MTH101 - Quiz 1A Questions

1. Let \((x_n)\) be a sequence defined by \(x_1 = 1, x_2 = 2\) and \(x_{n+2} = \frac{3}{4}x_n + \frac{1}{4}x_{n+1}\) for \(n \geq 1\).
   
   (a) Show that \((x_n)\) converges.
   
   (b) Find the limit of \((x_n)\). [3] [4]

2. Let \(g(x) = \sin \frac{1}{x}\) for \(x \neq 0\). Show that \(\lim_{x \to 0} g(x)\) does not exist.

3. Test the convergence/divergence of the series \(\sum_{n=1}^{\infty} (1 - n \sin \frac{1}{n})n^{\frac{1}{4}}\). [5]

Tentative Marking Scheme for Quiz 1A

1. (a) Observe that \(|x_{n+2} - x_{n+1}| = \frac{3}{4}|x_n - x_{n+1}|\).
   
   The sequence satisfies Cauchy criterion and hence it converges.
   
   (b) Observe that \(x_{n+2} + \frac{3}{4}x_{n+1} = x_{n+1} + \frac{3}{4}x_n\).
   
   Let \(x_n \to L\) for some \(L\). Then \(L + \frac{3}{4}L = 2 + \frac{3}{4}\).
   
   Hence \(L = \frac{11}{7}\). [3] [2] [2]

2. Note that for \(x_n = \frac{1}{2n\pi}, x_n \to 0\) but \(g(x_n) \to 0\),
   and for \(y_n = \frac{1}{2n\pi + \frac{1}{2}}, y_n \to 0\) but \(g(y_n) \to 1\). [3]

3. Let \(a_n = (1 - n \sin \frac{1}{n})n^{\frac{1}{4}}\) and \(b_n = \frac{n^{\frac{1}{4}}}{n^{\frac{1}{4}}}\).
   
   Now \(\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{x \to 0} \frac{\sin x}{x^{\frac{1}{4}}} = \frac{1}{6}\).
   
   Since \(\sum b_n\) converges, by LCT, \(\sum a_n\) converges. [2] [2] [1]
Questions-Quiz 1B

1. Let \( f(x) = \cos \frac{1}{x} \) for \( x \neq 0 \). Show that \( \lim_{x \to 0} f(x) \) does not exist. \[3\]

2. Test the convergence/divergence of the series \( \sum_{n=1}^{\infty} \sqrt{n}(1 - n \sin \frac{1}{n}) \). \[5\]

3. Let \((y_n)\) be a sequence defined by \( y_1 = 2, y_2 = 4 \) and \( y_{n+2} = \frac{1}{4}y_n + \frac{3}{4}y_{n+1} \) for \( n \geq 1 \).
   
   (a) Show that \((y_n)\) converges. \[3\]
   
   (b) Find the limit of \((y_n)\). \[4\]

Tentative Marking Scheme

1. Note that for \( x_n = \frac{1}{2n\pi}, x_n \to 0 \) but \( f(x_n) \to 1 \), and for \( y_n = \frac{1}{2n\pi + \frac{\pi}{2}}, y_n \to 0 \) but \( f(y_n) \to 0 \). \[3\]

2. Let \( a_n = \sqrt{n}(1 - n \sin \frac{1}{n}) \) and \( b_n = \frac{\sqrt{n}}{n^2} \). \[2\]

   Now \( \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{x \to 0} \frac{1 - \sin x}{x^2} = \frac{1}{6} \). \[2\]

   Since \( \sum b_n \) converges, by LCT, \( \sum a_n \) converges. \[1\]

3. (a) Observe that \( |y_{n+2} - y_{n+1}| = \frac{1}{4}|y_n - y_{n+1}| \). \[3\]

   The sequence satisfies Cauchy criterion and hence it converges.

   (b) Observe that \( y_{n+2} + \frac{1}{4}y_{n+1} = y_{n+1} + \frac{1}{4}y_n \).

   Let \( y_n \to L \) for some \( L \). Then \( L + \frac{1}{4}L = 4 + \frac{1}{4}2 \).

   Hence \( L = \frac{18}{5} \). \[2\]