements, strains and stresses in a body. This is the *second set of governing equations* when the deformation is small.

By substituting $\hat{\boldsymbol{n}}_{\mathbf{0}} = \hat{\boldsymbol{i}}$ in Eq. (2.148), we find that the component ε_{xx} represents the unit extension (*i.e.*, the change in length per unit length) along the direction which was initially along x-axis. Similarly, the components ε_{yy} and ε_{zz} denote the unit extensions along the directions which were respectively along y and z axes in the initial configuration. These three components, which represent the *deformation in linear dimension* along *three mutually perpendicular directions,* are called as *normal strain components.* By substituting $\hat{\boldsymbol{n}}_{\mathbf{01}} = \hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{n}}_{\mathbf{02}} = \hat{\boldsymbol{j}}$ in Eq. (2.149), we find that the component ε_{xy} represents *half* the shear (*i.e.*, half the change in angle) associated with the directions which were along x and y axes in the initial configuration. Similarly, the component ε_{yz} denotes half the shear associated with the directions which were initially along y and z axes. Further, the component ε_{zx} represents half the change in angle between the directions which were originally along z and x axes. These three components, which represent the *deformation in angular dimension* associated with the same *three mutually perpendicular directions,* are called as *shear strain components.* The sign convention for the strain components is as follows. A *normal strain* component is considered *positive* if there is *elongation* in that direction and *negative* if there is *compression*. A *shear strain* component is considered *positive* if the *angle decreases* and *negative* if the *angle increases*. Note that the sign convention for the shear strain components is different than what you might expect. However, it is chosen to ensure that a positive shear stress would cause a positive shear strain and *vice versa*.

2.4.1.3 Infinitesimal Rotation Tensor

At a point, define tensor $\boldsymbol{\omega}$ as the antisymmetric part of the displacement gradient tensor $\nabla_{\mathbf{0}}\boldsymbol{u}$:

$$\boldsymbol{\omega} = \frac{1}{2}\left(\nabla_{\mathbf{0}}\boldsymbol{u} - (\nabla_{\mathbf{0}}\boldsymbol{u})^{\mathrm{T}}\right). \tag{2.152}$$

In matrix notation, this can be written as

$$[\omega] = \frac{1}{2}\left([\nabla_0 u] - [\nabla_0 u]^{\mathrm{T}}\right), \tag{2.153}$$

whilst in index notation, it can be expressed as

$$\omega_{i,j} = \frac{1}{2}(u_{i,j} - u_{j,i}), \tag{2.154}$$

where it is understood that the comma indicates the derivatives with respect to the coordinates in the *initial* configuration. It can be shown that when components of the tensor $\nabla_{\mathbf{0}}\boldsymbol{u}$ are small compared to 1, the tensor $\boldsymbol{\omega}$ represents rotation of a neighborhood of the point. Note that when the components of $\nabla_{\mathbf{0}}\boldsymbol{u}$ are small, the components of $\boldsymbol{\omega}$ are also small. Thus, $\boldsymbol{\omega}$ represents the rotation only when it is small. We call $\boldsymbol{\omega}$ as the *infinitesimal rotation tensor*.

The diagonal components of $\boldsymbol{\omega}$, namely ω_{xx}, ω_{yy} and ω_{zz} are zero. The expressions for the *non-diagonal components* of $\boldsymbol{\omega}$ are as follows:

$$\begin{aligned}
\omega_{zy} &= -\omega_{yz} = \frac{1}{2}\left(\frac{\partial u_z}{\partial y_0} - \frac{\partial u_y}{\partial z_0}\right), \\
\omega_{xz} &= -\omega_{zx} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z_0} - \frac{\partial u_z}{\partial x_0}\right), \\
\omega_{yx} &= -\omega_{xy} = \frac{1}{2}\left(\frac{\partial u_y}{\partial x_0} - \frac{\partial u_x}{\partial y_0}\right).
\end{aligned} \tag{2.155}$$

The components ω_{zy}, ω_{xz} and ω_{yx} represent the angle of rotation respectively about x, y and z axes. They are considered *positive* if they are *counterclockwise* and *negative* if *clockwise*. Since, an antisymmetric tensor has only 3 non-zero components, one can always associate a vector with it. The vector which can be associated with $\boldsymbol{\omega}$ is given by

$$\begin{aligned}
\omega_{zy}\hat{\boldsymbol{i}} + \omega_{xz}\hat{\boldsymbol{j}} + \omega_{yx}\hat{\boldsymbol{k}} &= \frac{1}{2}\left[\left(\frac{\partial u_z}{\partial y_0} - \frac{\partial u_y}{\partial z_0}\right)\hat{\boldsymbol{i}} + \left(\frac{\partial u_x}{\partial z_0} - \frac{\partial u_z}{\partial x_0}\right)\hat{\boldsymbol{j}} + \left(\frac{\partial u_y}{\partial x_0} - \frac{\partial u_x}{\partial y_0}\right)\hat{\boldsymbol{k}}\right], \\
&= \frac{1}{2}\in_{ijk}\frac{\partial u_k}{\partial x_{0j}}\hat{\boldsymbol{i}}_{\boldsymbol{i}}, \\
&= \frac{1}{2}\nabla_{\mathbf{0}} \times \boldsymbol{u}.
\end{aligned} \tag{2.156}$$

This is consistent with the fact that only small rotation can be expressed as a vector.

***Example* 2.10:** For the beam of Figure 2.7, components of the displacement vector $\boldsymbol{u}$ at a point (x_0, y_0, z_0), with respect to (x, y, z) coordinate system, are given as

$$u_x = A\left\{\frac{1}{2}a^2 y_0 + \left(\frac{1}{2}x_0^2 - \ell x_0\right)y_0 - \frac{1}{4}\left(y_0^3 + y_0 z_0^2\right)\right\}, \tag{2.157a}$$

$$u_y = A\left\{\frac{1}{2}a^2 x_0 + \frac{1}{2}\ell x_0^2 - \frac{1}{6}x_0^3 + \frac{1}{4}(\ell - x_0)(y_0^2 - z_0^2)\right\}, \tag{2.157b}$$

$$u_z = A\left\{\frac{1}{2}(\ell - x_0)y_0 z_0\right\}, \tag{2.157c}$$

where

$$A = \frac{4F_y}{\pi E a^4}. \tag{2.157d}$$

Here, a, ℓ and F_y are as shown in Figure 2.7. Further, E is a material constant which is defined in Section 2.5.1

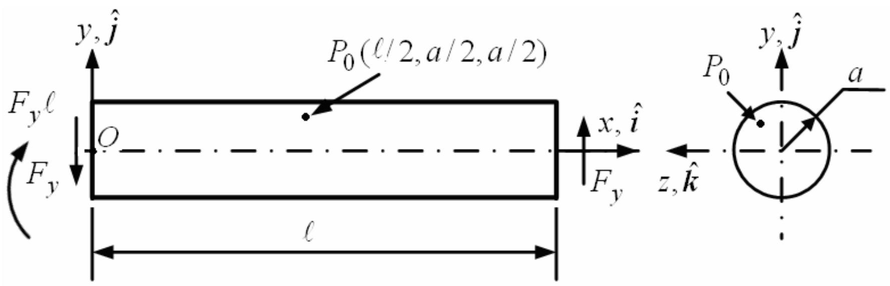


Figure 2.7. A beam of circular cross-section subjected to shear forces and bending moment. The point O is fixed against the translation and rotation. Further, since the deformation is small, the deformed and undeformed configurations almost overlap

(a) Find the components of the displacement gradient tensor $\nabla_{\mathbf{0}}\boldsymbol{u}$.

(b) Find the components of the linear strain tensor $\boldsymbol{\varepsilon}$ and the infinitesimal rotation tensor $\boldsymbol{\omega}$.

(c) Evaluate the strain components at point P_0 (Figure 2.7) whose coordinates are $(x_0, y_0, z_0) = (\ell/2, a/2, a/2)$. Further, at P_0, find the unit extension along the direction

$$\hat{\boldsymbol{n}}_{\mathbf{0}} = (1/\sqrt{3})(\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}). \tag{2.158}$$

and the shear associated with the directions:

$$\hat{\boldsymbol{n}}_{\mathbf{01}} = (1/5)(3\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}),\ \hat{\boldsymbol{n}}_{\mathbf{02}} = (1/5)(4\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}). \tag{2.159}$$

(d) Evaluate the non-diagonal components of $\boldsymbol{\omega}$ at P_0.

