Lecture 20: Peano Kernel Analysis

In the previous lecture, we proved an error bound for the trapezoid rule using the mean value theorem for integrals, and we stated a result for Simpson’s rule without proof. In this lecture, we present a general convergence theory that is applicable to a wide range of quadrature rules.

4.1.4. Error analysis for interpolatory quadrature. Consider a general quadrature rule of the form

\[ I(f) = \sum_{j=0}^{m} A_j f(x_j) \]

for \(a \leq x_0 < x_1 < \cdots < x_{m-1} < x_m \leq b\). The Newton–Cotes methods discussed in the previous lecture all fit this template. For example, the trapezoid rule has \(x_0 = a, x_1 = b\), and \(A_0 = A_1 = (b - a)/2\).

Define the error function for this quadrature rule as

\[ E(f) = \int_a^b f(x) \, dx - I(f). \]

The integral is linear \((f + g = \int f + \int g)\), and so we expect the quadrature rule \(I(f)\) to share this property. Indeed it does, and thus the error function \(E(f)\) is itself linear:

\[
E(\alpha f + g) = \int_a^b \alpha f(x) + g(x) \, dx - \sum_{j=0}^{m} A_j \left( \alpha f(x_j) + g(x_j) \right)
\]

\[
= \alpha \left( \int_a^b f(x) \, dx - \sum_{j=0}^{m} A_j f(x_j) \right) + \left( \int_a^b g(x) \, dx - \sum_{j=0}^{m} A_j g(x_j) \right)
\]

\[
= \alpha E(f) + E(g)
\]

for arbitrary \(f\) and \(g\), and any \(\alpha \in \mathbb{R}\).

Our goal is to describe \(E(f)\) using basic properties of the function \(f\) and the particular quadrature rule \(I(f)\). We will develop a general theory, then apply it to specific quadrature rules.

Assume that \(I\) exactly integrates all polynomials of degree \(n\) or less:

If \(p \in \mathcal{P}_n\), then \(I(p) = \int_a^b p(x) \, dx\).

For the trapezoid rule, we must take \(n = 1\). For Simpson’s rule, \(n = 3\).

Suppose further that \(f \in C^{n+1}[a, b]\), so that we can expand \(f\) in a Taylor series,

\[
f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)(x - a)^k}{k!} + \frac{1}{n!} \int_a^x f^{(n+1)}(t)(x - t)^n \, dt.
\]

\(^1\)Some quadrature rules require evaluations of derivatives of \(f\), such as \(f'(x_j)\) or \(f''(x_j)\), etc., as would occur if you integrate the Hermite polynomial interpolant. The theory we discuss here also works in that case, though for simplicity we don’t discuss those details here.
Here we have used the integral remainder term for the Taylor series.\footnote{For more on the integral remainder term, see Strichartz, The Way of Analysis, pp. 210–211.}

The first $n$ terms of the Taylor series form a polynomial of degree $n$ (or less) in $x$, which we’ll call $p_n$; denote the remainder term by $r_n$:

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)(x-a)^k}{k!} \in \mathcal{P}_n, \quad r_n(x) = \frac{1}{n!} \int_a^x f^{(n+1)}(t)(x-t)^n \, dt.$$  

Since $p_n$ is a polynomial of degree $n$ or less, it is exactly integrated by the quadrature rule, so $E(p_n) = 0$. Using the linearity of $E$, we obtain

$$E(f) = E(p_n + r_n) = E(p_n) + E(r_n) = E(r_n).$$

Thus, to describe $E(f)$ we need only derive an expression for

$$E(r_n) = \int_a^b r_n(x) \, dx - I(r_n) = \frac{1}{n!} \int_a^b \int_a^x f^{(n+1)}(t)(x-t)^n \, dt \, dx - I(r_n).$$

It will be convenient to remove the $x$ from the upper limit of the interior integral above. Toward this end we introduce the truncated power function

$$(x-t)^n_+ = \begin{cases} (x-t)^n & x \geq t; \\ 0 & x < t. \end{cases}$$

Below we compare $(x-t)^2_+$ (left) and $(x-t)^3_+$ (right) to standard power functions for fixed $t = \frac{1}{2}$.

