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**Text**

**Course Outline**

*Introduction*: Introduction to numerical methods and analysis.

*Error Analysis*: Approximations; Round off and Truncation errors. Error Analysis.


*Numerical Differentiation*: Introduction to finite difference approximations, truncation error analysis. Finite difference approximations on irregular grid. Richardson’s extrapolation.


*ODE, Boundary Value Problems*: Decomposition into Linear System of ODEs, Shooting Method, Direct Method.

Assignments

I. Nonlinear Equations

1. Find a simple root (other than \( x = 0 \)) of the equation: \( f(x) = \sin x - (x/2)^2 \) using Bisection method, Regula-Falsi method, Fixed Point method, Newton-Raphson’s method and Secant method. In each case, calculate true relative error and approximate relative error at each iteration (the true root may be taken as 1.93375496). Plot both of these errors as Log (%error) vs. iteration number for each of the methods. Terminate the iterations when the approximate relative error is less than 0.01 %. Use starting points for Bisection, Regula-Falsi and Secant methods as \( x = 1 \) and \( x = 2 \).

2. Find the root of the polynomial, \( x^4 - 2x^3 - 53x^2 + 54x + 504 \), by (a) Mueller’s method and (b) Bairstow’s method using \( \epsilon = 0.01\% \).

3. If \( \alpha \) is a zero of \( f(x) \) of multiplicity \( m > 1 \), show that
   a) Newton Raphson method given by \( x_{k+1} = x_k - f(x_k) / f'(x_k) \) is first order.
   b) If we modify the Newton Raphson method as \( x_{k+1} = x_k - mf(x_k) / f'(x_k) \), the method becomes at least 2nd order.

II. Linear Simultaneous Equations

1. Solve the following system of equations by Gauss Elimination, Doolittle’s method, Crout’s method and Cholesky decomposition:

\[
\begin{bmatrix}
9.3746 & 3.0416 & -2.4371 \\
3.0416 & 6.1832 & 1.2163 \\
-2.4371 & 1.2163 & 8.4429 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
9.2333 \\
8.2049 \\
3.9339 \\
\end{bmatrix}
\]

2. Solve the following system of equation using Thomas algorithm:

\[
\begin{bmatrix}
-2 & 1 & 0 & 0 \\
1 & -4 & 1 & 0 \\
0 & 1 & -4 & 1 \\
0 & 0 & 1 & -2 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix}
= 
\begin{bmatrix}
3 \\
1 \\
2 \\
-2 \\
\end{bmatrix}
\]

3. A set of equations is described as below:

\[
\begin{bmatrix}
10 & 7 & 8 & 7 \\
7 & 5 & 6 & 5 \\
8 & 6 & 10 & 9 \\
7 & 5 & 9 & 10 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix}
= 
\begin{bmatrix}
32 \\
23 \\
33 \\
31 \\
\end{bmatrix}
\]

An approximation to the x-values as \([-7.2, 14.6, -2.5, 3.1]\) yields the right hand side vector as \([31.9, 23.1, 32.9, 31.1]\). A very different set of x-values \([0.18, 2.36, 0.65, 1.21]\) also yields a very close right hand side vector as \([31.99, 23.01, 32.99, 31.01]\). It is not clear whether any of the x-values are close to the true solution. Use Crout’s decomposition and improve the solution starting from each of the above approximations of x-values.
III. Eigenvalues and Eigenvectors

1. Consider the following matrix:

\[
\begin{bmatrix}
  2 & -1 & 0 & 0 \\
  -1 & 4 & -1 & 0 \\
  0 & -1 & 4 & -1 \\
  0 & 0 & -1 & 2
\end{bmatrix}
\]

a) Find an eigenvalue and the corresponding eigenvector using the Power method.

b) Formulate the characteristic polynomial using Fadeev-Leverrier method. Solve the polynomial equation using the Bairstow’s method for all the eigenvalues of the matrix.

c) Obtain all the eigenvalues using QR algorithm and compare with those obtained in (b) above.

d) Using the Inverse Power Method with shift, compute the eigenvectors corresponding to each of the eigenvalues obtained in (c).