***Solution*:** (a) Differentiating Eqs. (2.157a-2.157c), we get the components of the displacement gradient tensor $\nabla_{\mathbf{0}}\boldsymbol{u}$:

$$\frac{\partial u_x}{\partial x_0} = A[(x_0 - \ell)y_0],$$
$$\frac{\partial u_x}{\partial y_0} = A\left[\frac{1}{2}a^2 + \left(\frac{1}{2}x_0^2 - \ell x_0\right) - \frac{1}{4}\left(3y_0^2 + z_0^2\right)\right],$$
$$\frac{\partial u_x}{\partial z_0} = A\left[-\frac{1}{2}y_0 z_0\right],$$
$$\frac{\partial u_y}{\partial x_0} = A\left[\frac{1}{2}a^2 + \ell x_0 - \frac{1}{2}x_0^2 - \frac{1}{4}\left(y_0^2 - z_0^2\right)\right],$$
$$\frac{\partial u_y}{\partial y_0} = A\left[\frac{1}{2}(\ell - x_0)y_0\right],$$
$$\frac{\partial u_y}{\partial z_0} = A\left[-\frac{1}{2}(\ell - x_0)z_0\right],$$
$$\frac{\partial u_z}{\partial x_0} = A\left[-\frac{1}{2}y_0 z_0\right],$$
$$\frac{\partial u_z}{\partial y_0} = A\left[\frac{1}{2}(\ell - x_0)z_0\right],$$
$$\frac{\partial u_z}{\partial z_0} = A\left[\frac{1}{2}(\ell - x_0)y_0\right]. \tag{2.160}$$

(b) Substituting the expressions of the displacement derivatives of part (a) into the strain-displacement relations (Eq. 2.151), we get the components of the linear strain tensor $\boldsymbol{\varepsilon}$:

$$
\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u_x}{\partial x_0} = A(x_0 - \ell)y_0, \\
\varepsilon_{yy} &= \frac{\partial u_y}{\partial y_0} = \frac{A}{2}(\ell - x_0)y_0, \\
\varepsilon_{zz} &= \frac{\partial u_z}{\partial z_0} = \frac{A}{2}(\ell - x_0)y_0, \\
\varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u_x}{\partial y_0} + \frac{\partial u_y}{\partial x_0}\right) = \frac{A}{2}\left(a^2 - y_0^2\right), \\
\varepsilon_{yz} &= \frac{1}{2}\left(\frac{\partial u_y}{\partial z_0} + \frac{\partial u_z}{\partial y_0}\right) = 0, \\
\varepsilon_{zx} &= \frac{1}{2}\left(\frac{\partial u_z}{\partial x_0} + \frac{\partial u_x}{\partial z_0}\right) = -\frac{A}{2} y_0 z_0.
\end{aligned}
\tag{2.161}
$$

Again substituting the expressions of the displacement derivatives of part (a) into the rotation-displacement relations (Eq. 2.155), we get the non-diagonal components of the infinitesimal rotation tensor $\boldsymbol{\omega}$:

$$
\begin{aligned}
\omega_{zy} &= -\omega_{yz} = \frac{1}{2}\left(\frac{\partial u_z}{\partial y_0} - \frac{\partial u_y}{\partial z_0}\right) = \frac{A}{2}(\ell - x_0)z_0, \\
\omega_{xz} &= -\omega_{zx} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z_0} - \frac{\partial u_z}{\partial x_0}\right) = 0, \\
\omega_{yx} &= -\omega_{xy} = \frac{1}{2}\left(\frac{\partial u_y}{\partial x_0} - \frac{\partial u_x}{\partial y_0}\right) = \frac{A}{2}\left[x_0(2\ell - x_0) + \frac{1}{2}\left(y_0^2 + z_0^2\right)\right].
\end{aligned}
\tag{2.162}
$$

The diagonal components of $\boldsymbol{\omega}$, namely ω_{xx} , ω_{yy} and ω_{zz} , are of course zero.

(c) We obtain values of the strain components at point P_0 by substituting $(x_0, y_0, z_0) = (\ell/2, a/2, a/2)$ in the expressions of the strain components of part (b). Then, the strain matrix at point P_0 becomes:

$$
[\varepsilon] = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{zx} \\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{yz} & \varepsilon_{zz} \end{bmatrix} = \frac{Aa}{8}\begin{bmatrix} -2\ell & 3a & -a \\ 3a & \ell & 0 \\ -a & 0 & \ell \end{bmatrix}. \tag{2.163}
$$

To get the unit extension along the direction $\hat{\boldsymbol{n}}_{\mathbf{0}}$ at point P_0, we substitute the above equation along with the components of $\hat{\boldsymbol{n}}_{\mathbf{0}}$ (Eq. 2.158) in Eq. (2.148):

$$\varepsilon_n = \{n_0\}^{\mathrm{T}}[\varepsilon]\{n_0\} = \begin{Bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{Bmatrix} \frac{Aa}{8} \begin{bmatrix} -2\ell & 3a & -a \\ 3a & \ell & 0 \\ -a & 0 & \ell \end{bmatrix} \begin{Bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{Bmatrix} = \frac{1}{6} Aa^2. \tag{2.164a}$$

To get the shear associated with the directions $\hat{\boldsymbol{n}}_{\mathbf{01}}$ and $\hat{\boldsymbol{n}}_{\mathbf{02}}$ at point P_0, we substitute Eq. (2.163) along with the components of $\hat{\boldsymbol{n}}_{\mathbf{01}}$ and $\hat{\boldsymbol{n}}_{\mathbf{02}}$ (Eq. 2.159) in Eq. (2.149):

$$\begin{aligned} \gamma_{n1n2} &= 2\{n_{01}\}^{\mathrm{T}}[\varepsilon]\{n_{02}\} = 2\begin{Bmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \end{Bmatrix} \frac{Aa}{8} \begin{bmatrix} -2\ell & 3a & -a \\ 3a & \ell & 0 \\ -a & 0 & \ell \end{bmatrix} \begin{Bmatrix} \frac{4}{5} \\ \frac{3}{5} \\ 0 \end{Bmatrix}, \\ &= -\frac{1}{100} Aa(36\ell + 21a). \end{aligned} \tag{2.164b}$$

(d) We obtain values of the non-diagonal rotation components at point P_0, by substituting $(x_0, y_0, z_0) = (\ell/2, a/2, a/2)$ in the rotation-displacement equations of part (b). We get

$$\begin{aligned} \omega_{zy} &= \frac{A}{2}(\ell - x_0)z_0 = \frac{1}{8}A\ell a, \\ \omega_{xz} &= 0, \\ \omega_{yx} &= \frac{A}{2}\left[x_0(2\ell - x_0) + \frac{1}{2}\left(y_0^2 + z_0^2\right)\right] = \frac{1}{8}A\left(3\ell^2 + a^2\right). \end{aligned} \tag{2.165}$$

For the following values of geometric, force and material parameters:

$$\ell = 200\,\mathrm{mm}, \ a = 10\,\mathrm{mm}, \ F_y = 100\,\mathrm{N}, \ E = 2\times10^5\,\mathrm{N/mm^2}. \tag{2.166a}$$

we get

$$A = \frac{4F_y}{\pi E a^4} = 6.34 \times 10^{-8} \, . \tag{2.166b}$$

Then, we obtain

$$\varepsilon_n = \frac{1}{6} A a^2 = 1.06 \times 10^{-6}, \; \gamma_{n1n2} = -\frac{1}{100} A a (36\ell + 21a) = -47.17 \times 10^{-6} \, \text{rad} \, , \tag{2.167a}$$

$$\omega_{zy} = \frac{A\ell a}{8} = 1.59 \times 10^{-5} \, \text{rad}, \; \omega_{yx} = \frac{1}{8} A\left(3\ell^2 + a^2\right) = 9.51 \times 10^{-4} \, \text{rad} \, , \tag{2.167b}$$

Thus, for a typical situation, the deformation and rotation are quite small.

2.4.2 Analysis of Strain at a Point

As stated earlier, in this section, we carry out the analysis of strain at a point to discuss the concepts of principal strains and principal directions, principal invariants, maximum shear, volumetric strain and the hydrostatic and deviatoric parts of strain.

