HONG KONG BAPTIST UNIVERSITY
SEMESTER 1 EXAMINATION, 1999–2000

Subject Code: **MATH 2140** Section No.: **00001** Time Allowed: **2** Hour(s)

Subject Title: **Numerical Methods I *WITH ANSWERS*** Total Number of Pages: **5**

INSTRUCTIONS:

i. Answer ALL of the following questions.

ii. The full mark for this examination is 100.

iii. Calculators are allowed, but they must not be pre-programmed or have stored text.

1. (15 marks)
Answer each of the following short questions (justification not required):

   a. What is the order of the truncation error of the trapezoidal rule as function of n, the number of trapezoids? (5 marks)

   Answer: $\mathrm{O}(n^{-2})$

   b. Scaled partial pivoting is used in solving linear systems of equations to reduce what kind of error? (5 marks)

   Answer: round-off error

   c. What is the order of time required to fit a spline curve to n points? (5 marks)

   Answer: $\mathrm{O}(n)$

2. (20 marks)
Consider the function $\sin(x)$.

   a. Compute the quadratic Taylor polynomial approximation to $\sin(x)$ expanded about the point $x = \pi/4$. (5 marks)

   Answer:

$$\begin{aligned} P_T(x) &= \sin(\pi/4) + \cos(\pi/4)(x - \pi/4) - \frac{1}{2}\sin(\pi/4)(x - \pi/4)^2 \\ &= \frac{\sqrt{2}}{2}[1 + (x - \pi/4) - \frac{1}{2}(x - \pi/4)^2] \end{aligned}$$

   and

b. Give an upper bound on the error of this Taylor polynomial for $x \in [0, \pi/2]$. (5 marks)

Answer:

$$\sin(x) = P_T(x) + E_T(x),$$

where

$$E_T(x) = -\frac{\sin(\xi)}{6}(x - \pi/4)^3.$$

Since $|\sin(x)| \leq 1$ *and* $\left|(x - \pi/4)^3\right| \leq (\pi/4)^3$ *for* $x \in [0, \pi/2]$, *it follows that*

$$|E_T(x)| = \frac{\pi^3}{6 \times 64} = 0.08075.$$

c. Compute the polynomial that interpolates $\sin(x)$ at the points $x = 0, \pi/4, \pi/2$. (5 marks)

Answer:

$$P_I(x) = \left(\frac{\sqrt{2}}{2}\right)\frac{x(x - \pi/2)}{-\pi^2/16} + \frac{x(x - \pi/4)}{\pi^2/8}$$

d. Give an upper bound on the error of this interpolating polynomial for $x \in [0, \pi/2]$. Which of the two polynomials have smaller maximum error on $x \in [0, \pi/2]$? (5 marks)

Answer:

$$\sin(x) = P_I(x) + E_I(x),$$

where

$$E_I(x) = -\frac{\sin(\zeta)}{6}x(x - \pi/4)(x - \pi/2).$$

Again, note that $|\sin(x)| \leq 1$. *The function*

$$x(x - \pi/4)(x - \pi/2) = (x - \pi/4)^3 - (\pi/4)^2(x - \pi/4)$$

has local optima at $x - \pi/4 = \pm\pi/(4\sqrt{3})$. *So*

$$|x(x - \pi/4)(x - \pi/2)| \leq [\pi/(4\sqrt{3})]^3\,|1 - 3| = \pi^3/(96\sqrt{3})$$

and it follows that

$$|E_I(x)| = \frac{\pi^3}{6 \times 96\sqrt{3}} = 0.0310789.$$

The error of the interpolating polynomial is smaller than that of the Taylor polynomial.

