MTH101A - Quiz 1A, 23.08.2013(17:25-17:45 hrs), Maximum Marks: 15

NAME: ———————————— **Roll No.**—————— **Section:** ————

1. Let (x_n) be a sequence defined by

$$x_n = n^{\alpha}(1+\beta)^{-n}\sin n$$

for all $n \in \mathbb{N}$ where α and β are fixed positive real numbers. Show that (x_n) converges. [Do not try with the L'Hospital Rule]. [4]

2. Using the mean value theorem, show that $\frac{1}{2\sqrt{n+1}} < \sqrt{n+1} - \sqrt{n} < \frac{1}{2\sqrt{n}}$ for all $n \in \mathbb{N}$. [4]

3. Let $f : [0, 12] \to \mathbb{R}$ be continuous and $f(0) = f(12)$. Show that there exist $x_1, x_2, x_3, x_4 \in [0, 12]$ such that $x_2 - x_1 = 6,\ x_4 - x_3 = 3,\ f(x_1) = f(x_2)$ and $f(x_3) = f(x_4)$. [7]

Tentative Marking Scheme

1. Let $y_n = \frac{n^{\alpha}}{(1+\beta)^n}$. Then $\frac{y_{n+1}}{y_n} = \left(\frac{n+1}{n}\right)^{\alpha}\frac{1}{1+\beta} \to \frac{1}{1+\beta} < 1$. [2]

   Therefore $y_n \to 0$. [1]

   Since $|x_n| \le |y_n|$ (or $(\sin n)$ is a bounded sequence), $x_n \to 0$. [1]

2. By the MVT $\sqrt{n+1} - \sqrt{n} = \frac{1}{2\sqrt{c}}$ for some $c \in (n, n+1)$. [3]

   Since $c \in (n, n+1)$, $\frac{1}{2\sqrt{n+1}} < \frac{1}{2\sqrt{c}} < \frac{1}{2\sqrt{n}}$. [1]

3. Define $g(x) = f(x+6) - f(x)$. [1]

   Then $g(0) = f(6) - f(0)$ and $g(6) = f(12) - f(6) = f(0) - f(6)$.

   By IVP, $\exists\ x_1 \in [0, 6]$ such that $g(x_1) = 0$. [2]

   This implies that $f(x_1) = f(x_2)$ where $x_2 = x_1 + 6$.

   Define $g_1(x) = f(x+3) - f(x)$. [2]

   Then $g_1(x_1) = f(x_1+3) - f(x_1)$ and $g_1(x_1+3) = f(x_1+6) - f(x_1+3)$ $= f(x_1) - f(x_1+3)$. [2]

   By IVP, $\exists\ x_3 \in [x_1, x_1+3]$ such that $g_1(x_3) = 0$.

   This implies that $f(x_3) = f(x_4)$ where $x_4 = x_3 + 3$.

MTH101A - Quiz 1B, 23.08.2013(17:25-17:45 hrs), Maximum Marks: 15

NAME: ______________________ **Roll No.**__________ **Section:** ______

1. Using mean value theorem, show that $\frac{x-1}{x} < \ln x < x - 1$ for $x > 1$. [4]

2. Let (x_n) be a sequence defined by

$$x_n = (1+\alpha)^{-n} n^{\beta} \cos n$$

for all $n \in \mathbb{N}$ where α and β are fixed positive real numbers. Show that (x_n) converges. [Do not try with the L'Hospital Rule]. [4]

3. Let $f : [0,8] \to \mathbb{R}$ be continuous and $f(0) = f(8)$. Show that there exist $x_1, x_2, x_3, x_4 \in [0,8]$ such that $x_2 - x_1 = 4,\ x_4 - x_3 = 2,\ f(x_1) = f(x_2)$ and $f(x_3) = f(x_4)$. [7]

<u>Tentative Marking Scheme</u>

1. By the MVT, there exists $c \in (1,x)$ such that $\ln x - \ln 1 = \frac{1}{c}(x-1)$. [3]

   Since $c \in (1,x)$, $\frac{x-1}{x} < \frac{1}{c}(x-1) < x - 1$. [1]

2. Let $y_n = \frac{n^{\beta}}{(1+\alpha)^n}$. Then $\frac{y_{n+1}}{y_n} = \left(\frac{n+1}{n}\right)^{\beta} \frac{1}{1+\alpha} \to \frac{1}{1+\alpha} < 1$. [2]

   Therefore $y_n \to 0$. [1]

   Since $|x_n| \le |y_n|$ (or $(\cos n)$ is a bounded sequence), $x_n \to 0$. [1]

3. Define $g(x) = f(x+4) - f(x)$. [1]

   Then $g(0) = f(4) - f(0)$ and $g(4) = f(8) - f(4) = f(0) - f(4)$.

   By IVP, $\exists\ x_1 \in [0,4]$ such that $g(x_1) = 0$. [2]

   This implies that $f(x_1) = f(x_2)$ where $x_2 = x_1 + 4$.

   Define $g_1(x) = f(x+2) - f(x)$. [2]

   Then $g_1(x_1) = f(x_1+2) - f(x_1)$ and $g_1(x_1+2) = f(x_1+4) - f(x_1+2)$
   $= f(x_1) - f(x_1+2)$. [2]

   By IVP, $\exists\ x_3 \in [x_1, x_1+2]$ such that $g_1(x_3) = 0$.

   This implies that $f(x_3) = f(x_4)$ where $x_4 = x_3 + 2$.