2.4.2.1 Principal Strains, Principal Directions and Principal Invariants

There exist at least 3 mutually perpendicular directions (in the initial configuration) such that the shear (γ_{n1n2}) associated with these directions is zero. It means these directions remain perpendicular in the deformed configuration also. These directions are called as the *principal directions* (of strain). The unit extensions (ε_n) along these directions are called as the *principal strains.* We denote the principal strains as ε_1, ε_2 and ε_3 and the unit vectors along the principal directions (of strain) as $\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$ and $\hat{\boldsymbol{e}}_3$. Recall that the same notation has been used for the unit vectors along the principal directions (of stress). However, whether we are referring to the principal directions of stress or strain will be clear from the context. Further, the principal directions of stress exist in the deformed configuration whereas the principal directions of strain exist in the initial configuration. We arrange the principal strains as

$$\varepsilon_1 \geq \varepsilon_2 \geq \varepsilon_3 \, . \tag{2.168}$$

The senses of the unit vectors along the principal directions are so chosen that they always form a right-sided system. Thus,

$$\hat{\boldsymbol{e}}_1 \cdot \hat{\boldsymbol{e}}_2 \times \hat{\boldsymbol{e}}_3 = +1 \, . \tag{2.169}$$

Since the unit extension along a principal direction i $(i = 1,2,3)$ is ε_i and the shear associated with these principal directions is zero, the components of the linear strain tensor, with respect to the principal directions as the coordinate system, become

$$[\varepsilon]^p = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix}. \tag{2.170}$$

It can be easily verified that, at a point, *maximum* value of the *unit extension* (ε_n) with respect to the orientation of the direction $\hat{\boldsymbol{n}}_{\mathbf{0}}$ is ε_1. Further, the *minimum* value is ε_3.

It can be shown that the principal strains are the eigen values or the principal values and the unit vectors along the principal directions are the eigen vectors of the linear strain tensor. The principal strains are determined as the roots of the following equation:

$$\lambda^3 - I_\varepsilon \lambda^2 - II_\varepsilon \lambda - III_\varepsilon = 0, \tag{2.171}$$

where

$$I_\varepsilon = \varepsilon_{ii}, \tag{2.172}$$

$$II_\varepsilon = \frac{1}{2}(\varepsilon_{ij}\varepsilon_{ij} - \varepsilon_{ii}\varepsilon_{jj}), \tag{2.173}$$

$$III_\varepsilon = \in_{ijk} \varepsilon_{1i}\varepsilon_{2j}\varepsilon_{3k}. \tag{2.174}$$

Here, I_ε, II_ε and III_ε are the three *principal invariants* of the linear strain tensor. After finding the principal strains, the unit vectors $\hat{\boldsymbol{e}}_{\boldsymbol{i}}$ along the principal directions are found from an equation similar to Eq. (2.85)

$$\left([\varepsilon] - \varepsilon_i[1] \right)\{e_i\} = \{0\} \text{ (no sum over } i\text{)}. \tag{2.175}$$

2.4.2.2 Maximum Shear

It can be shown that, at a point, *maximum* value of the *shear* (γ_{n1n2}) with respect to the orientation of the directions $\hat{\boldsymbol{n}}_{\mathbf{01}}$ and $\hat{\boldsymbol{n}}_{\mathbf{02}}$ is

$$\left| \gamma_{n1n2} \right|_{\max} = \varepsilon_1 - \varepsilon_3. \tag{2.176}$$

Further, the directions associated with the maximum shear are given by

$$\hat{\boldsymbol{n}}_{\mathbf{01}} = \pm\frac{1}{\sqrt{2}}(\hat{\boldsymbol{e}}_{\mathbf{1}} + \hat{\boldsymbol{e}}_{\mathbf{3}}),\ \hat{\boldsymbol{n}}_{\mathbf{02}} = \pm\frac{1}{\sqrt{2}}(\hat{\boldsymbol{e}}_{\mathbf{1}} - \hat{\boldsymbol{e}}_{\mathbf{3}})\,. \tag{2.177}$$

2.4.2.3 Volumetric Strain, Decomposition into the Hydrostatic and Deviatoric Parts

The change in volume per unit volume of a small element around a point is called as the *volumetric strain* and is denoted by ε_v. It can be shown that when the deformation is small (*i.e.*, when the components of the tensor $\nabla_{\mathbf{0}}\boldsymbol{u}$ are small compared to 1), ε_v is given by [2,4]

$$\varepsilon_v = tr\boldsymbol{\varepsilon} = \varepsilon_{ii}\,. \tag{2.178}$$

Similar to the decomposition of the stress tensor $\boldsymbol{\sigma}$ (Eq. 2.99), the linear strain tensor $\boldsymbol{\varepsilon}$ also can be decomposed as

$$\boldsymbol{\varepsilon} = \left(\frac{1}{3}tr\boldsymbol{\varepsilon}\right)\mathbf{1} + \boldsymbol{\varepsilon}',\ \ \left(tr\boldsymbol{\varepsilon}' = 0\right). \tag{2.179}$$

In index notation, this can be written as

$$\varepsilon_{ij} = \left(\frac{1}{3}\varepsilon_{kk}\right)\delta_{ij} + \varepsilon'_{ij},\ (\varepsilon'_{ii} = 0). \tag{2.180}$$

Note that, since $\boldsymbol{\varepsilon}$ is a symmetric tensor, $\boldsymbol{\varepsilon}'$ is also a symmetric tensor. The first part of Eqs. (2.179-2.180) is called as the *hydrostatic* part of $\boldsymbol{\varepsilon}$ while the second part is called as the *deviatoric* part of $\boldsymbol{\varepsilon}$. Since, $tr\boldsymbol{\varepsilon}$ is volumetric strain, the hydrostatic part of $\boldsymbol{\varepsilon}$ represents a deformation in which there is only change in volume (or size) but no change in shape. Such a deformation is called as *dilatation.* Since $tr\boldsymbol{\varepsilon}'$ is zero, the deviatoric part of $\boldsymbol{\varepsilon}$ represents a deformation in which there is no change in volume but only change in shape. Such a deformation is called as *distortion.*

As stated earlier, in *isotropic materials*, the hydrostatic part of stress tensor causes only dilation type of deformation while the deviatoric part causes only the distortion type of deformation. The yielding consists of only the distortion type of deformation. Therefore, as we shall see in Chapter 3, in isotropic ductile materials yielding is caused only by the deviatoric part of stress tensor.

***Example* 2.11:** Components of the linear strain tensor $\boldsymbol{\varepsilon}$ at a point with respect to the (x, y, z) coordinate system are given by

$$[\varepsilon] = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix} 10^{-4} . \tag{2.181}$$

(a) Find the principal invariants of $\boldsymbol{\varepsilon}$.
(b) Find the principal strains ε_i and the unit vectors $\hat{\boldsymbol{e}}_i$ along the principal directions. Arrange ε_i such that $\varepsilon_1 \geq \varepsilon_2 \geq \varepsilon_3$. Further, choose the senses of $\hat{\boldsymbol{e}}_i$ such that $\hat{\boldsymbol{e}}_1 \cdot \hat{\boldsymbol{e}}_2 \times \hat{\boldsymbol{e}}_3 = +1$.
(c) Find the maximum shear (γ_{n1n2}) and the directions $\hat{\boldsymbol{n}}_{01}$ and $\hat{\boldsymbol{n}}_{02}$ associated with maximum shear.
(d) Using the tensor transformation relation, find the components of $\boldsymbol{\varepsilon}$ with respect to the principal directions as the coordinate system.