IV. Approximation of Function, Curve Fitting, Interpolation

1. The following data have been measured in an experiment:

<table>
<thead>
<tr>
<th>$k$</th>
<th>$x_k$</th>
<th>$y_k$</th>
<th>$z_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>23.000</td>
<td>22.000</td>
<td>10800.000</td>
</tr>
<tr>
<td>2</td>
<td>35.000</td>
<td>21.999</td>
<td>162010.797</td>
</tr>
<tr>
<td>3</td>
<td>71.000</td>
<td>22.012</td>
<td>831492.000</td>
</tr>
<tr>
<td>4</td>
<td>103.000</td>
<td>22.078</td>
<td>2234520.000</td>
</tr>
<tr>
<td>5</td>
<td>111.000</td>
<td>22.622</td>
<td>4062960.000</td>
</tr>
<tr>
<td>6</td>
<td>109.000</td>
<td>25.536</td>
<td>5918854.000</td>
</tr>
<tr>
<td>7</td>
<td>100.000</td>
<td>36.094</td>
<td>7510450.000</td>
</tr>
<tr>
<td>8</td>
<td>86.000</td>
<td>57.113</td>
<td>8512614.000</td>
</tr>
<tr>
<td>9</td>
<td>71.000</td>
<td>76.565</td>
<td>8764492.000</td>
</tr>
<tr>
<td>10</td>
<td>59.000</td>
<td>85.632</td>
<td>8416764.000</td>
</tr>
<tr>
<td>11</td>
<td>47.000</td>
<td>86.572</td>
<td>7701761.000</td>
</tr>
<tr>
<td>12</td>
<td>39.000</td>
<td>82.884</td>
<td>6800436.000</td>
</tr>
<tr>
<td>13</td>
<td>32.000</td>
<td>76.928</td>
<td>5841266.500</td>
</tr>
<tr>
<td>14</td>
<td>28.000</td>
<td>70.121</td>
<td>4901137.000</td>
</tr>
<tr>
<td>15</td>
<td>24.000</td>
<td>63.270</td>
<td>4022114.000</td>
</tr>
<tr>
<td>16</td>
<td>22.000</td>
<td>56.796</td>
<td>3222201.250</td>
</tr>
<tr>
<td>17</td>
<td>22.000</td>
<td>50.913</td>
<td>2534144.000</td>
</tr>
<tr>
<td>18</td>
<td>22.000</td>
<td>45.663</td>
<td>1966323.250</td>
</tr>
<tr>
<td>19</td>
<td>22.000</td>
<td>41.076</td>
<td>1504742.000</td>
</tr>
<tr>
<td>20</td>
<td>22.000</td>
<td>37.144</td>
<td>1135166.000</td>
</tr>
</tbody>
</table>

It is proposed to approximate the data by an expression of the form

\[ \hat{z}_k = A[Bx_k + (1 - B)y_k] + C \]

where (\(\hat{\cdot}\)) denotes “estimated”. Determine a least-squares approximation of the form given by the above equation. (note that there are two independent variables, \(x\) and \(y\))

2. Consider the approximation of the function \( f(t) = e^{\frac{t^2}{32}} \) in the interval \([-2\pi, 2\pi]\). First map the \(t\)-domain to the \(x\)-domain in such a way that \([-2\pi, 2\pi]\) (in \(t\)-domain) maps into \([-1,1]\) (in \(x\)-domain).
Approximate the function by employing a Legendre basis \( \{P_j(x)\}_{j=0}^{4} \). Graphically compare the function to be approximated with the resulting approximants.

3. The water level in the North Sea is mainly determined by the so-called M\(_2\)-tide, whose period is about 12 hours and thus has the form
\[
H(t) = h_0 + a_1 \sin \frac{2\pi t}{12} + a_2 \cos \frac{2\pi t}{12},
\]
where \( t \) is in hours. One has made the following measurements:

<table>
<thead>
<tr>
<th>( t ), hours</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H(t) ), meters</td>
<td>1.0</td>
<td>1.6</td>
<td>1.4</td>
<td>0.6</td>
<td>0.2</td>
<td>0.8</td>
</tr>
</tbody>
</table>

Fit \( H(t) \) to the series of measurements using the method of least squares and determine \( h_0, a_1 \) and \( a_2 \).

4. Estimate the value of the function at \( x = 4 \) from the table of data given below, using, (a) Lagrange interpolating polynomial of 2\(^{nd}\) order; (b) Newton’s interpolating polynomial.

\[
\begin{array}{c|c}
 x & f(x) \\
 1 & 1 \\
 2 & 12 \\
 3 & 54 \\
 5 & 375 \\
 6 & 756 \\
\end{array}
\]

V. Differentiation and Integration

1. Consider the function \( f(x) = \sin x / x^3 \)
   (a) Obtain finite difference approximations of \( f' \) with first order backward difference, second order central difference and 4\(^{th}\) order central difference. Evaluate \( f' \) by the three methods at 20 equally spaced points in the interval \([1, 2\pi]\). Also evaluate the true value of \( f' \) at the same points. Plot \( f' \) vs. \( x \) and graphically compare the true values with the three approximations you have obtained, all in the same plot. Show them by different styles of lines.

   (b) Start with \( h = 1 \) and do repeated interval halving for 10 times. For each \( h \) value, obtain the approximate derivative at \( x = 4 \). Also calculate the true derivative at \( x = 4 \). Now, compute the absolute value of the error for each \( h \)-value. Now, plot \( \ln[\text{error}] \) vs. \( \ln[h] \) and obtain the slope of the line. Repeat this procedure for each of the three methods mentioned in 1(a). What are the slopes of these lines?