3. (20 marks)
Suppose that one estimates a quantity I by its numerical approximation, $Q(n)$ for $n = 10, 30, 90, \ldots$. It is known that $|I - Q(n)| = \mathrm{O}(n^{-4})$ as $n \to \infty$.

a. Derive a formula to **estimate** the error $|I - Q(90)|$, based on knowing $Q(30)$ and $Q(90)$. (10 marks)

Answer: Let $E(n) = I - Q(n)$. By the assumption $E(n) \approx Kn^{-4}$. It follows that

$$Q(n) - Q(n/3) = E(n/3) - E(n) \approx K(n/3)^{-4} - Kn^{-4} = (3^4 - 1)Kn^{-4} \approx 80E(n)$$

So $E(n) \approx [Q(n) - Q(n/3)]/80 = \hat{E}(n)$, i.e. $E(90) \approx [Q(90) - Q(30)]/80 = \hat{E}(90)$.

b. Derive a formula to **extrapolate** the $Q(n)$ to get, $\tilde{Q}(n)$, a better estimate of I. (5 marks)

Answer: Let $\tilde{Q}(n) = Q(n) + \hat{E}(n)$.

c. Suppose that $Q(30) = 20.052$ and $Q(90) = 23.613$. Use your results in the previous two sections to estimate the error of $Q(90)$ and find an extrapolated approximation. (5 marks)

Answer:

$$\hat{E}(90) = [Q(90) - Q(30)]/80 = [23.613 - 20.052]/80 = 0.045$$
$$\tilde{Q}(90) = Q(90) + \hat{E}(90) = 23.613 + 0.045 = 23.658$$

4. (16 marks)
Given a nonsingular $n \times n$ matrix $\mathbf{A}$, and $n \times 1$ vectors $\mathbf{x}, \hat{\mathbf{x}}, \mathbf{b} = \mathbf{A}\mathbf{x}$, and $\hat{\mathbf{b}} = \mathbf{A}\hat{\mathbf{x}}$, recall that

$$\frac{\|\mathbf{x} - \hat{\mathbf{x}}\|}{\|\mathbf{x}\|} \times \frac{\|\mathbf{b}\|}{\left\|\mathbf{b} - \hat{\mathbf{b}}\right\|} \leq \operatorname{cond}(\mathbf{A}).$$

a. Suppose that $\mathbf{b}$ is an eigenvector of $\mathbf{A}$ with eigenvalue λ, and $\mathbf{b} - \hat{\mathbf{b}}$ is an eigenvector of $\mathbf{A}$ with eigenvalue ω. Find a better bound for

$$\frac{\|\mathbf{x} - \hat{\mathbf{x}}\|}{\|\mathbf{x}\|} \times \frac{\|\mathbf{b}\|}{\left\|\mathbf{b} - \hat{\mathbf{b}}\right\|}$$

in this special case. (8 marks)

Answer:

$$\frac{\|\mathbf{x} - \hat{\mathbf{x}}\|}{\|\mathbf{x}\|} \times \frac{\|\mathbf{b}\|}{\left\|\mathbf{b} - \hat{\mathbf{b}}\right\|} = \frac{\|\mathbf{x} - \hat{\mathbf{x}}\|}{\left\|\mathbf{b} - \hat{\mathbf{b}}\right\|} \times \frac{\|\mathbf{b}\|}{\|\mathbf{x}\|} = \frac{\left\|\mathbf{A}^{-1}(\mathbf{b} - \hat{\mathbf{b}})\right\|}{\left\|\mathbf{b} - \hat{\mathbf{b}}\right\|} \times \frac{\|\mathbf{b}\|}{\|\mathbf{A}^{-1}\mathbf{b}\|}$$
$$= \frac{\left\|\omega^{-1}(\mathbf{b} - \hat{\mathbf{b}})\right\|}{\left\|\mathbf{b} - \hat{\mathbf{b}}\right\|} \times \frac{\|\mathbf{b}\|}{\|\lambda^{-1}\mathbf{b}\|} = \frac{|\omega|^{-1}\left\|(\mathbf{b} - \hat{\mathbf{b}})\right\|}{\left\|\mathbf{b} - \hat{\mathbf{b}}\right\|} \times \frac{\|\mathbf{b}\|}{|\lambda|^{-1}\|\mathbf{b}\|} = \frac{|\lambda|}{|\omega|}$$

b. If the eigenvalues of $\mathbf{A}$ are $100, -50, 3 \pm 5\sqrt{-1}$, then find a lower bound on $\text{cond}(\mathbf{A})$ using the result of part a. and under the assumptions of part a. (8 marks)

