Department of Mathematics and Statistics, I.I.T. Kanpur

MTH101N -First Mid-Semester Examination - 30.8.2008

Maximum Marks: 40 Time: 8:00-9:00 a.m

Answer all questions.

1. (a) Let $x_1 = 1$ and $x_{n+1} = \frac{1}{2+x_n^2}$. Show that the sequence (x_n) converges. [4]

   (b) Let $x_n = (\frac{a^n+b^n+c^n}{3})^{\frac{1}{n}} + n^a a^{1-n}$ where $1 < a < b < c$. Show that the sequence (x_n) converges and find its limit. [6]

2. (a) Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Show that there exists $x_0 \in [0,1]$ such that $f(x_0) = \frac{1}{3}(f(\frac{1}{4}) + f(\frac{1}{2}) + f(\frac{3}{4}))$. [5]

   (b) Let $f : [0,1] \to \mathbb{R}$ be differentiable, $f(\frac{1}{2}) = \frac{1}{2}$ and $0 < \alpha < 1$ for some α. Suppose $\mid f'(x) \mid \leq \alpha$ for all $x \in [0,1]$. Show that $\mid f(x) \mid < 1$ for all $x \in [0,1]$. [5]

3. (a) Show that the equation $7x^{17} - e^{-x} - 5 = 0$ has exactly one real root. [4]

   (b) Sketch the graph of the function $f(x) = \frac{2x^2-1}{x^2-1}$ after locating the intervals where the function is increasing and decreasing, convexity and concavity, the local maxima and minima, and the asymptotes. [6]

4. (a) Show that $\sum_{n=2}^{\infty} \frac{1}{n(logn)^p}$ converges if and only if $p > 1$. (Do not use the integral test !). [5]

   (b) Show that for $x > 0$, $\mid \log(x+1) - (x - \frac{x^2}{2} + \frac{x^3}{3}) \mid \leq \frac{x^4}{4}$. [5]