(1) Find the supremum of the set \( X = \{ \pi - 1, \pi - \frac{1}{2}, \pi - \frac{1}{3}, \ldots \} \). [5]

Solution: Clearly, \( \pi \) is an upper bound for the set \( X \). Let \( \epsilon > 0 \).
By the Archimedean property, there exists a natural number \( n_\epsilon \) such that \( \frac{1}{n_\epsilon} < \epsilon \). [2]
\[
\pi - \epsilon < \pi + \frac{1}{n_\epsilon} < \pi.
\] [2]
Hence, Supremum of the set \( X \) is \( \pi \). [1]

(2) Let \( f : \mathbb{R} \to \mathbb{R} \) be a function defined as
\[
f(x) = \begin{cases} 
x^2 \sin \frac{1}{x^2}, & x \neq 0 \\
0, & x = 0
\end{cases}
\]
Compute \( f'(x) \), for all \( x \in \mathbb{R} \). [5]

Solution: \( f'(x) = 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2} \), at each \( x \neq 0 \).
\[
f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h}, \text{ if the limit exists.}
\]
\[
\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} h \sin \frac{1}{h^2}
\]
\[
-h \leq h \sin \frac{1}{h^2} \leq h
\]
Taking limits as \( h \to 0 \), we get \( f'(0) = 0 \). [1]

(3) Let \( f : \mathbb{R} \to \mathbb{R} \) be a function defined as
\[
f(x) = \begin{cases} 
0, & x \text{ is rational} \\
-1, & x \text{ is irrational}
\end{cases}
\]
Show that \( f \) is not continuous at any point in \( \mathbb{R} \). What can you say about the continuity of \(|f|\)? Justify. [5]

Solution: Let \( x \in \mathbb{Q} \) and \( \{x_n\} \) be an irrational sequence converging to \( x \).
\[
\{f(x_n)\} = \{-1\} \to -1 \neq f(x) = 0.
\] [1]
Hence \( f \) is not continuous at any rational point.
By a similar argument \( f \) is not continuous at any irrational point.
\[
|f|(x) = \begin{cases} 
0, & x \text{ is rational} \\
1, & x \text{ is irrational}
\end{cases}
\]
By the same argument as the one above, we see that \(|f|\) is discontinuous at each point of \( \mathbb{R} \). [1]