Solution: (a) Substituting the values of ε_{ij} from Eq. (2.181) and the values of $\in_{ijk}$ from Eq. (2.14) in the expressions (2.172-2.174), we get

$$I_\varepsilon = \varepsilon_{ii} = (0+0+0)\times 10^{-4} = 0 , \tag{2.182a}$$

$$II_\varepsilon = \frac{1}{2}(\varepsilon_{ij}\varepsilon_{ij} - \varepsilon_{ii}\varepsilon_{jj}) = \frac{1}{2}\left[3(0)^2 + 6(2)^2 - (0)(0)\right]\times 10^{-8} = 12\times 10^{-8} , \tag{2.182b}$$

$$\begin{aligned} III_\varepsilon &= \in_{ijk} \varepsilon_{1i}\varepsilon_{2j}\varepsilon_{3k} , \\ &= \varepsilon_{11}(\varepsilon_{22}\varepsilon_{33} - \varepsilon_{23}\varepsilon_{32}) + \varepsilon_{12}(\varepsilon_{23}\varepsilon_{31} - \varepsilon_{21}\varepsilon_{33}) + \varepsilon_{13}(\varepsilon_{21}\varepsilon_{32} - \varepsilon_{22}\varepsilon_{31}) \\ &= 0(0\times 0 - 2\times 2) + 2(2\times 2 - 2\times 0) + 2(2\times 2 - 0\times 2) \\ &= 16\times 10^{-12}. \end{aligned} \tag{2.182c}$$

(b) Substituting the values of I_ε, II_ε and III_ε from part (a), the cubic equation for λ (Eq. 2.171) becomes

$$\lambda^3 - 0\lambda^2 - 12\times 10^{-8}\lambda - 16\times 10^{-12} = 0 . \tag{2.183}$$

The roots of this equation are: $\lambda = 4\times 10^{-4}, -2\times 10^{-4}, -2\times 10^{-4}$. Thus, we have a double eigen value. Arranging the roots in decreasing order, we get the following values of the principal strains:

$$\varepsilon_1 = 4\times 10^{-4},\ \varepsilon_2 = -2\times 10^{-4},\ \varepsilon_3 = -2\times 10^{-4} , \tag{2.184}$$

To find the unit vectors $\hat{\boldsymbol{e}}_i$ along the principal directions, we follow the procedure of Example 2.8(b). Thus, corresponding to the first eigen value $\varepsilon_1 = 4\times10^{-4}$, we get the following expression for the first unit vector:

$$\hat{\boldsymbol{e}}_1 = \frac{1}{\sqrt{3}}(\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}) . \tag{2.185a}$$

While finding the eigenvector corresponding to the eigen value $\varepsilon_2 = -2\times10^{-4}$, it is observed that only one scalar equation out of the three equations (Eq. 2.175) satisfied by the components of $\hat{\boldsymbol{e}}_2$ is linearly independent. This means the eigenvector has no unique direction. In fact, it can be shown that every vector in the plane perpendicular to $\hat{\boldsymbol{e}}_1$ is an eigenvector of $\varepsilon_2 = -2\times10^{-4}$. This happens because it is a double eigen value. Therefore, we choose any pair of orthonormal vectors (*i.e.*, any two unit vectors perpendicular to each other) in the plane perpendicular to $\hat{\boldsymbol{e}}_1$ as the vectors $\hat{\boldsymbol{e}}_2$ and $\hat{\boldsymbol{e}}_3$. We make the following choice:

$$\hat{\boldsymbol{e}}_2 = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}) , \tag{2.185b}$$

$$\hat{\boldsymbol{e}}_3 = \frac{1}{\sqrt{6}}(\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}) . \tag{2.185c}$$

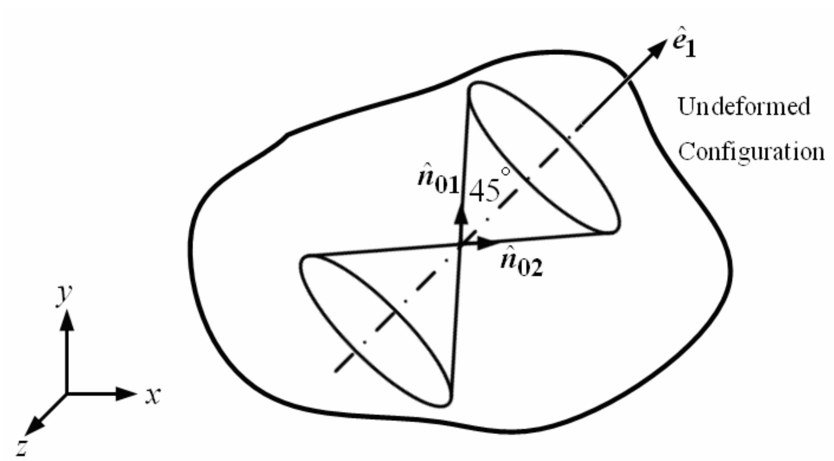


Figure 2.8. Conical surface on which the directions $\hat{n}_{01}$ and $\hat{n}_{02}$ associated with maximum shear lie when the second and third principal stresses are equal. The vector $\hat{e}_1$ represents the first principal direction

(c) Maximum shear is given by Eq. (2.176). Substituting the values of ε_1 and ε_3 from part (b) in this equation, we get

$$\left|\gamma_{n1n2}\right|_{\max} = \varepsilon_1 - \varepsilon_3 = [4-(-2)]\times 10^{-4} = 6\times 10^{-4}\,. \tag{2.186}$$

The directions $\hat{\boldsymbol{n}}_{\mathbf{01}}$ and $\hat{\boldsymbol{n}}_{\mathbf{02}}$ associated with $\left|\gamma_{n1n2}\right|_{\max}$ are given by Eq. (2.177). So we can obtain them by substituting the expressions for $\hat{\boldsymbol{e}}_{\mathbf{1}}$ and $\hat{\boldsymbol{e}}_{\mathbf{3}}$ from part (b) into this equation. However, the vector $\hat{\boldsymbol{e}}_{\mathbf{3}}$ has no unique direction. As stated earlier, it can have any direction in the plane perpendicular to $\hat{\boldsymbol{e}}_{\mathbf{1}}$. (Eq. 2.185c is just one such direction). Therefore, the directions $\hat{\boldsymbol{n}}_{\mathbf{01}}$ and $\hat{\boldsymbol{n}}_{\mathbf{02}}$ associated with $\left|\gamma_{n1n2}\right|_{\max}$ are also not unique. Expression (2.177) shows that whereas $\hat{\boldsymbol{n}}_{\mathbf{01}}$ (with $+$ sign) makes an angle of 45^0 with both $\hat{\boldsymbol{e}}_{\mathbf{1}}$ and $\hat{\boldsymbol{e}}_{\mathbf{3}}$ directions, $\hat{\boldsymbol{n}}_{\mathbf{02}}$ (with $+$ sign) makes an angle of 45^0 with $\hat{\boldsymbol{e}}_{\mathbf{1}}$ direction but 135^0 with $\hat{\boldsymbol{e}}_{\mathbf{3}}$ direction. Thus, the directions $\hat{\boldsymbol{n}}_{\mathbf{01}}$ and $\hat{\boldsymbol{n}}_{\mathbf{02}}$ lie on the surface of a cone whose axis is along $\hat{\boldsymbol{e}}_{\mathbf{1}}$ and semi-cone angle is 45^0 (Figure 2.8).

(d) To find the components of $\boldsymbol{\varepsilon}$ with respect to the principal directions, we first evaluate the transformation matrix $[Q]$. For that purpose, we substitute the direction cosines of the principal directions as given by Eqs. (2.185a-2.185c) into the expression (Eq. 2.54) for $[Q]$. Thus, we get

$$[Q] = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix}. \tag{2.187}$$

Using the tensor transformation relation (Eq. 2.55), we get the following matrix of the components of the strain tensor with respect to the principal directions:

$$[\varepsilon]^p = [Q][\varepsilon][Q]^{\mathrm{T}},$$

$$= \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix} \times 10^{-4} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix},$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \times 10^{-4}. \tag{2.188}$$

Note that if we choose any other pair of orthonormal vectors (in the plane perpendicular to $\hat{\boldsymbol{e}}_1$) as the second and third principal directions, then also the tensor transformation relations will lead to the same expression for $[\varepsilon]^p$.

***Example* 2.12:** Components of the linear strain tensor $\boldsymbol{\varepsilon}$ at point O of the thin plate of Figure 2.9, with respect to (x, y, z) coordinate system, are given as:

$$[\varepsilon] = \frac{\sigma_0}{E} \begin{bmatrix} 1-\nu & 0 & 0 \\ 0 & 1-\nu & 0 \\ 0 & 0 & -2\nu \end{bmatrix}. \tag{2.189}$$

Here, σ_0 is the maximum value of the parabolically varying tensile stresses acting on the edges of the plate (Figure 2.9). Further, E and ν are the material constants which are defined in Section 2.5.1.

(a) Find the volumetric strain ε_v at point O.

(b) Find the hydrostatic and deviatoric parts of $\boldsymbol{\varepsilon}$ at point O.

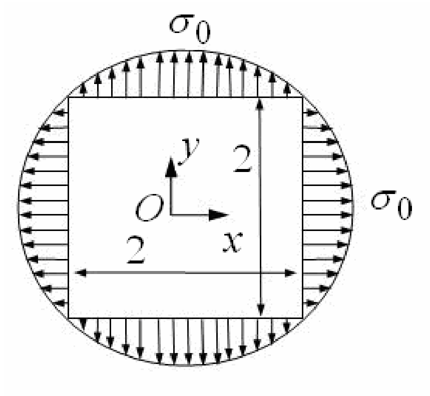


Figure 2.9. A thin square plate subjected to in-plane stresses. Since the deformation is small, the initial and deformed configurations almost overlap.