2. Find \( \int_1^8 \frac{\log x}{x} dx \) numerically using 5 points in the interval by a) Trapezoidal rule, b) Simpson’s rule, c) Gaussian Quadrature. Compute the % error in each of the three cases.

3. The following table is given for the values of \( e^x \):

<table>
<thead>
<tr>
<th>( x )</th>
<th>0.00</th>
<th>0.25</th>
<th>0.50</th>
<th>0.75</th>
<th>1.00</th>
<th>1.25</th>
<th>1.50</th>
<th>1.75</th>
<th>2.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e^x )</td>
<td>1.0000</td>
<td>1.2840</td>
<td>1.6487</td>
<td>2.1170</td>
<td>2.7183</td>
<td>3.4903</td>
<td>4.4817</td>
<td>5.7546</td>
<td>7.3891</td>
</tr>
</tbody>
</table>
a) Compute \( \frac{de^x}{dx} \) using central difference scheme with \( h = 0.25, 0.50 \) and 1.00.

b) Using the values computed in (a), obtain an estimate with maximum possible accuracy for the derivative by successive application of Richardson’s extrapolation.

c) Compute absolute values of the true relative error for each computed value of the derivative.

4. We are interested in fitting a piecewise Lagrange Polynomial through a set of \( N+1 \) equispaced (regular grid) discrete points by taking three points at a time. The grid points are denoted as \( x_0, x_1, x_2, \ldots, x_n \) and the corresponding functional values as \( f_0, f_1, f_2, \ldots, f_n \). Consider any three consecutive grid points \( x_i, x_{i+1} \) and \( x_{i+2} \) where corresponding functional values are \( f_i, f_{i+1} \) and \( f_{i+2} \), respectively.

a) Write the expression for the Lagrange Polynomial \( \hat{f}(x) \) through these three points.

b) Using (a), obtain the expressions for \( \hat{f}'(x) \) and \( \hat{f}''(x) \) at point \( x_i \).

c) What is the order of truncation errors for the expressions obtained in (b)?

d) Compare these expressions with the well-known Finite Difference expressions.

VI. ODE: Initial Value Problems

1. Solve the differential equation \( \frac{dy}{dt} = -100 y - 99 e^{-t} \) with the initial condition \( y(0)=2 \) using, (a) Euler’s forward (explicit) method, and (b) Euler backward (implicit) method, to obtain the value of \( y \) at \( t=0.1 \). Use time steps of 0.01, 0.02 and 0.025. Find the analytical solution and compare the errors for these time steps.

2. Solve the differential equation \( \frac{dy}{dx} = x^2 y - 2y \) with \( y(0)=1 \) over the interval \( x=0 \) to 0.5, using (a) Heun’s method without iteration with \( h=0.25 \) and 0.125, (b) Heun’s method with iteration (with \( h=0.25 \) and stopping criterion 1\%), (c) Classical 4th order Runge-Kutta method with \( h=0.125 \) and 0.25. Obtain the exact value of \( y \) at \( x=0.5 \) and perform an error analysis.

3. Solve the differential equation \( \frac{dy}{dx} = 10 \sin(\pi x) \) with the initial condition \( y(0)=0 \) and step length of 0.2 using (a) the 4th order R-K method, (b) the Milne’s method and (c) 4th order Adams method to obtain the value of \( y \) at \( t=0.2, 0.4, 0.6, 0.8 \) and 1.0. (For the multi-step methods use the values obtained from the R-K method for start-up.)

VII. System of ODEs and Boundary Value Problems

1. Solve the differential equation \( \frac{d^2y}{dx^2} - 2y + 2x = 3 \) with the boundary conditions \( y(0)=0 \) and \( y(0.5)=0.6967 \) using the direct method (use \( \Delta x = 0.25 \)).

2. Consider a simple pendulum consisting of mass \( m \) attached to a string of length \( l \). The equation of motion for the mass is \( \ddot{\theta} = -\frac{g}{l} \sin \theta \) where positive \( \theta \) is counterclockwise. For small angles \( \theta \), \( \sin \theta \approx \theta \) and the linearized equation of motion is \( \ddot{\theta} = -\frac{g}{l} \theta \). The acceleration due to gravity is \( g = 9.81 \) \( \text{m/sec}^2 \), and \( l = 0.6 \) \( \text{m} \). Assume that the pendulum starts from rest with \( \theta (t = 0) = 10^\circ \).
Solve the linearized equation for $0 = t = 2.0$ using a time step $h = 0.2$ by:

a) Analytical method and obtain true solutions.

b) Euler explicit method and compute true absolute error at each time step.

c) Euler implicit method and compute true absolute error at each time step.

d) Trapezoidal method and compute true absolute error at each time step.

e) Graphically compare your results of (b), (c) and (d) with (a) and discuss in terms of the accuracy and stability of these numerical schemes. Which of the above three methods is most accurate for this equation and why?

