Lecture 18: Minimax Approximation, Optimal Interpolation, Chebyshev Polynomials

3.4.2. Optimal interpolation points.

As an application of the minimax approximation procedure, we consider how best to choose interpolation points $\{x_j\}_{j=0}^n$ to minimize

$$\|f - p_n\|_{L^\infty},$$

where $p_n \in \mathcal{P}_n$ in the interpolant to f at the specified points.

Recall the interpolation error bound developed in Lecture 7: If $f \in C^{n+1}[a,b]$, then

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\prod_{j=0}^{n}(x - x_j)$$

for some $\xi \in [a,b]$. Taking absolute values and maximizing over $[a,b]$ yields the bound

$$\|f - p_n\|_{L^\infty} = \max_{\xi\in[a,b]} \frac{|f^{(n+1)}(\xi)|}{(n+1)!} \max_{x\in[a,b]} \Big|\prod_{j=0}^{n}(x - x_j)\Big|.$$

For Runge's example, $f(x) = 1/(1+x^2)$ for $x \in [-5,5]$, we observed that $\|f - p_n\|_{L^\infty} \to \infty$ as $n \to \infty$ if the interpolation points $\{x_j\}$ are uniformly spaced over $[-5,5]$. However, Marcinkiewicz's theorem (Lecture 7) guarantees there is always some scheme for assigning the interpolation points such that $\|f - p_n\|_{L^\infty} \to 0$ as $n \to \infty$. In the case of Runge's function, we observed that the choice

$$x_j = 5\cos(j\pi/n), \qquad j = 0, \ldots, n$$

is one such scheme. For general functions $f \in C[a,b]$, there is no *a priori* method for picking interpolation points to ensure convergence. However, we can get a good estimate of optimal interpolation points by choosing those $\{x_j\}_{j=0}^n$ that *minimize the error bound*. That is, we want to solve

$$\min_{x_0,\ldots,x_n} \max_{x\in[a,b]} \Big|\prod_{j=0}^{n}(x - x_j)\Big|. \tag{1}$$

Notice that

$$\begin{aligned}\prod_{j=0}^{n}(x - x_j) &= x^{n+1} - x^n\sum_{j=0}^{n} x_j + x^{n-1}\sum_{j=0}^{n}\sum_{k=0}^{n} x_j x_k - \cdots + (-1)^{n+1}\prod_{j=0}^{n} x_j \\ &= x^{n+1} - r(x),\end{aligned}$$

where $r \in \mathcal{P}_n$ is a degree-n polynomial depending upon the interpolation notes $\{x_j\}_{j=0}^n$.

To find the optimal interpolation points according to (1), we should solve

$$\min_{r\in\mathcal{P}_n} \max_{x\in[a,b]} |x^{n+1} - r(x)| \;=\; \min_{r\in\mathcal{P}_n} \|x^{n+1} - r(x)\|_{L^\infty}.$$

Here the goal is to approximate an $n+1$-degree polynomial, x^{n+1}, with an n-degree polynomial. The method of solution is somewhat indirect: we will produce a class of polynomials of the form $x^{n+1} - r(x)$ that satisfy the requirements of the oscillation theorem, and thus $r(x)$ must be the minimax polynomial. We focus on the particular interval $[a,b] = [-1,1]$.

Definition. The degree n *Chebyshev polynomial* is defined for $x \in [-1, 1]$ by the formula

$$T_n(x) = \cos(n \cos^{-1} x).$$

At first glance, this formula may not appear to define a polynomial at all, since it involves trigonometric functions.† But computing the first few examples, we find

$$\begin{aligned} n = 0&: \quad T_0(x) = \cos(0 \cos^{-1} x) = \cos(0) = 1 \\ n = 1&: \quad T_1(x) = \cos(\cos^{-1} x) = x \\ n = 2&: \quad T_2(x) = \cos(2 \cos^{-1} x) = 2\cos^2(\cos^{-1} x) - 1 = 2x^2 - 1. \end{aligned}$$

For $n = 2$, we employed the identity $\cos 2\theta = 2\cos^2\theta - 1$, substituting $\theta = \cos^{-1} x$. More generally, we have the identity

$$\cos(n + 1)\theta = 2\cos\theta \cos n\theta - \cos(n - 1)\theta.$$

This formula implies, for $n \geq 2$,

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x),$$

a formula related to the three term recurrence used to construct orthogonal polynomials. (In fact, Chebyshev polynomials are orthogonal polynomials on $[-1, 1]$ with respect to the inner product $ipf, g = \int_a^b f(x)g(x)(1 - x^2)^{-1/2}$.