Answer: Since under the assumptions of part a.,

$$\frac{|\lambda|}{|\omega|} = \frac{\|\mathbf{x} - \hat{\mathbf{x}}\|}{\|\mathbf{x}\|} \times \frac{\|\mathbf{b}\|}{\left\|\mathbf{b} - \hat{\mathbf{b}}\right\|} \leq \text{cond}(\mathbf{A})$$

it follows that

$$\text{cond}(\mathbf{A}) \geq \frac{\textit{largest absolute eigenvalue of } \mathbf{A}}{\textit{smallest absolute eigenvalue of } \mathbf{A}} = \frac{100}{\left|3 + 5\sqrt{-1}\right|} = \frac{100}{\sqrt{34}}$$

5. (14 marks)
Two iterative methods for solving linear systems of algebraic equations, $\mathbf{Ax} = \mathbf{b}$, are Jacobi and Gauss-Seidel. For each of these methods give necessary and sufficient conditions on $\mathbf{A}$ so that the method will give the exact answer in *one* iteration, regardless of $\mathbf{b}$ and the starting value $\mathbf{x}^{(0)}$.

Answer: Write $\mathbf{A} = \mathbf{L} + \mathbf{D} + \mathbf{U}$, *where* $\mathbf{L}$ *is a strictly lower triangular matrix (zero along the diagonal),* $\mathbf{D}$ *is a diagonal matrix (zero along the diagonal), and* $\mathbf{U}$ *is a strictly upper triangular matrix (zero along the diagonal). Then one may write*

$$\begin{aligned} \textit{Jacobi} \quad & \mathbf{D}\mathbf{x}^{(k+1)} = \mathbf{b} - (\mathbf{L} + \mathbf{U})\mathbf{x}^{(k)} \\ \textit{Gauss-Seidel} \quad & (\mathbf{D} + \mathbf{L})\mathbf{x}^{(k+1)} = \mathbf{b} - \mathbf{U}\mathbf{x}^{(k)} \end{aligned}$$

The Jacobi method will give the exact answer in one iteration for all initial guesses iff $\mathbf{L} + \mathbf{U} = \mathbf{0}$, *i.e.,* $\mathbf{A}$ *is a nonsingular diagonal matrix. The Gauss-Seidel method will give the exact answer in one iteration for all initial guesses iff* $\mathbf{U} = \mathbf{0}$, *i.e.,* $\mathbf{A}$ *is a nonsingular lower triangular matrix.*

6. (15 marks)
Consider the tri-diagonal matrix

$$\mathbf{A} = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

To find eigenvalues one uses a QR algorithm involving successive iterations of Givens rotations. Apply one complete iteration of Givens rotations to this matrix.

Answer: First, we form the matrix

$$\mathbf{G}_1 = \frac{1}{\sqrt{4^2 + 2^2}} \begin{pmatrix} 4 & 2 & 0 \\ -2 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0.8944 & 0.4472 & 0 \\ -0.4472 & 0.8944 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then we multiply,

$$\mathbf{G}_1\mathbf{A} = \begin{pmatrix} 4.4721 & 2.6833 & 0 \\ 0 & 0.8944 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Next, we form the matrix

$$\mathbf{G}_2 = \frac{1}{\sqrt{0.8944^2 + 1^2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.8944 & 1 \\ 0 & -1 & 0.8944 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.6667 & 0.7454 \\ 0 & -0.7454 & 0.6667 \end{pmatrix}.$$

Then we multiply,

$$\mathbf{G}_2\mathbf{G}_1\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.6667 & 0.7454 \\ 0 & -0.7454 & 0.6667 \end{pmatrix} \begin{pmatrix} 4.4721 & 2.6833 & 0 \\ 0 & 0.8944 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 4.4721 & 2.6833 & 0.4472 \\ 0 & 1.3416 & 1.3416 \\ 0 & 0 & 0 \end{pmatrix}$$

Finally, we multiply on the right by the Given's rotations

$$\mathbf{G}_2\mathbf{G}_1\mathbf{A}\mathbf{G}_1' = \begin{pmatrix} 5.2000 & 0.4000 & 0.4472 \\ 0.6000 & 1.2000 & 1.3416 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{G}_2\mathbf{G}_1\mathbf{A}\mathbf{G}_1'\mathbf{G}_2' == \begin{pmatrix} 5.2000 & 0.6000 & 0 \\ 0.6000 & 1.8000 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Just for your information, the eigenvalues are $5.3028, 1.6972, 0$.

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