MTH101N, 2011-12 , Quiz 1A

Max Marks 20 Time 17:20-17:50

Name Roll No.

1. A girl flies a kite at a height of 300 ft, the wind carrying the kite horizontally away from her at a rate of 20 ft/sec. How fast must she let out the string when the kite is 500 ft away from her? [6]

   Sol:

   - See figure. The relation between x and y is given by
   $$y^2 = x^2 + 300^2$$
   [2]

   - Therefore, $2y\frac{dy}{dt} = 2x\frac{dx}{dt}$. [1]

   - Given is $dx/dt = 20$ ft/sec. To find dy/dt at the point when $y = 500$ i.e., at $x = 400$. [2]
   - Answer is $\frac{dy}{dt}|_{y=500} = \frac{x}{y}\frac{dx}{dt} = \frac{400}{500} \times 20 = 16$ ft/sec. [1]

2. Circle the correct option for the following question (**-3 marks will be awarded for a wrong answer**): The number of points where the function $f(x) = \max\{\cos x, x^2 - 1\}$ is not differentiable in the interval $(-\pi, \pi)$ is
   $$0,\ 1,\ 2\ ,3\ ,4.$$
   Note: $F(x) = \max\{g(x), h(x)\}$ is defined as the maximum of the values $g(x)$ and $h(x)$ at each point x.

   [6]

   Sol: See figure. Answer is 2. **Remember to deduct 3 marks for a wrong answer or more than one answer.**

3. Let $x_1 = 1$. Define $x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$ for all $n \in \mathbb{N}$. Prove that the sequence $\{x_n\}$ is convergent and find its limit.

   [8]

   Sol:

   - Note that $x_n \geq 0, \forall n$.
   As $x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \geq \sqrt{x_n \frac{2}{x_n}} = \sqrt{2}$ (by the A.M -G.M. inequality). [3]

- Thus $x_{n+1} \geq \sqrt{2}$, $\forall\ n$. Hence, (x_n) is bounded below.
  Further $x_{n+1} - x_n = \frac{1}{2}(x_n + \frac{2}{x_n}) - x_n \leq 0$. Thus (x_n) is a decreasing sequence which is also bounded below and is therefore convergent. [3]

- Let $\lim\limits_{n \longrightarrow \infty} x_n = \ell$. $\ell = \frac{1}{2}(\frac{\ell^2+2}{\ell}) \Rightarrow \ell = \pm\sqrt{2}$. As x_n are nonnegative, the limit is also non negative and therefore $\ell = \sqrt{2}$. [2]
  Note: Here if they just right $\ell = \sqrt{2}$, and do not rule out the case $\ell = \sqrt{-2}$, still give (2) marks.