VIII. Partial Differential Equation

1. Temperature distribution in a plate is governed by the following equation:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

in $x \in (0,1), y \in (0,1)$, subject to the boundary conditions $T(0,y) = T(1,y) = T(x,0) = 0$ and $T(x,1) = \sin \pi x$. The exact solution of the problem is given by $T(x, y) = \sin \pi x \frac{\sinh \pi y}{\sinh \pi}$. Develop a computer code for the numerical solution of the problem using central difference approximations and graphically compare the numerical solution with the exact solution at $x = 0.5$ for $\Delta x = \Delta y = 0.1$.

2. Consider the following inhomogeneous heat equation:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + (\pi^2 - 1)e^{-t} \sin \pi x$$

with initial and boundary conditions $T(0,t) = T(1,t) = 0$ and $T(x,0) = \sin \pi x$

a) Write a computer program to solve the equation using Euler explicit-Central difference approximations, for $\alpha = 1, \Delta x = 0.05$ and $\Delta t = 0.001$. Plot $T(x)$ vs. $x$ at $t = 0.0, 0.5, 1.0, 1.5$ and $2.0$ in one plot.

b) Take new $\Delta t = 0.0015$ and solve the equation for the same $\alpha$ and $\Delta x$. Plot $T(x)$ vs. $x$ in the 2nd plot at $t = (0.0, 0.15, 0.153, 0.1545, 0.156)$, if you are using mainframe for computation and $t = (0.0, 0.075, 0.0915, 0.093, 0.0945)$, if you are using PC/Linux.

c) Explain the results obtained in (a) and (b).
1. The computation of the expression
\[ G(\varepsilon) = \frac{\sqrt{1 + 8\varepsilon^2} - 1}{2} \]
involves the difference of small numbers when \( \varepsilon \ll 1 \). Obtain the value of \( G \) for \( \varepsilon = 0.001 \) and estimate the corresponding relative error, performing operations by rounding all mantissas to six decimals (use double precision calculation to obtain the true solution that is needed to calculate the relative error). Improve your computation of \( G \) by employing a Taylor series expansion, and estimate the corresponding relative error.

2. One has measured two sides and the included angle of a triangle to be \( a = 100.0 \pm 0.1, b = 101.0 \pm 0.1, \) and the angle \( C = 1.00^\circ \pm 0.01^\circ \). How accurately is it possible to give the third side \( c \)?

3. Let the function \( f(x) \) be four times continuously differentiable and have a simple zero at \( \xi \). Successive approximations \( x_n \) for the root \( \xi \) are computed from
\[ x_{n+1} = \frac{x_n + x'_n}{2} \]
where,
\[ x'_n = x_n - \frac{f(x_n)}{f'(x_n)} \]
\[ x''_n = x_n - \frac{u(x_n)}{u'(x_n)}, \quad u(x) = \frac{f(x)}{f'(x)} \]

Show that if the sequence \( \{x_n\} \) converges to \( \xi \), then the rate of convergence is cubic.

4. Find the root of \( x - \cos x = 0 \) using Bisection, False position, Fixed Point Iteration, Newton and Secant Methods. Also perform error analysis. Take the true value of the root as 0.739097 and stop when the errors are less than 0.01%.

5. Find a root of the following equation using Mueller’s method to an approximate error of \( \varepsilon_n \leq 0.1\% \):
\[ x^4 \sin x - e^x = 0 \]
Take three starting values as 1, 2 and 3.

6. Find the root of the polynomial using Bairstow’s method using \( \varepsilon = 0.01\% \).
\[ x^3 - 21x^2 + 129x - 234 = 0 \]

7. Consider the following set of equations:
\[
\begin{bmatrix}
10^{-5} & 10^{-5} & 1 \\
10^{-5} & -10^{-5} & 1 \\
1 & 1 & 2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= \begin{bmatrix} 2 \times 10^{-5} \\
-2 \times 10^{-5} \\
1
\end{bmatrix}
\]

a) Solve the system using Gaussian elimination, without pivoting, using 3-digit floating-point arithmetic with round-off.
b) Perform complete pivoting and carry out Gaussian elimination steps once again using 3-digit floating-point arithmetic with round-off. Explain the results.
c) Rewrite the set of equations after scaling according to \( x'_3 = 10^5 \times x_3 \) and equilibration.
8. For the circuit shown in the figure, find the currents through the elements using (a) Gauss elimination (b) Gauss Jordan and (c) Gauss Seidel methods. (Use \( R_1 = 10 \) Ohm; \( R_2 = 20 \) Ohm; \( R_3 = 40 \) Ohm; \( V_A - V_C = 200 \) Volt; \( V_B - V_C = 100 \) Volt)

![Circuit Diagram]

9. In the above problem, how much will be the change in current in \( R_3 \) due to unit change in voltage difference \((V_A - V_C)\)? Which of the currents is the most sensitive to a change in voltage at \( B \)?