***Solution*:** (a) Using the values of ε_{ij} from Eq. (2.189), we get

$$\varepsilon_{ii} = \frac{\sigma_0}{E}[(1-\nu)+(1-\nu)+(-2\nu)] = \frac{\sigma_0}{E}2(1-2\nu)\,. \tag{2.190}$$

Substituting this expression into Eq. (2.178) for the volumetric strain, we get

$$\varepsilon_v = \varepsilon_{ii} = \frac{\sigma_0}{E}2(1-2\nu)\,. \tag{2.191}$$

(b) As per Eq. (2.180), components of the hydrostatic part are given by $[(1/3)\varepsilon_{kk}]\delta_{ij}$. Since $\varepsilon_{kk} = \frac{\sigma_0}{E}2(1-2\nu)$ from part (a), the matrix of the hydrostatic part of $\boldsymbol{\varepsilon}$ becomes:

$$\frac{\sigma_0}{E}\begin{bmatrix} \frac{2}{3}(1-2\nu) & 0 & 0 \\ 0 & \frac{2}{3}(1-2\nu) & 0 \\ 0 & 0 & \frac{2}{3}(1-2\nu) \end{bmatrix}. \tag{2.192}$$

Using $\varepsilon_{kk} = \frac{\sigma_0}{E}2(1-2\nu)$ and Eq. (2.180), components of the deviatoric part can be expressed as:

$$\varepsilon'_{ij} = \varepsilon_{ij} - \frac{\sigma_0}{E}\frac{2}{3}(1-2\nu)\delta_{ij}\,. \tag{2.193}$$

Using the values of ε_{ij} from Eq. (2.189), we get the following expression for the matrix of the deviatoric part:

$$[\varepsilon]' = \frac{\sigma_0}{E}\begin{bmatrix} 1-\nu & 0 & 0 \\ 0 & 1-\nu & 0 \\ 0 & 0 & -2\nu \end{bmatrix} - \frac{\sigma_0}{E}\begin{bmatrix} \frac{2}{3}(1-2\nu) & 0 & 0 \\ 0 & \frac{2}{3}(1-2\nu) & 0 \\ 0 & 0 & \frac{2}{3}(1-2\nu) \end{bmatrix},$$

$$= \frac{\sigma_0}{E}\begin{bmatrix} \frac{1+\nu}{3} & 0 & 0 \\ 0 & \frac{1+\nu}{3} & 0 \\ 0 & 0 & -\frac{2(1+\nu)}{3} \end{bmatrix}. \tag{2.194}$$

2.4.3 Compatibility Conditions

Suppose the linear strain tensor at a point is known as a function of initial coordinates (x_0, y_0, z_0) of the point and we wish to find the displacement vector $\boldsymbol{u}$ at that point by integrating the strain-displacement relations (Eqs. 2.151). Then, we have six scalar equations to solve but only three scalar unknowns to be determined. These unknowns are the components (u_x, u_y, u_z) of the displacement vector. Is it possible to get a single-valued solution in this case? The *necessary condition* to get the single-valued displacements, in this case, is that the strain components should satisfy the following constraints [2, 3, 4]:

$$
\begin{aligned}
\boldsymbol{E}_1 &\equiv \frac{\partial^2 \varepsilon_{\boldsymbol{xx}}}{\partial \boldsymbol{y}_0^2} + \frac{\partial^2 \varepsilon_{\boldsymbol{yy}}}{\partial \boldsymbol{x}_0^2} - 2\frac{\partial^2 \varepsilon_{\boldsymbol{xy}}}{\partial \boldsymbol{x}_0 \partial \boldsymbol{y}_0} = 0,\\
\boldsymbol{E}_2 &\equiv \frac{\partial^2 \varepsilon_{\boldsymbol{yy}}}{\partial \boldsymbol{z}_0^2} + \frac{\partial^2 \varepsilon_{\boldsymbol{zz}}}{\partial \boldsymbol{y}_0^2} - 2\frac{\partial^2 \varepsilon_{\boldsymbol{yz}}}{\partial \boldsymbol{y}_0 \partial \boldsymbol{z}_0} = 0,\\
\boldsymbol{E}_3 &\equiv \frac{\partial^2 \varepsilon_{\boldsymbol{zz}}}{\partial \boldsymbol{x}_0^2} + \frac{\partial^2 \varepsilon_{\boldsymbol{xx}}}{\partial \boldsymbol{z}_0^2} - 2\frac{\partial^2 \varepsilon_{\boldsymbol{zx}}}{\partial \boldsymbol{z}_0 \partial \boldsymbol{x}_0} = 0,\\
\boldsymbol{E}_4 &\equiv \frac{\partial^2 \varepsilon_{\boldsymbol{xx}}}{\partial \boldsymbol{y}_0 \partial \boldsymbol{z}_0} - \frac{\partial}{\partial \boldsymbol{x}_0}\left[-\frac{\partial \varepsilon_{\boldsymbol{yz}}}{\partial \boldsymbol{x}_0} + \frac{\partial \varepsilon_{\boldsymbol{zx}}}{\partial \boldsymbol{y}_0} + \frac{\partial \varepsilon_{\boldsymbol{xy}}}{\partial \boldsymbol{z}_0} \right] = 0,\\
\boldsymbol{E}_5 &\equiv \frac{\partial^2 \varepsilon_{\boldsymbol{yy}}}{\partial \boldsymbol{z}_0 \partial \boldsymbol{x}_0} - \frac{\partial}{\partial \boldsymbol{y}_0}\left[\frac{\partial \varepsilon_{\boldsymbol{yz}}}{\partial \boldsymbol{x}_0} - \frac{\partial \varepsilon_{\boldsymbol{zx}}}{\partial \boldsymbol{y}_0} + \frac{\partial \varepsilon_{\boldsymbol{xy}}}{\partial \boldsymbol{z}_0} \right] = 0,\\
\boldsymbol{E}_6 &\equiv \frac{\partial^2 \varepsilon_{\boldsymbol{zz}}}{\partial \boldsymbol{x}_0 \partial \boldsymbol{y}_0} - \frac{\partial}{\partial \boldsymbol{z}_0}\left[\frac{\partial \varepsilon_{\boldsymbol{yz}}}{\partial \boldsymbol{x}_0} + \frac{\partial \varepsilon_{\boldsymbol{zx}}}{\partial \boldsymbol{y}_0} - \frac{\partial \varepsilon_{\boldsymbol{xy}}}{\partial \boldsymbol{z}_0} \right] = 0.
\end{aligned}
\tag{2.195}
$$

These conditions are called as the strain compatibility conditions or integrability conditions.

While finding three unknowns from six equations, it would seem that only three constraints are needed. But, we have six conditions. However, it can be shown that only three out of the six compatibility conditions are independent [2].

It can be shown that conditions (Eq. 2.195) are also *sufficient* for getting the single-valued displacements, but only for *simply-connected regions* [2, 3, 4]. For multiply-connected regions, additional compatibility conditions are required. Further, when the conditions (Eq. 2.195) are satisfied in a simply-connected region, only the *non-rigid part of the displacement vector* becomes single-valued. Uniqueness of the *rigid part of the displacement vector* depends on the displacement boundary conditions of the problem.

***Example* 2.13:** Components of the linear strain tensor $\boldsymbol{\varepsilon}$ at a point (x_0, y_0, z_0), with respect to (x, y, z) coordinate system, are given as

$$
\begin{aligned}
\varepsilon_{xx} &= a\left(x_0^2 + y_0^2\right),\\
\varepsilon_{yy} &= b\left(x_0^2 + y_0^2\right),\\
\varepsilon_{xy} &= c x_0 y_0,\\
\varepsilon_{xz} &= \varepsilon_{yz} = \varepsilon_{zz} = 0,
\end{aligned}
\tag{2.196}
$$

where a, b and c are constants. Check whether this state of strain is compatible.