Chebyshev polynomials have a wealth of interesting properties, of which we mention just three.

Proposition. Let T_n be the degree-n Chebyshev polynomial, $T_n = \cos(n \cos^{-1} x)$ for $x \in [-1, 1]$.

- $|T_n(x)| \leq 1$ for $x \in [-1, 1]$;
- The roots of T_n are the $\xi_j = \cos \frac{(2j-1)\pi}{2n}$ for $j = 1, \ldots, n$;
- For $n \geq 1$, T_n attains its maximum on $[-1, 1]$ at the points $\eta_j = \cos(j\pi/n)$ for $j = 1, \ldots, n$:

$$T_n(\eta_j) = (-1)^j.$$

Proof. These results follow from direct calculations. For $x \in [-1, 1]$, $T_n(x) = \cos(n \cos^{-1}(x))$ cannot exceed one in magnitude because cosine can't exceed one in magnitude. To verify the formula for the roots, compute

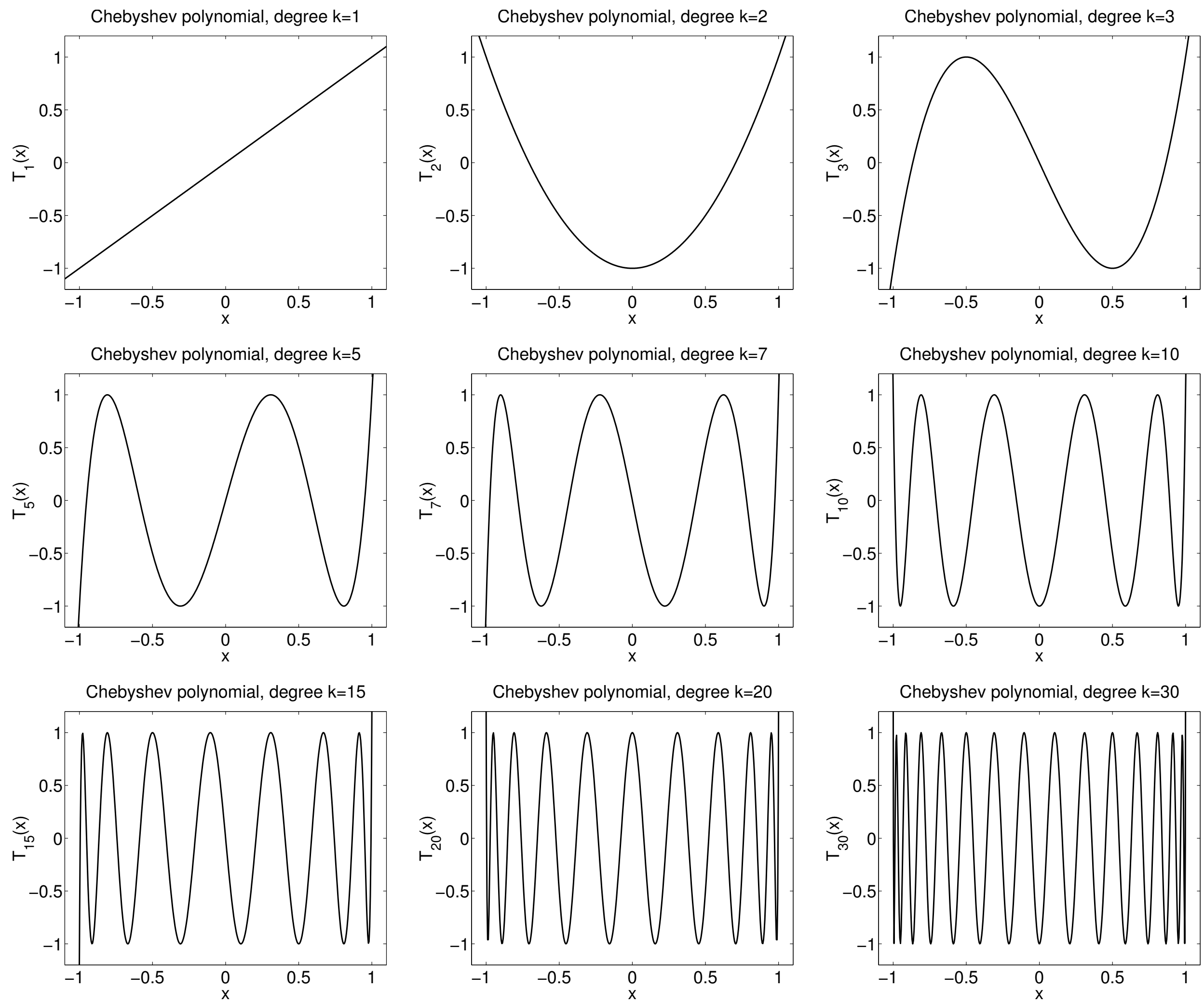
$$T_n(\xi_j) = \cos\Big(n \cos^{-1} \cos\Big(\frac{(2j-1)\pi}{2n}\Big)\Big) = \cos\Big(\frac{(2j-1)\pi}{2}\Big) = 0,$$