Since $(x-t)^n_+ = 0$ for $x < t$, we have

$$r_n(x) = \frac{1}{n!} \int_a^x f^{(n+1)}(t)(x-t)^n \, dt = \frac{1}{n!} \int_a^b f^{(n+1)}(t)(x-t)^n_+ \, dt,$$

and so

$$E(r_n) = E\left(\frac{1}{n!} \int_a^b f^{(n+1)}(t)(x-t)^n_+ \, dt\right) = \frac{1}{n!} E\left(\int_a^b f^{(n+1)}(t)(x-t)^n_+ \, dt\right).$$
Now we simplify this expression for \( E(r_n) \) even further:

\[
\begin{align*}
    n!E \left( \int_a^b f^{(n+1)}(t)(x - t)^n_+ \, dt \right) &= \int_a^b \int_a^b f^{(n+1)}(t)(x - t)^n_+ \, dt \, dx - I \left( \int_a^b f^{(n+1)}(t)(x - t)^n_+ \, dt \right) \\
    &= \int_a^b \int_a^b f^{(n+1)}(t)(x - t)^n_+ \, dt \, dx - \sum_{j=0}^{m} A_j \int_a^b f^{(n+1)}(t)(x_j - t)^n_+ \, dt \\
    &= \int_a^b f^{(n+1)}(t) \int_a^b (x - t)^n_+ \, dx \, dt - \int_a^b f^{(n+1)}(t) \sum_{j=0}^{m} A_j (x_j - t)^n_+ \, dt \\
    &= \int_a^b f^{(n+1)}(t) \left( \int_a^b (x - t)^n_+ \, dx - \sum_{j=0}^{m} A_j (x_j - t)^n_+ \right) \, dt \\
    &= \int_a^b f^{(n+1)}(t) E((x - t)^n_+) \, dt.
\end{align*}
\]

Perhaps this final expression still looks a little tedious, but it actually is quite powerful in practice. For Newton–Cotes formulas, it will allow us to express the quadrature error for arbitrary \( C^{n+1} \) functions in terms of the quadrature for one special class of functions, the truncated power functions. We have just proved the Peano kernel theorem. (In fact, the full theorem is a bit more general than what we proved here, though our development is sufficient for Newton–Cotes analysis.)

**Theorem (Peano kernel theorem).** Suppose \( f \in C^{n+1}[a, b] \) and let \( I(f) = \sum_{j=0}^{m} A_j f(x_j) \) be a quadrature rule that exactly integrates all polynomials of degree-\( n \) or less on \([a, b]\). Then

\[
E(f) = \frac{1}{n!} \int_a^b f^{(n+1)}(t) K(t) \, dt,
\]

where \( K(t) = E((x - t)^n_+) \).

This may appear quite tedious at first, but it is actually quite a beautiful idea: the error in the quadrature of \( f \) is related to the integral of a canonical function \( K(t) \) that is independent of \( f \). In typical circumstances, the mean value theorem for integrals is applied to this formula for \( E(f) \) to extract \( f \) from inside the integral. This allows \( E(f) \) to be described as a constant (dependent upon \( f \)) times the integral of the error of the kernel function. To appreciate how this operates in practice, we will work several examples.

**Application: trapezoid rule.** Recall that the formula for the trapezoid rule is

\[
I(f) = \frac{b - a}{2} (f(a) + f(b)).
\]

This quadrature rule exactly integrates linear polynomials, so take \( n = 1 \). In this case,

\[
K(t) = E((x - t)_+) = \int_a^b (x - t)_+ \, dx - I((x - t)_+)
\]

\[
= \int_t^b (x - t) \, dx - I((x - t)_+)
\]

\[
= \left[ \frac{(x - t)^2}{2} \right]_{x=t}^{b} - \frac{b - a}{2} ((a - t)_+ + (b - t)_+)
\]

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\[ (b - t)^2 = \frac{(b - a)(b - t)}{2} \]
\[ = \frac{(b - t)(a - t)}{2}, \]
where we have used the fact that \((a - t)_+ = 0\) since \(a \leq t\) for all \(t \in [a, b]\), and \((b - t)_+ = b - t\) since \(b \geq t\) for all \(t \in [a, b]\).

Note that for \(t \in [a, b]\), \(b - t \geq 0\) and \(a - t \leq 0\), so \(K(t) \leq 0\) for all \(t \in [a, b]\). In particular, \(K(t)\) does not change signs on \([a, b]\), so we can apply the mean value theorem for integrals.\(^8\)

Thus for the trapezoid rule, we have

\[
E(f) = \frac{1}{11!} \int_a^b f''(t) K(t) \, dt \\
= f''(\xi) \int_a^b K(t) \, dt
\]
for some \(\xi \in [a, b]\). Furthermore, it is simple to verify that

\[
\int_a^b K(t) \, dt = \int_a^b \frac{(b - t)(a - t)}{2} \, dt = -\frac{1}{12} (b - a)^3.
\]

Thus, we conclude that the error in the trapezoid rule is given by

\[
E(f) = -\frac{1}{12} f''(\xi)(b - a)^3,
\]
which is exactly the same result we derived in the previous lecture via simpler means. The real power of the Peano kernel approach becomes apparent when studying higher-order quadrature formulas.