***Solution*:** Note that the strain components ε_{xz}, ε_{yz} and ε_{zz} are zero. Further, the components ε_{xx}, ε_{yy} and ε_{xy} are independent of z_0. Therefore, the last 5 compatibility conditions (Eq. 2.195) are identically satisfied. Substituting the expressions (Eq. 2.196) for ε_{xx} , ε_{yy} and ε_{xy} in the first compatibility condition, we get

$$E_1 \equiv \frac{\partial^2 \varepsilon_{xx}}{\partial y_0^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x_0^2} - 2\frac{\partial^2 \varepsilon_{xy}}{\partial x_0 \partial y_0} = 2a + 2b - 2c \, . \tag{2.197}$$

Therefore, the given state of strain is compatible if

$$a + b = c \, . \tag{2.198}$$

Note that when the strain components ε_{xz} , ε_{yz} and ε_{zz} are zero at a point, the state of deformation is called as the *state of plane strain (at a point) in* $x-y$ *plane.* When these strain components are zero at every point of the body and if, additionally, the remaining strain components ε_{xx} , ε_{yy} and ε_{xy} are independent of z_0 , it is called as the *state of plane strain (in a body) in* $x-y$ *plane.* It is seen that the state of strain described by Eq. (2.196) is of this type.

2.5 Material Behavior

Relations which characterize various responses (like mechanical, thermal, electrical *etc*.) of a material are called as the *constitutive equations*. It is these relations which differentiate one material from another. These relations are based on experimental observation.

In this section, we shall consider only mechanical response. It is possible that a mechanical response may be caused by non-mechanical stimuli like a change in temperature or an application of electromagnetic field. But, we shall consider only *purely mechanical response*, that is, a mechanical response caused by a mechanical stimulus. Constitutive equation for such a response is usually expressed as a relation between the applied forces and the resulting deformation. In order to eliminate effects of the shape and size of the body and the nature and point of application of the loading, normally the constitutive equation is formulated for a material particle rather than for the whole body. For a purely mechanical response, such an equation is expressed as a relation between the stress and a measure of deformation (strain) and/or a measure of rate of deformation (strain rate).

There are various types of mechanical responses. The basic responses are : (i) elastic response, (ii) plastic response and (iii) viscous response. Sometimes, the response consists of a combination of the basic responses. Further, a material may exhibit different types of responses over different ranges of deformation. For example, metals behave elastically at small deformation but exhibit plastic behavior at a large deformation. As a result, it is quite difficult to express the complete mechanical behavior of a material over the entire range of deformation through just one single equation. Therefore, we simplify the constitutive equation by restricting ourselves to only *small deformation*. As stated earlier, metals behave elastically at small deformation. Therefore, in this section, we shall develop constitutive equation for *elastic* behavior of metals at small deformation. In elastic response, the stress depends on the instantaneous value of strain. Further, this relation is one-to-one. It means, if the external forces acting on the body are removed (*i.e.*, if the stress is reduced to the value zero), the strain will also attain the value zero, thereby bringing the body to the original undeformed configuration.

2.5.1 Elastic Stress-Strain Relations for Small Deformation

For small deformation, the linear strain tensor $\boldsymbol{\varepsilon}$ can be used as a measure of the deformation. Therefore, for small deformation, the constitutive equation becomes a relation between $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$.

2.5.1.1 One Dimensional Experimental Observations

As stated earlier, constitutive equations are based on experimental observation. Therefore, let us first see what the experimental observations are about the relation between $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$. The simplest experiment is the *tension test*. In tension test, a rod of uniform cross-section is subjected to an (axial) tensile force F_x as shown in Figure 2.10. The geometry and loading are such that, it is reasonable to assume that the state of stress is one-dimensional and homogeneous in the region away from the ends. That is, the only non-zero stress component is σ_{xx} and it is constant. Further, the state of strain also can be assumed to be homogeneous in the region away from the ends. But, the number of non-zero strain components is not one. Only the shear strain components can be assumed to be zero. Thus, there are three non-zero strain components, namely ε_{xx}, ε_{yy} and ε_{zz} and all are constant.

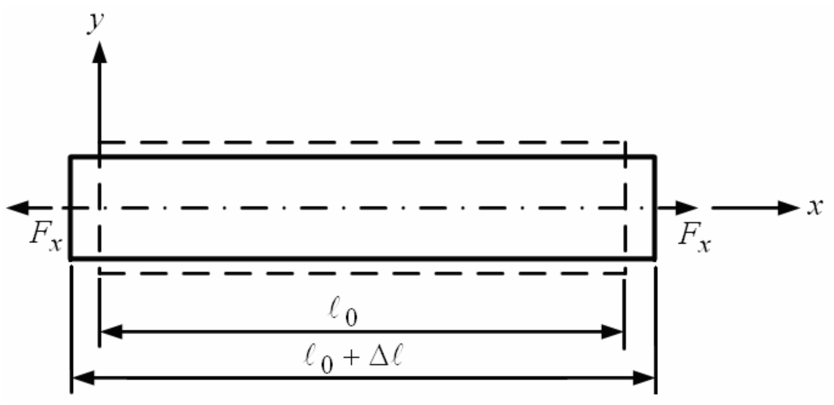


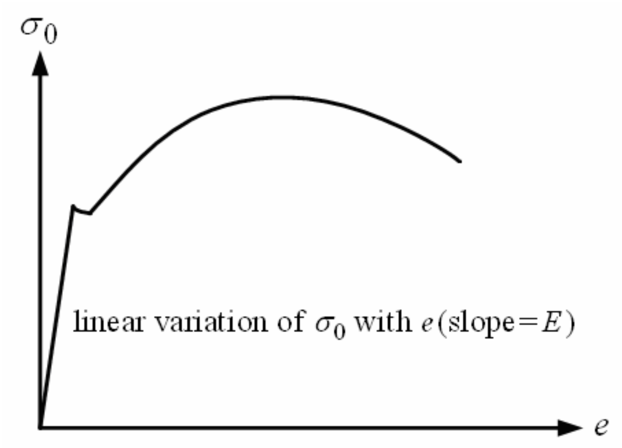
Figure 2.10. Rod subjected to axial tensile forces. The dashed lines indicate the undeformed configuration

For the rod of Figure 2.10, we define the following

$$\sigma_0 = \frac{F_x}{A_0}, \tag{2.199}$$

$$e = \frac{\Delta \ell}{\ell_0}, \tag{2.200}$$

where, A_0 is the initial area of the cross-section of the rod, ℓ_0 is the initial length of the rod and $\Delta\ell$ is the change in length corresponding to the (axial) tensile force F_x. Note that, when the deformation is small (*i.e.*, when the area A_0 does not change much), σ_0 is almost equal to σ_{xx} component of the stress tensor. However, when the change in area is large, σ_0 does not represent the *true stress*.



F**igure 2.11.** Variation of engineering stress with engineering strain for a ductile material in tension test

Therefore, we call σ_0 as *engineering* or *nominal stress*. Again, when the deformation is small (*i.e.*, when the change in length $\Delta \ell$ is small), e is equal to $\partial u/\partial x$ and thus represents ε_{xx} component of the *linear* or *infinitesimal strain* tensor. But, when the change in length is large, ε_{xx} or $\partial u/\partial x$ does not become equal to e . Therefore, we call e as the *engineering strain*.

Figure 2.11 shows the variation of σ_0 with e upto fracture for a typical metal (mild steel). The figure shows that σ_0 varies linearly with e when the deformation is small. But, for small deformation, σ_0 is same as σ_{xx} and e is equal to ε_{xx} . Therefore, for small deformation, σ_{xx} varies linearly with ε_{xx} .

It should be noted that the stress-strain relations need not be linear for all elastic materials. For a material like rubber, which is elastic in nature, the stress-strain relations are non-linear.

2.5.1.2 Generalization to Three Dimensional Case

One can generalize the one-dimensional experimental observation of Figure 2.11 (for small deformation) as follows. For small deformation, one can assume that each stress component depends linearly on all the components of the linear strain tensor. Thus,

$$\begin{aligned} \sigma_{xx} &= C_{xxxx}\varepsilon_{xx} + C_{xxxy}\varepsilon_{xy} + C_{xxxz}\varepsilon_{xz} + C_{xxyx}\varepsilon_{yx} + \ldots\ldots\ldots\ldots + C_{xxzz}\varepsilon_{zz}, \\ \sigma_{xy} &= C_{xyxx}\varepsilon_{xx} + C_{xyxy}\varepsilon_{xy} + C_{xyxz}\varepsilon_{xz} + C_{xyyx}\varepsilon_{yx} + \ldots\ldots\ldots\ldots + C_{xyzz}\varepsilon_{zz}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ \sigma_{zz} &= C_{zzxx}\varepsilon_{xx} + C_{zzxy}\varepsilon_{xy} + C_{zzxz}\varepsilon_{xz} + C_{zzyx}\varepsilon_{yx} + \ldots\ldots\ldots\ldots\ldots\ldots + C_{zzzz}\varepsilon_{zz}. \end{aligned} \tag{2.201}$$

The stress-strain relations given by Eq. (2.201) have 81 material constants. These constants characterize the elastic response of the metal at small deformation. These constants need to be determined by experiments.