10. Solve the following set of equations using (a) Doolittle and Crout decomposition (b) Thomas algorithm and (c) Cholesky decomposition

\[
\begin{align*}
4x_1 + x_2 &= 6 \\
x_1 + 4x_2 + x_3 &= 12 \\
x_2 + 4x_3 &= 14
\end{align*}
\]

11. Let \( A \) be a given nonsingular \( n \times n \) matrix, and \( X_0 \) an arbitrary \( n \times n \) matrix. We define a sequence of matrices by

\[ X_{k+1} = X_k + X_k(I - AX_k), \quad k = 0, 1, 2, \ldots \]

Prove that \( \lim_{k \to \infty} X_k = A^{-1} \) if and only if \( \rho(I - AX_0) < 1 \).

12. Solve the following equations using (a) fixed-point iteration and (b) Newton-Raphson method, starting with an initial guess of \( x=1 \) and \( y=1 \)

\[
\begin{align*}
x^2 - x + y - 0.5 &= 0 \\
x^2 - 5xy - y &= 0
\end{align*}
\]

13. Consider the following Matrix:

\[
\begin{bmatrix}
7 & -2 & 1 \\
-2 & 10 & -2 \\
1 & -2 & 7
\end{bmatrix}
\]

a) Obtain the Equation of the Characteristic polynomial using Fadeev-Leverrier Method.
b) Perform one complete iteration of the Bairstow’s method with the starting values as \( r = 18 \) and \( s = -10 \). Compute the approximate eigenvalues of the matrix after this iteration.

14. Compute the approximate eigenvalues of the matrix using QR-decomposition. Perform only two iterations of the QR algorithm.

\[
\begin{bmatrix}
3 & 4 & 1 \\
3 & 5 & 1 \\
2 & 2 & 1
\end{bmatrix}
\]
15. The cost of fuel consumed by a truck was assumed to be linearly related to the travel distance and the load carried. Over a certain period, the following data was recorded by the driver. Obtain the underlying relationship (add the constraint that there is no cost when both the distance and the load are zero). How good is the assumption that the relationship is linear?

<table>
<thead>
<tr>
<th>Distance (km)</th>
<th>88</th>
<th>210</th>
<th>320</th>
<th>88</th>
<th>210</th>
<th>320</th>
<th>245</th>
<th>65</th>
</tr>
</thead>
<tbody>
<tr>
<td>Load Factor</td>
<td>0.33</td>
<td>0.42</td>
<td>0.50</td>
<td>0.17</td>
<td>0.28</td>
<td>0.67</td>
<td>0.32</td>
<td>1.00</td>
</tr>
<tr>
<td>Cost (Rs.)</td>
<td>140</td>
<td>270</td>
<td>400</td>
<td>110</td>
<td>250</td>
<td>450</td>
<td>280</td>
<td>225</td>
</tr>
</tbody>
</table>

16. It is known that bacteria swim against the concentration gradient of the food. The following data were measured for the concentration gradient of the food and the corresponding speed of swim:

<table>
<thead>
<tr>
<th>Conc. Grad., x, (µmol/cm)</th>
<th>0.1</th>
<th>0.2</th>
<th>0.5</th>
<th>1.3</th>
<th>4.2</th>
<th>9.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Speed, y, (µm/min)</td>
<td>2.85</td>
<td>4.00</td>
<td>6.00</td>
<td>10.00</td>
<td>15.00</td>
<td>20.00</td>
</tr>
</tbody>
</table>

It was decided to fit the equation, \( y = \frac{a\sqrt{x}}{1 + b\sqrt{x}} \) to the data. a) Obtain least square estimates of \( a \) and \( b \); b) Calculate \( r^2 \) for the non-linear fit; c) Graphically compare the data values with the fitted curve.

17. The mass of a radioactive substance is measured at 2-day intervals till 8 days. Unfortunately, the reading could not be taken at 6 days due to equipment malfunction. The following table shows the other readings:

<table>
<thead>
<tr>
<th>Time (d)</th>
<th>Mass (g)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.000</td>
</tr>
<tr>
<td>2</td>
<td>0.7937</td>
</tr>
<tr>
<td>4</td>
<td>0.6300</td>
</tr>
<tr>
<td>8</td>
<td>0.3968</td>
</tr>
</tbody>
</table>

(a) Estimate the mass at 6 days using (i) Newton’s divided difference, (ii) Lagrange polynomials, (iii) Cubic spline and (iv) second-order polynomial regression

(b) Using this table and the value obtained in (i) and (ii) above, estimate the half-life of the substance using least squares regression after linearising the equation.

