Assignment 13 - Solutions

1. Use spherical coordinates. Let \( x = \rho \cos \theta \sin \phi \), \( y = \rho \sin \theta \sin \phi \) and \( z = \rho \cos \phi \), where \( 0 \leq \rho \leq 1 \), \( 0 \leq \theta \leq 2\pi \) and \( 0 \leq \phi \leq \pi \).

\[
\iiint_{W} dzdydx \sqrt{1 + x^2 + y^2 + z^2} = \frac{\pi}{\rho} \int_{0}^{1} \int_{0}^{2\pi} \int_{0}^{\pi} \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi = 2\pi (\sqrt{2} - \ln(1 + \sqrt{2})).
\]

2. In the cylinder there are three surfaces \( S_1, S_2 \) and \( S_3 \) where

(a) \( S_1 \) : The base of the cylinder, i.e., \( z = 0 \),
(b) \( S_2 \) : The top of the cylinder i.e., \( z = h \),
(c) \( S_3 \) : The curved surface of the cylinder.

(a) On \( S_1 \), the integral is zero.

(b) The surface integral over \( S_2 \) is

\[
\iint_{S_2} x^2 \, z \, d\sigma = \frac{ah^4}{4}.
\]

(c) A parametric representation of \( S_3 \) is

\[
r(u, v) = (a \cos u, a \sin u, v), \quad 0 \leq u \leq 2\pi, 0 \leq v \leq h.
\]

The surface integral over \( S_3 \) is

\[
= \int_{0}^{h} \int_{0}^{2\pi} (a \cos u)^2 \, v \sqrt{E-G-F^2} \, du \, dv,
\]

where \( E = r_u \cdot r_u \), \( G = r_v \cdot r_v \) and \( F = r_u \cdot r_v \).

Note that \( \sqrt{E-G-F^2} = a \). Therefore, \( \iint_{S_3} x^2 \, z \, d\sigma = \frac{ah^2 \pi}{2} \).

Hence, the required integral is

\[
\frac{ah^4}{4} + \frac{ah^2 \pi}{2}.
\]

Over the entire volume, the integral is

\[
V = \int_{0}^{2\pi} \int_{0}^{a} \int_{0}^{h} (r \cos \theta)^2 \cdot z r \, dr \, d\theta \, dz = \frac{h^2 \pi a^4}{8}.
\]

3. \( \oint_{C} (y, -x, 1) \cdot dR = \int_{0}^{2\pi} \left( (\sin t)(-\sin t) dt - \cos t \cos t + \frac{1}{2\pi} \right) dt \).

4. Take \( C = R(t) = (cost, sint), \quad 0 \leq t \leq 2\pi \). Then

\[
\int_{C} T \cdot dR = \int_{0}^{2\pi} T(t) \cdot R'(t) dt = \int_{0}^{2\pi} \frac{R'(t)}{\| R'(t) \|} \cdot R'(t) dt = 2\pi
\]

5. If \( F = yzi + (xz + 1)j + xyk \), then \( F = \nabla \varphi \), where \( \varphi(x, y, z) = xyz + y \). Hence, by the 2nd fundamental theorem of calculus for line integrals, the problem follows.

Assignment 14 - Solutions
1. \( M = 2x^2 - y^2 \) and \( N = x^2 + y^2 \). By Green’s Theorem

\[
\int_C (2x^2 - y^2)\,dx + (x^2 + y^2)\,dy = \int_0^1 \int_0^{\sqrt{1-x^2}} (N_x - M_y)\,dy\,dx
\]

\[
= \int_0^1 \int_0^{\sqrt{1-x^2}} 2(x + y)\,dy\,dx = \frac{4}{3}.
\]

2. Let \( F = -y^3\vec{i} + x^3\vec{j} - z^3\vec{k} \). By Stoke’s Theorem, \( \int_{\partial S} F\cdot\vec{n}\,d\sigma = \iint_S (\text{curl} \, F)\cdot\vec{\mathbf{n}}\,d\sigma \).

Note that \( \nabla \times F = 3(x^2 + y^2)\vec{k} \). Hence, \( \int_{\partial S} F\cdot\vec{n}\,d\sigma = \iint_D 3(x^2 + y^2)\,dxdy = \frac{3\pi}{2} \).

3. Note that \( \text{div} \, F = 0 \). By divergence theorem

\[
\iint_S F\cdot\vec{n}\,d\sigma = \iint_{S_\rho} F\cdot\vec{n}\,d\sigma
\]

where \( S_\rho \) is a sphere of (small) radius \( \rho \) with center at origin. On \( S_\rho \), \( n = \frac{1}{\rho}(xi + yj + zk) \) and hence \( F\cdot n = \frac{1}{\rho^2} \). Therefore,

\[
\iint_{S_\rho} F\cdot\vec{n}\,d\sigma = \frac{1}{\rho^2} \iint_{S_\rho} d\sigma = \frac{1}{\rho^2} 4\pi \rho^2 = 4\pi.
\]

4. \( \text{div} \, F = 2x + 2y + 2z \). By the divergence theorem,

\[
\int_{\partial D} F\cdot\vec{n}\,d\sigma = \int \int_D 2(x + y + z)\,dV = 2 \int \int_{x^2 + y^2 \leq 1} (\int_0^{x+y+z} (x + y + z)\,dz)\,dxdy = \frac{19\pi}{4}
\]