since cosine is zero at half-integer multiples of π. Similarly,

$$T_n(\eta_j) = \cos\Big(n \cos^{-1} \cos\Big(\frac{j\pi}{n}\Big)\Big) = \cos(j\pi) = (-1)^j. \qquad \blacksquare$$

Below we plot several Chebyshev polynomials.

†Furthermore, it doesn't apply if $|x| > 1$. In that case, one can define the Chebyshev polynomials using hyperbolic trigonometric functions, $T_n(x) = \cosh(n \cosh^{-1} x)$. Indeed, using hyperbolic trigonometric identities, one can show that this expression generates the same polynomials we get from the standard trigonometric identities.



The punchline. Finally, we are ready to solve the key minimax problem that will reveal optimal interpolation points. Looking at the above plots of Chebyshev polynomials, with their equi-oscillation properties, maybe you have already guessed it yourself.

We defined the Chebyshev polynomials so that

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

with $T_0(x) = 1$ and $T_1(x) = x$. Thus T_{n+1} has the leading coefficient 2^n for $n \geq 0$. Define

$$\widehat{T}_{n+1} = 2^{-n}T_{n+1}$$

for $n \geq 0$, with $\widehat{T}_0 = 1$. These *normalized* Chebyshev polynomials are *monic*, i.e., the leading term in $\widehat{T}_{n+1}$ is x^{n+1}, rather than $2^n x^{n+1}$ as for T_{n+1}. Thus, we can write

$$\widehat{T}_{n+1}(x) = x^{n+1} - q_n(x)$$

for some polynomial $q_n(x) = x^{n+1} - \widehat{T}_{n+1}(x) \in \mathcal{P}_n$.

For $n \geq 0$, polynomials $\widehat{T}_{n+1}(x)$ oscillate between $\pm 2^{-n}$ for $x \in [-1,1]$, with the maximal values attained at

$$\eta_j = \cos\Big(\frac{j\pi}{n+1}\Big)$$

for $j = 0, \ldots, n+1$. In particular,

$$\widehat{T}_{n+1}(\eta_j) = (\eta_j)^{n+1} - q_n(\eta_j) = (-1)^j 2^{-n}.$$

Thus, we have found a polynomial $q_n \in \mathcal{P}_n$, together with $n+2$ distinct points, $\eta_j \in [-1,1]$ where the maximum error

$$\max_{x \in [-1,1]} |x^{n+1} - q_n(x)| = 2^{-n}$$

is attained with alternating sign. Thus, by the oscillation theorem, we have found the minimax approximation to x^{n+1}!

Theorem (Optimal approximation of x^{n+1}). The optimal approximation to x^{n+1} from $\mathcal{P}_n$ on the interval $x \in [-1,1]$ is given by

$$q_n(x) = x^{n+1} - \widehat{T}_{k+1}(x) = x^{n+1} - 2^{-n}\widehat{T}_{k+1}(x) \in \mathcal{P}_n.$$

Thus, the optimal interpolation points are those $n+1$ roots of $x^{n+1} - q_n$, which are the roots of the degree-$n+1$ Chebyshev polynomial,

$$\xi_j = \cos\Big(\frac{(2j-1)\pi}{2n+2}\Big)$$

for $j = 1, \ldots, n$.

It turns out that similar properties hold if interpolation is performed at the points

$$\eta_j = \cos\Big(\frac{j\pi}{n+1}\Big),$$

which are also called *Chebyshev points*, for $j = 0, \ldots, n+1$. (These are essentially the points that yielded convergence for Runge's function.)

For generic intervals $[a, b]$, a change of variable demonstrates that the same points, appropriately scaled, will be optimal.

Because of the central role Chebyshev polynomials play in this field, minimax approximation is sometimes known as *Chebyshev approximation*."