18. Consider the following polynomial \( \{p_n(x)\} \):

\[
p_n(x) = \frac{\sin(n+1)\phi}{\sin\phi} \quad \text{where} \ x = \cos \phi
\]

a) Show that this polynomial satisfies the same recursion formula as the Tchebycheff polynomial.

b) What are the first three polynomials, \( p_0(x) \), \( p_1(x) \) and \( p_2(x) \) ?

c) Show that \( \{p_n(x)\} \) forms an orthogonal system of polynomials with the weight function \( w(x) = (1 - x^2)^{1/2} \), \( x \in [-1, 1] \).
19. Derive a finite difference approximation for \( f_j'' \) in terms of \( f_j, f_{j+1} \) and \( f_{j+2} \). What is the order of accuracy of this approximation?

20. The location of an object at various times was measured as follows:

<table>
<thead>
<tr>
<th>Time (min)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance (cm)</td>
<td>0</td>
<td>3</td>
<td>14</td>
<td>39</td>
<td>84</td>
<td>155</td>
<td>258</td>
<td>399</td>
<td>584</td>
<td>819</td>
</tr>
</tbody>
</table>

Estimate the speed and acceleration of the object at 5 minutes using (a) Forward difference, \( O(h^2) \); (b) Backward difference, \( O(h^2) \); (c) Central difference, \( O(h^2) \); and (d) Richardson extrapolation, \( O(h^6) \) using three central differences of \( O(h^2) \). (e) Perform an error analysis for the results obtained in the above problem, using the fact that the distance is given by \( x = t + t^2 + t^3 \).

21. The flow rate through a circular pipe is given by \( Q = \int_0^r 2\pi r v dr \), where \( v \) is the velocity at a distance of \( r \) from the centre of pipe and \( r_0 \) is the radius of the pipe. If the velocity is approximated by \( v = 2 \left( 1 - \frac{r}{r_0} \right)^{1/3} \) (in m/s), and the pipe radius is 12 cm, compute \( Q \) using (a) trapezoidal rule with \( h = 2 \) cm; (b) Simpson’s one-third rule with \( h = 3 \) cm; (c) Simpson’s three-eighth rule with \( h = 4 \) cm; and (d) 3-point Gauss-Legendre quadrature. (e) Perform an error analysis for the results obtained in the above problem, using the true value of the flow rate as 0.0738902 m\(^3\)/s.

22. Velocity profile in an open channel flow is shown below.

The depth of the channel is \( d \). The velocity at any depth is given by an arbitrary function \( u(z) \), graphically shown above. The mean velocity of the channel \( \bar{u} \) is given by \( \bar{u} = \frac{1}{d} \int_0^d u(z)dz \).

a) Find two constants \( c_1 \) and \( c_2 \) such that \( z_1 = c_1d, z_2 = c_2d \), and \( \bar{u} = \frac{u(z_1) + u(z_2)}{2} \).

b) The velocities \( u_0, u_1, u_2 \) and \( u_3 \) have been measured at depths of 0.2\( d \), 0.4\( d \), 0.6\( d \) and 0.8\( d \), respectively. Compute \( \omega_0, \omega_1, \omega_2 \) and \( \omega_3 \) such that the velocity at 0.5\( d \) can be expressed as \( \omega_0u_0 + \omega_1u_1 + \omega_2u_2 + \omega_3u_3 \).
23. Show that the trapezoidal rule, with \( h = \frac{2\pi}{n+1} \) is exact for \textit{trigonometric polynomials} of period \( 2\pi \), i.e., for functions of the form \( \sum_{k=-n}^{n} c_k e^{ikt} \); \( k \) integer, when it is used for integration over a whole period.

24. A common problem is that of solving the Fredholm integral equation:

\[
 f(x) = \phi(x) + \int_{a}^{b} K(x,t)\phi(t)dt
\]

where the function \( f(x) \) and \( K(x,t) \) are given and the problem is to obtain \( \phi(x) \). Given \( f(x) = \pi x^2 \), \( K(x,t) = -3(0.5 \sin 3x - tx^2) \), \( a = 0 \) and \( b = \pi \).

a) Obtain \( \phi(x) \) at \( x = 0, \pi/2 \) and \( \pi \) and compare with the true solution given by \( \phi(x) = \sin 3x \). Comment on the amount of error with reasoning. Use 3-point Simpson’s rule approximation for the integral evaluation.

b) The above equation becomes the Volterra integral equation when \( b = x \). For the same \( f(x) \) and \( K(x,t) \) and \( a \) as given above for the Fredholm equation, compute \( \phi(x) \) at \( x = 0, \pi/2 \) and \( \pi \) for the Volterra equation. Use Trapezoidal rule for the evaluation of the integral with \( h = \pi/2 \).