In index notation, Eq. (2.201) can be written as

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}. \tag{2.202}$$

Note that, C_{ijkl} are the components of a *fourth order tensor* $\boldsymbol{C}$ which is called as the *Elasticity tensor*. In three dimensions, a fourth order tensor has $3^4 = 81$ components.

2.5.1.3. Restrictions on Elasticity Tensor ***C***

One can reduce the number of constants in the stress-strain relation as follows. Since, σ_{ij} and ε_{kl} are *symmetric* tensors, that is,

$$\sigma_{ij} = \sigma_{ji}, \ \varepsilon_{kl} = \varepsilon_{lk}, \tag{2.203}$$

the components C_{ijkl} must satisfy the following relations:

$$C_{ijkl} = C_{jikl}, C_{ijkl} = C_{ijlk}. \tag{2.204}$$

These relations imply that the tensor $\boldsymbol{C}$ has only 36 independent components.

Further simplification can be achieved by using conservative nature of the internal forces generated by elastic response. For a certain class of elastic materials, work done by the internal forces, during deformation, is path-independent. As a result, the work of deformation (per unit volume) during an infinitesimal deformation can be expressed as an exact differential of a scalar quantity which has the dimensions of energy per unit volume (called as the *strain energy density*). The work of deformation (per unit volume) during an infinitesimal deformation is $\sigma_{ij} d\varepsilon_{ij} = C_{ijkl}\varepsilon_{kl} d\varepsilon_{ij}$. For this expression to be an exact differential, the tensor $\boldsymbol{C}$ must be symmetric in the first two and the last two indices:

$$C_{ijkl} = C_{klij}. \tag{2.205}$$

Equations (2.204-2.205) imply that the tensor $\boldsymbol{C}$ has only 21 independent components.

For *isotropic* materials, the number of independent components of $\boldsymbol{C}$ can be reduced further. For isotropic material, the response of the material is same in every direction. Mathematically it means, the constants in the stress-strain relations remain invariant with a change in the coordinate system. Equation (2.202) represents the stress-strain relations in (x, y, z) coordination system. Let σ'_{ij} and ε'_{kl} represent respectively the stress and strain components in (x', y', z') coordinate system. Then, for isotropic materials, the stress-strain relations in (x', y', z') coordinate system can be written as

$$\sigma'_{ij} = C_{ijkl}\varepsilon'_{kl}. \tag{2.206}$$

Note that, since the material is isotropic, the constants C_{ijkl} appearing in the stress-strain relations are same both in (x, y, z) and (x', y', z') coordinate systems. Note that, since $\boldsymbol{\sigma}$ is a second order tensor, its components σ'_{ij} and σ_{mn} with respect to two coordinate systems are related by the tensor transformation relation (Eq. 2.56). Rewriting this relation with the change of indices, we get

$$\sigma'_{ij} = Q_{im}\sigma_{mn}Q^{\mathrm{T}}_{nj}, \tag{2.207}$$

where the matrix $[Q]$ (Eq. 2.54) represents the transformation from (x, y, z) coordinate system to (x', y', z') system. Since $\boldsymbol{\varepsilon}$ is also a second order tensor, its components ε'_{kl} and ε_{pq} are also related by a similar relation:

$$\varepsilon'_{kl} = Q_{kp}\varepsilon_{pq}Q^{\mathrm{T}}_{ql}. \tag{2.208}$$

Substituting the relations (2.207-2.208) in Eq. (2.206) and using the orthogonality of matrix $[Q]$ (Eq. 2.61), we get

$$\sigma_{mn} = \left(Q^{\mathrm{T}}_{mi}\, Q_{jn}\, C_{ijkl}\, Q_{kp}\, Q^{\mathrm{T}}_{ql}\right)\varepsilon_{pq}. \tag{2.209}$$

In changed indices, Eq. (2.202) can be rewritten as

$$\sigma_{mn} = C_{mnpq} \varepsilon_{pq} . \tag{2.210}$$

Comparing Eqs. (2.209) and (2.210), we get the following restriction on the components of $\boldsymbol{C}$ due to isotropy:

$$C_{mnpq} = Q_{im} \, Q_{jn} \, C_{ijkl} \, Q_{kp} \, Q_{lq} . \tag{2.211}$$

Equation (2.211) must hold for all rotations of a coordinate system, *i.e.* for all orthogonal matrices whose determinant is $+1$.

Equations (2.204) and (2.211) imply that the six components C_{1122}, C_{1133}, C_{2211}, C_{2233}, C_{3311} and C_{3322} are equal. Further, these equations imply that the twelve components C_{1212}, C_{1221}, C_{2112}, C_{2121}, C_{2323}, C_{2332}, C_{3223}, C_{3232}, C_{3131}, C_{3113}, C_{1331} and C_{1313} are also equal but their value is different than the value of the first set of components. Additionally, these equations imply that the three components C_{1111}, C_{2222} and C_{3333} are also equal and their value is related to the values of the first and second sets of components. If the value of the first set is λ and that of the second set is μ, then the value of the third set is $\lambda + 2\mu$. Thus, we have the following relations between the 21 components of the tensor $\boldsymbol{C}$:

$$\begin{aligned} &C_{1122} = C_{1133} = C_{2211} = C_{2233} = C_{3311} = C_{3322} = \lambda, \\ &C_{1212} = C_{1221} = C_{2112} = C_{2121} = C_{2323} = C_{2332} = C_{3223} = C_{3232} \\ &\qquad = C_{3131} = C_{3113} = C_{1331} = C_{1313} = \mu, \\ &C_{1111} = C_{2222} = C_{3333} = \lambda + 2\mu. \end{aligned} \tag{2.212}$$

Finally, these equations imply that the remaining 60 components of the tensor $\boldsymbol{C}$ are zero. Thus, for isotropic materials, there are only 2 independent components of the tensor $\boldsymbol{C}$ [2,4].

2.5.1.4 Stress-Strain Relations for Isotropic Materials

Substituting the values of 21 components of $\boldsymbol{C}$ from expressions (2.212) in Eq. (2.202) and setting the remaining components of $\boldsymbol{C}$ to zero, the stress-strain relations for isotropic materials become:

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} . \tag{2.213}$$

In tensor notation, they can be expressed as

$$\boldsymbol{\sigma} = \lambda (tr\boldsymbol{\varepsilon}) \mathbf{1} + 2\mu \boldsymbol{\varepsilon} . \tag{2.214}$$

Further, in component forms, they can be written as

$$\begin{aligned}
\sigma_{xx} &= \lambda(\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz})+2\mu\varepsilon_{xx},\\
\sigma_{yy} &= \lambda(\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz})+2\mu\varepsilon_{yy},\\
\sigma_{zz} &= \lambda(\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz})+2\mu\varepsilon_{zz},\\
\sigma_{xy} &= 2\mu\varepsilon_{xy},\\
\sigma_{yz} &= 2\mu\varepsilon_{yz},\\
\sigma_{zx} &= 2\mu\varepsilon_{zx}.
\end{aligned} \tag{2.215}$$

Expressions for the remaining three shear stress components are not needed because of the symmetry of the stress tensor. The constants λ and μ are called as the Lame's constants.

As stated in the introduction, there are 3 sets of equations which govern the displacements, strains and stresses in a body. This is the *third set of governing equations* when the deformation is small and the material is linearly elastic.

