MTH101 - Quiz 1A Questions

1. Let (x_n) be a sequence defined by $x_1 = 1, x_2 = 2$ and $x_{n+2} = \frac{3}{4}x_n + \frac{1}{4}x_{n+1}$ for $n \geq 1$.

   (a) Show that (x_n) converges. [3]

   (b) Find the limit of (x_n). [4]

2. Let $g(x) = \sin\frac{1}{x}$ for $x \neq 0$. Show that $\lim_{x\to 0} g(x)$ does not exist. [3]

3. Test the convergence/divergence of the series $\sum_{n=1}^{\infty}(1 - n\sin\frac{1}{n})n^{\frac{1}{4}}$. [5]

Tentative Marking Scheme for Quiz 1A

1. (a) Observe that $|x_{n+2} - x_{n+1}| = \frac{3}{4}|x_n - x_{n+1}|$. [3]
   The sequence satisfies Cauchy criterion and hence it converges.

   (b) Observe that $x_{n+2} + \frac{3}{4}x_{n+1} = x_{n+1} + \frac{3}{4}x_n$. [2]
   Let $x_n \to L$ for some L. Then $L + \frac{3}{4}L = 2 + \frac{3}{4}$. [2]
   Hence $L = \frac{11}{7}$.

2. Note that for $x_n = \frac{1}{2n\pi}$, $x_n \to 0$ but $g(x_n) \to 0$,
   and for $y_n = \frac{1}{2n\pi + \frac{\pi}{2}}$, $y_n \to 0$ but $g(y_n) \to 1$. [3]

3. Let $a_n = (1 - n\sin\frac{1}{n})n^{\frac{1}{4}}$ and $b_n = \frac{n^{\frac{1}{4}}}{n^2}$. [2]
   Now $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{x\to 0}\frac{1-\frac{\sin x}{x}}{x^2} = \frac{1}{6}$. [2]
   Since $\sum b_n$ converges, by LCT, $\sum a_n$ converges. [1]

Questions-Quiz 1B

1. Let $f(x) = \cos \frac{1}{x}$ for $x \neq 0$. Show that $\lim_{x\to 0} f(x)$ does not exist. [3]

2. Test the convergence/divergence of the series $\sum_{n=1}^{\infty} \sqrt{n}(1 - n \sin \frac{1}{n})$. [5]

3. Let (y_n) be a sequence defined by $y_1 = 2, y_2 = 4$ and $y_{n+2} = \frac{1}{4}y_n + \frac{3}{4}y_{n+1}$ for $n \geq 1$.

   (a) Show that (y_n) converges. [3]

   (b) Find the limit of (y_n). [4]

Tentative Marking Scheme

1. Note that for $x_n = \frac{1}{2n\pi}$, $x_n \to 0$ but $f(x_n) \to 1$,
   and for $y_n = \frac{1}{2n\pi + \frac{\pi}{2}}$, $y_n \to 0$ but $f(y_n) \to 0$. [3]

2. Let $a_n = \sqrt{n}(1 - n \sin \frac{1}{n})$ and $b_n = \frac{\sqrt{n}}{n^2}$. [2]

   Now $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{x\to 0} \frac{1 - \frac{\sin x}{x}}{x^2} = \frac{1}{6}$. [2]

   Since $\sum b_n$ converges, by LCT, $\sum a_n$ converges. [1]

3. (a) Observe that $|y_{n+2} - y_{n+1}| = \frac{1}{4}|y_n - y_{n+1}|$. [3]
   The sequence satisfies Cauchy criterion and hence it converges.

   (b) Observe that $y_{n+2} + \frac{1}{4}y_{n+1} = y_{n+1} + \frac{1}{4}y_n$. [2]
   Let $y_n \to L$ for some L. Then $L + \frac{1}{4}L = 4 + \frac{1}{4}2$. [2]
   Hence $L = \frac{18}{5}$.