25. The amount of lowering of water level, \( s \), in a well at a time \( t \), due to pumping from groundwater is governed by an equation of the form \( s = A W(u) \), where \( A \) is a constant (proportional to the discharge), \( W \) is called the Well Function, and \( u \) is inversely proportional to \( t \). The well function is given by the equation \( \text{d}W(u)/\text{d}u = -\exp(-u)/u \). If the value of \( W(1) \) is 0.2194, find the value of \( W(0.5) \) using (a) Romberg integration algorithm with accuracy O(\( h^6 \)) (b) Modified Euler with \( h = -0.5 \) (c) Heun’s method with \( h = -0.25 \) and (d) Fourth-order Runge-Kutta method with \( h = -0.25 \).

26. Using a series expansion for \( \exp(-u) \), integrating term by term, and minimizing the error, an approximate expression for \( W(u) \) for \( u \) less than 1 is obtained as: \( W(u) = -\ln u - 0.57722 + 0.99999 u - 0.24991 u^2 + 0.05519 u^3 - 0.00976 u^4 + 0.00108 u^5 \). Perform an error analysis for the results obtained in the previous problem, using the true value obtained from this expression.

27. Solve the differential equation \( \frac{\text{d}y}{\text{d}t} = -100y + 99 \exp(-t) \) with the initial condition \( y(0) = 2 \) using the Euler’s method to obtain the value of \( y \) at \( t = 0.1 \). Use time steps of (a) 0.01 (b) 0.02 and (c) 0.025. Find the analytical solution and compare the errors for these time steps.

28. Repeat the previous problem using the Implicit Euler’s method for all three step sizes.

29. Solve the differential equation \( \frac{\text{d}y}{\text{d}t} = -y + \exp(-t) \) with the initial condition \( y(0) = 1 \) using (a) the Milne’s method and (b) 4\textsuperscript{th} order Adams method to obtain the value of \( y \) at \( t = 0.8 \) and 1.0. (To start the computations for these multi-step methods, find the analytical solution and use the true values of \( y \) and \( \frac{\text{d}y}{\text{d}t} \) at \( t = 0, 0.2, 0.4, \) and 0.6.)

30. Solve the differential equation \( \frac{\text{d}^2y}{\text{d}x^2} - \frac{\text{d}y}{\text{d}x} - 2y + 2x = 3 \) with the boundary conditions \( y(0) = 0 \) and \( y(0.5) = 0.6967 \) using the shooting method (use \( \Delta x = 0.25 \)).

31. The diagram shows a body of conical section fabricated from stainless steel immersed in air at zero temperature. It is of circular cross section that varies with \( x \). The large end is located at \( x = 0 \) and is held at temperature \( T_A = 5 \). The small end is located at \( x = L = 2 \) and is insulated (i.e., the temperature gradient is zero). Conservation of energy can be used to develop a heat balance equation.
at any cross section of the body. When the body is not insulated along its length and the system is at steady state, its temperature satisfies the following ODE:

\[ \frac{d^2 T}{dx^2} + a(x) \frac{dT}{dx} + b(x) T = f(x) \]

where \( a(x) \), \( b(x) \), and \( f(x) \) are functions of the cross-sectional area, heat transfer coefficients, and the heat sinks inside the body. In the present case, they are given by

\[ a(x) = -\frac{x + 3}{x + 1}, \quad b(x) = \frac{x + 3}{(x + 1)^2}, \quad \text{and} \quad f(x) = 2(x + 1) + 3b(x). \]

![Diagram](image)

a) Discretize the above equation using 2\textsuperscript{nd} order finite difference approximation and formulate the set of three linear simultaneous equation. Incorporate the boundary conditions such that the accuracy of the scheme is preserved. Use \( \Delta x = 0.5 \).

b) Solve the system of equations using Thomas Algorithm and draw the temperature profile indicating the values at the nodes.

32. The following scheme has been proposed for solving \( \frac{dy}{dt} = f(y) \):

\[ y_{n+1} = y_n + \omega_1 k_1 + \omega_2 k_2 \]

where, \( k_1 = hf(y_n), \ k_0 = hf(y_n + \beta_0 k_1), \) and \( k_2 = hf(y_n + \beta_1 k_0), \) with \( h \) being the time step.

(a) Determine the coefficients \( \omega_1, \ \omega_2, \ \beta_0 \) and \( \beta_1 \) that would maximize the order of accuracy of this method. [Hint: Perform all the Taylor series expansions for \( y_{n+1}, k_0, k_1 \) and \( k_2 \) up to \( h^3 \) exactly and represent the residual as \( O(h^4) \).]

(b) Applying this method to \( y' = \alpha y \), what is the maximum step size \( h \) for purely imaginary \( \alpha \)?

(c) Applying this method to \( y' = \alpha y \), what is the maximum step size \( h \) when \( \alpha \) is real and negative?