Sometimes, we need inverse relations. That is, we need expressions for the strain components in terms of the stress components. They can be obtained by inverting Eq. (2.215). When we do that, we get the following relations:

$$\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}\left[-\nu(\sigma_{xx}+\sigma_{yy}+\sigma_{zz})+(1+\nu)\sigma_{xx}\right],\\
\varepsilon_{yy} &= \frac{1}{E}\left[-\nu(\sigma_{xx}+\sigma_{yy}+\sigma_{zz})+(1+\nu)\sigma_{yy}\right],\\
\varepsilon_{zz} &= \frac{1}{E}\left[-\nu(\sigma_{xx}+\sigma_{yy}+\sigma_{zz})+(1+\nu)\sigma_{zz}\right],\\
\varepsilon_{xy} &= \frac{(1+\nu)}{E}\sigma_{xy},\\
\varepsilon_{yz} &= \frac{(1+\nu)}{E}\sigma_{yz},\\
\varepsilon_{zx} &= \frac{(1+\nu)}{E}\sigma_{zx},
\end{aligned} \tag{2.216}$$

where

$$E=\frac{\mu(3\lambda+2\mu)}{\lambda+\mu},\ \nu=\frac{\lambda}{2(\lambda+\mu)}. \tag{2.217}$$

In index notation, Eq. (2.216) can be expressed as

$$\varepsilon_{ij}=\frac{1}{E}\left[-\nu\sigma_{kk}\delta_{ij}+(1+\nu)\sigma_{ij}\right], \tag{2.218}$$

and, in tensor notation, it can be written as

$$\boldsymbol{\varepsilon} = \frac{1}{E}[-\nu(tr\boldsymbol{\sigma})\mathbf{1} + (1+\nu)\boldsymbol{\sigma}]. \tag{2.219}$$

It can be shown that the constant E is the slope of the straight portion of the one-dimensional stress-strain curve (Figure 2.11):

$$E = \frac{\sigma_{xx}}{\varepsilon_{xx}}. \tag{2.220}$$

It is called as the *Young's modulus*. Further, the constant ν can be shown to be negative of the ratio of the transverse normal strain to the axial or longitudinal normal strain in tension test. Thus,

$$\nu = -\frac{\varepsilon_{yy}}{\varepsilon_{xx}} = -\frac{\varepsilon_{zz}}{\varepsilon_{xx}}. \tag{2.221}$$

It is called as the *Poisson's ratio*. Equations (2.215) or (2.216) are called as the *generalized Hooke's law.*

Elimination of λ from two parts of Eq. (2.217) gives the following expression for μ :

$$\mu = \frac{E}{2(1+\nu)}. \tag{2.222}$$

Similarly, elimination of μ from two parts of Eq. (2.217) gives the following expression for λ :

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}. \tag{2.223}$$

2.5.1.5 Alternate Form of Stress-Strain Relations for Isotropic Materials

If we substitute the decompositions of stress and strain tensors (Eqs. 2.100 and 2.180) in the stress-strain relations (Eq. 2.213) and equate the hydrostatic and deviatoric parts on each side, we get the following relations:

$$\left(\frac{1}{3}\sigma_{kk}\right) = (3\lambda + 2\mu)\left(\frac{1}{3}\varepsilon_{kk}\right), \tag{2.224}$$

$$\sigma'_{ij} = 2\mu\varepsilon'_{ij}. \tag{2.225}$$

In tensor notation, they become:

$$\left(\frac{1}{3}tr\boldsymbol{\sigma}\right) = (3\lambda + 2\mu)\left(\frac{1}{3}tr\boldsymbol{\varepsilon}\right). \tag{2.226}$$

$$\boldsymbol{\sigma}' = 2\mu\boldsymbol{\varepsilon}' . \tag{2.227}$$

This is the third form of the stress-strain relations. It relates the hydrostatic and deviatoric parts of stress and strain tensors separately. This is possible only in isotropic materials. Equation (2.226) is a scalar equation. Because of the symmetry of $\boldsymbol{\sigma}'$ and $\boldsymbol{\varepsilon}'$, the tensor equation (2.227) represents 6 scalar equations. So, it appears that this form of the stress-strain relations consists of 7 scalar relations. However, it is not so. Because of the constraints $tr\boldsymbol{\sigma}' = 0$ (Eq. 2.99) and $tr\boldsymbol{\varepsilon}' = 0$ (Eq. 2.179), only 5 out of 6 equations from the set (2.227) are independent.

Equation (2.225) or (2.227) shows that, in isotropic materials, the elastic constant μ relates the deviatoric parts of stress and strain tensors. Therefore, it is called as the *shear modulus*. These equations imply that, in isotropic materials, the change in shape (without change in volume) is caused only by the deviatoric part of stress tensor. It also means, in isotropic materials, the hydrostatic part of stress tensor causes only the change in volume (without change in shape).

Besides the 4 elastic constants λ, μ, E and ν, there is one more elastic constant that is often used. It is called as the *bulk modulus* and is denoted by K. It is defined as the ratio of the hydrostatic part of stress to the volumetric strain. In small deformation, the volumetric strain is given by $tr\boldsymbol{\varepsilon} = \varepsilon_{ll}$ (Eq. 2.178). Thus, for small deformation, K is defined as

$$K = \frac{(1/3)\sigma_{kk}}{\varepsilon_{ll}} = \frac{(1/3)tr\boldsymbol{\sigma}}{tr\boldsymbol{\varepsilon}} . \tag{2.228}$$

This shows that, when the deformation is small, the bulk modulus K relates the hydrostatic parts of stress and strain tensors. Combining Eqs. (2.224) and (2.228) we get the following expression for the bulk modulus in terms of λ and μ:

$$K = \frac{(3\lambda + 2\mu)}{3} . \tag{2.229}$$

By taking the trace of Eq. (2.219) and using the expression (2.228) for K, we get

$$K = \frac{E}{3(1-2\nu)} . \tag{2.230}$$

Using the sign conventions for stress and strain components described in Sections 2.3.1.4 and 2.4.1.2, experimental observations in real materials show that the signs of E, ν, μ, λ and K are all positive. Equation (2.230) shows that for compressible materials (finite K), ν has to be less that (1/2). For incompressible materials ($K \to \infty$), ν must be (1/2).

***Example* 2.14:** Using the stress-strain relations (Eq. 2.215), find the expressions for the stress components corresponding to the strain expressions of Example 2.10 (Eq. 2.161).

***Solution*:** Using the strain expressions of Example 2.10 (Eq. 2.161), we get

$$\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = A(x_0 - \ell)y_0 \left[1 - \frac{1}{2} - \frac{1}{2}\right] = 0 . \tag{2.231}$$

Note that Eq. (2.231) implies that the volumetric strain ε_v is zero. This is expected, since the material is incompressible. Further, it implies that the hydrostatic part of the strain tensor is zero. Thus, the whole strain tensor is identical to its deviatoric part.

Substituting the strain expressions of Example 2.10 (Eq. 2.161) along with Eq. (2.231) in the stress-strain relations (Eq. 2.215), we get

$$\begin{aligned}
\sigma_{xx} &= \lambda(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) + 2\mu\varepsilon_{xx} = 0 + 2\mu A(x_0 - \ell)y_0, \\
\sigma_{yy} &= \lambda(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) + 2\mu\varepsilon_{yy} = 0 + \mu A(\ell - x_0)y_0, \\
\sigma_{zz} &= \lambda(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) + 2\mu\varepsilon_{zz} = 0 + \mu A(\ell - x_0)y_0, \\
\sigma_{xy} &= 2\mu\varepsilon_{xy} = \mu A\left(a^2 - y_0^2\right), \\
\sigma_{yz} &= 2\mu\varepsilon_{yz} = 0, \\
\sigma_{zx} &= 2\mu\varepsilon_{zx} = -\mu A y_0 z_0.
\end{aligned} \tag{2.232}$$

2.6 Summary

In this chapter, first, the index notation and the associated summation convention which have been used throughout the book have been explained. Then, the equations which govern the displacements, strains and stresses in a deformable body have been developed for the case of small deformation of linearly elastic materials. These equations have been developed in the following stages. First, the concept of stress at a point has been discussed. Since the stress at a point is a tensor (a second order tensor to be precise), a simple definition of tensor has been provided. The analysis of stress at a point has been carried out to provide a background material for developing the theory of plasticity in Chapter 3. The equations of motion which the stress components satisfy have also been discussed. Next, the linear strain tensor at a point, which is a measure of small deformation, has been developed. The associated strain-displacement relations have been presented. The linear strain tensor is not applicable for the analysis of plastic deformation. However, it does provide an insight into the deformation of solids which would be useful while developing a measure of plastic deformation in the next chapter. Analysis of the linear strain at a point has also been carried out similar to the analysis of stress at a point. Finally, the stress-strain relations, for the case of small deformation of linearly elastic solids, have been developed. These relations provide an introduction to the material behavior and therefore, provide a useful foundation for developing the plastic stress-strain relations of Chapter 3.

2.7 References

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