Tentative marking scheme: I mid semester exam. (2009-10 I sem.)

1. (a) Note that $x_{n+1} - x_n = \frac{1}{3}\left(2x_n + \frac{a}{x_n^2}\right) - x_n = \frac{a - x_n^3}{3x_n^2}$ ...... (2).
   and by AM-GM inequality $x_{n+1} = \frac{1}{3}(x_n + x_n + \frac{a}{x_n^2}) \geq a^{\frac{1}{3}}$ .... (3)
   Since the sequence is decreasing and bounded below, it converges. ...... (1)

   (b) Note that $x_n = \frac{1}{1+p} + \frac{n^s}{(1+p)^{n+1}}$. .... (1)
   If we let $a_n = \frac{n^s}{(1+p)^{n+1}}$ then $\frac{a_{n+1}}{a_n} \to \frac{1}{1+p} < 1$ and hence $a_n \to 0$. ...... (2)
   Therefore $x_n \to \frac{1}{1+p}$. .... (1)

2. (a) Note that $p(0) = -1$. ...... (1)
   Since $p(x) = x^n\left(1 + \frac{a_{n-1}}{x} + ... - \frac{1}{x}\right)$ for $x \neq 0$,
   $p(x) \to \infty$ as $x \to \infty$ and $x \to -\infty$. ..... (1+1)
   By IVP, there is a root in $(-\infty, 0)$ and also in $(0, \infty)$. .... (1+1)

   (b) Let $p(x) = a_1x + a_2x^2 + a_3x^3 + ... + a_nx^n$ ...... (3)
   Then $p(0) = 0$ and $p(1) = 0$ .... (1)
   By Rolle's theorem $p'(x) = g(x)$ has a real root. ..... (1)

3. (a) For the existence of $f'(0)$, consider $\lim_{h\to 0}\frac{f(h)-f(0)}{h}$ ....
   By MVT $\exists$ c_x between 0 and h such that $\lim\limits_{h\to 0}\frac{f(h)-f(0)}{h} = \lim\limits_{h\to 0} f'(c_x)$ ...... (3)
   Therefore, $f'(0) = 1$. ...... (1)

   (b) Let $a_n = \left(\frac{n}{n+1}\right)^{n^2}$. Then $a_n^{\frac{1}{n}} = \frac{1}{(1+\frac{1}{n})^n} \to \frac{1}{e} < 1$.
   By root test the series $\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}$ converges. ...... (3)
   Let $b_n = \left(\frac{n}{n+1}\right)^{n^2}(sin\frac{1}{n} - 2)$. Note that $\mid b_n \mid \leq 2a_n$ or $3a_n$. ...... (1)
   Therefore $\sum \mid b_n \mid$ converges and hence $\sum b_n$ converges. ..... (2)

4. Note that for $x \neq 1$, $f(x) = x + 2 - \frac{3}{x-1}$, $\frac{df}{dx} = 1 + \frac{3}{(x-1)^2}$ and $\frac{d^2f}{dx^2} = \frac{-6}{(x-1)^3}$.

   (a) The asymptotes are $x = 1$ and $y = x + 2$. ...... (2)

   (b) The function is increasing on $(-\infty, 1) \cup (1, \infty)$. ...... (2)

   (c) The function is convex for $x < 1$ and concave for $x > 1$. ..... (2)

   (d) There is no point of local maximum and local minimum. ...... (2)

   (e) The graph is given below. .... (2)