33. The fourth order Runge-Kutta formula for the initial value problem \( \frac{dy}{dt} = f(y, t) \) can be written in the following form,

\[ y_{n+1} = y_n + \Delta \left[ \frac{1}{6} f(y_n, t_n) + \frac{1}{3} f(y_{n+1}^*, t_{n+1/2}) + \frac{1}{3} f(y_{n+1/2}^*, t_{n+1/2}) + \frac{1}{6} f(y_{n+1}, t_{n+1}) \right] \]

where,
\[ y_{n+1}^* = y_n + \frac{\Delta t}{2} f(y_n, t_n) \]
\[ y_{n+1}^{**} = y_n + \frac{\Delta t}{2} f\left(y_{n+1}^*, t_{n+1}\right) \]
\[ y_{n+1} = y_n + \Delta f\left(y_{n+1}^{**}, t_{n+1}\right) \]
\[ t_{n+1} = t_n + \frac{\Delta t}{2} \]

a) Obtain the leading order term in the truncation error.

b) Obtain the equation for the stability region of this method and plot it in the complex \( \mu \)-plane.
(Note: You will need to numerically solve a polynomial equation for complex root. You can write a function or subroutine for Bairstow’s method or you can use some ‘canned’ function or subroutine.)

34. For a plate of size \( L_x=2 \) cm and \( L_y=4 \) cm, with the boundary conditions as \( T=0 \) for \( y=0 \) and \( y=4 \); \( T=100 \) for \( x=2 \); and insulated boundary at \( x=0 \), find the steady state temperature at the centre using the Finite Difference Method (use \( \Delta x=1 \) cm and \( \Delta y=1 \) cm).

35. Toxic pollutant transport in a river is governed by the following equation:
\[
\frac{\partial c}{\partial t} + v \frac{\partial c}{\partial x} - D \frac{\partial^2 c}{\partial x^2} + kc = 0; \quad 0 \leq x \leq 1; \quad c(0,t) = c_0; \quad \left. \frac{\partial c}{\partial x}\right|_{(0,t)} = 0; \quad c(x,0) = x^2 e^{-x}
\]

Set-up the matrix equations for the solution of the above equation in terms of Courant Number (or CFL number) and Grid Peclet Number for general \( \Delta x \) and \( \Delta t \):

Courant or CFL No. \( C = \frac{v\Delta t}{\Delta x} \)

Grid Peclet No. \( P_g = \frac{v\Delta x}{D} \)

36. Consider the following hyperbolic dimensionless wave equation:
\[
\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \text{ in } x \in [0,1] \text{ for } t = 0
\]
subject to \( u(0,t) = \sin 2\pi t \), \( u_x(1,t) = 0 \), \( u(x,0) = 0 \), \( u_t(x,0) = 2\pi \cos 2\pi x \).

a) Using \( \Delta x = 0.25 \), write down the discretized equation in the matrix form using the \( O(\Delta x^2, \Delta t^2) \) accurate implicit scheme:
\[
\frac{\delta^2 t u^n_j}{\Delta t^2} = \frac{\delta^2 t u^{n+1}_j}{4\Delta x^2} + 2 \delta^2 t u^n_j + \delta^2 t u^{n-1}_j
\]
where \( \delta^2 t u^n_j = u^{n+1}_j - 2u^n_j + u^{n-1}_j \) and \( \delta^2 t u^n_j = u^{n+1}_j - 2u^n_j + u^{n-1}_j \). Incorporate the time derivative initial condition and the space derivative boundary condition in such a manner which preserves the accuracy of the scheme.

b) Write an algorithm for the solution of this equation.
37. The following numerical method has been proposed to solve \( \frac{\partial u}{\partial t} = \frac{c}{\partial x} : \)
\[
\frac{1}{\Delta t} \left[ u_{j}^{n+1} - \frac{1}{2} \left( u_{j+1}^{n} + u_{j-1}^{n} \right) \right] = \frac{c}{2\Delta x} \left[ u_{j+1}^{n} - u_{j-1}^{n} \right]
\]

a) Find the range of CFL number for which the method is stable using Von Neuman Analysis.

b) Is the method consistent (i.e., does it reduce to the original PDE as \( \Delta x, \Delta t \to 0 \))?

38. The 1-D convection diffusion equation in problem 35 becomes a pure convection process when \( \alpha = 0. \)
\[
\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} = 0
\]

For the solution of the pure convection problems, the following scheme (known as Lax-Wendroff scheme) is often used:
\[
T_{j}^{n+1} = T_{j}^{n} - \frac{\gamma}{2} \left( T_{j+1}^{n} - T_{j-1}^{n} \right) + \frac{\gamma^2}{2} \left( T_{j+1}^{n} - 2T_{j}^{n} + T_{j-1}^{n} \right) \quad \text{where,} \quad \gamma = \frac{u\Delta t}{\Delta x}
\]

a) What is the order of accuracy of this scheme? Is the scheme consistent?

b) Obtain the stability criteria for this scheme.