**Application: Simpson’s rule.** For Simpson’s rule, we have

\[
I(f) = \frac{b - a}{6} \left( f(a) + 4 f\left(\frac{1}{3}(a + b)\right) + f(b) \right).
\]

You will prove on your homework that Simpson’s rule is exact for cubic polynomials.\(^4\) Thus, the Peano kernel theorem gives that

\[
E(f) = \frac{1}{3!} \int_a^b f^{(3)}(t)K(t) \, dt,
\]
where \(K(t) = E((x - t)^3)\).

\(^8\)The mean value theorem for integrals states that if \(h, g \in C[a, b]\) and \(h\) does not change sign on \([a, b]\), then there exists some \(\xi \in [a, b]\) such that \(\int_a^b g(t)h(t) \, dt = g(\xi) \int_a^b h(t) \, dt\). (See, e.g., Kincaid & Cheney, p. 19.) The requirement that \(g\) not change sign is essential. For example, if \(g(t) = h(t) = t\) then \(\int_{-1}^1 g(t)h(t) \, dt = \int_{-1}^1 t^2 \, dt = 2/3\), yet \(\int_{-1}^1 h(t) \, dt = \int_{-1}^1 t \, dt = 0\), so for all \(\xi \in [-1, 1]\), \(g(\xi) \int_{-1}^1 h(t) \, dt = 0 \neq \int_{-1}^1 g(t)h(t) \, dt = 2/3\).

\(^4\)Note that one needs to know in advance the maximum degree polynomial for which the integration rule is exact. This is needed to apply the Peano kernel theorem; it is not a result of that theorem.
As with the trapezoid rule analysis, we begin by writing $K(t)$ in a more explicit fashion,

$$K(t) = E((x - t)^3_+) = \int_a^b (x - t)^3_+ dt - I((x - t)^3_+)$$

$$= \left. \frac{(x - t)^4}{4} \right|_t^b - \frac{b - a}{6} \left( (a - t)^3_+ + 4\left(\frac{1}{2}(a + b) - t\right)^3_+ + (b - t)^3_+ \right)$$

$$= \left. \frac{(b - t)^4}{4} \right|_t^b - \frac{b - a}{6} \left( 4\left(\frac{1}{2}(a + b) - t\right)^3_+ + (b - t)^3_+ \right),$$

where we have used the fact that $(a - t)_+ = 0$ for $t \in [a, b]$. One can show that $K(t)$ does not change sign for $t \in [a, b]$, and thus the mean value theorem for integrals guarantees the existence of some $\xi \in [a, b]$ such that

$$E(f) = \frac{1}{3!} \int_a^b f^{(4)}(t)K(t) dt = \frac{1}{3!} f^{(4)}(\xi) \int_a^b K(t) dt.$$

Thus, we can integrate the ‘simplified’ formula for $K(t)$ we just derived to get a clean expression for $E(f)$:

$$\int_a^b K(t) dt = \int_a^b \frac{(b - t)^4}{4} dt - \frac{b - a}{6} \int_a^b \left( 4\left(\frac{1}{2}(a + b) - t\right)^3_+ + (b - t)^3_+ \right) dt$$

$$= \left. \frac{(b - t)^5}{20} \right|_a^b - \frac{b - a}{6} \left[ 4 \int_a^{(a+b)/2} \left(\frac{1}{2}(a + b) - t\right)^3 dt + \int_a^b (b - t)^3 dt \right]$$

$$= \left. \frac{(b - t)^5}{20} \right|_a^b + \frac{b - a}{6} \left[ \frac{(a - b)}{2}^4 - \frac{(b - a)^4}{4} \right]$$

$$= -\frac{(b - a)^5}{480}.$$

Inserting this into the expression for $E(f)$, we obtain

$$E(f) = -\frac{1}{90} \left(\frac{b - a}{2}\right)^5 f^{(4)}(\xi).$$

When performing such analysis yourself, it is a good idea to use a symbolic math program, such as Maple or Mathematica, to help you avoid minor calculus/algebra mistakes.