Department of Mathematics and Statistics, I.I.T. Kanpur

MTH101N -First Mid-Semester Examination - 29.8.2009

Maximum Marks: 40 Time: 11:00-12:00

Answer all questions

1. (a) Let $a > 0, x_1 > 0$ and $x_{n+1} = \frac{1}{3}\left(2x_n + \frac{a}{x_n^2}\right)$. Show that the sequence (x_n) is monotone and converges.
(Hint: One can use the AM-GM inequality: $\frac{x_1+x_2+x_3}{3} \geq (x_1x_2x_3)^{\frac{1}{3}}$ for $x_1, x_2, x_3 > 0$. In fact this sequence converges to $\sqrt[3]{a}$) [6]

   (b) Let $p > 0, s > 0$ and $x_n = \frac{(1+p)^n+n^s}{(1+p)^{n+1}}$. Show that the sequence (x_n) converges and find its limit. (Do not use L'Hospital's rule !) [4]

2. (a) Let $p(x)$ be a polynomial
$$p(x) = a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a_0.$$
Suppose n is even ($n \neq 0$), $a_n = 1$ and $a_0 = -1$. Show that $p(x)$ has at least two real roots. [5]

   (b) Assume that $a_1, a_2, ..., a_n$ are real numbers such that $a_1 + a_2 + ... + a_n = 0$. Show that the polynomial
$$q(x) = a_1 + 2a_2x + 3a_3x^2 + ... + na_nx^{n-1}$$
has at least one real root. [5]

3. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and f' exist for all $x \neq 0$. Suppose $\lim_{x\to 0} f'(x) = 1$. Show that f is differentiable at 0 and $f'(0) = 1$. [4]

   (b) Show that the series $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$ converges. Further, discuss the convergence/divergence of the series $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2} (sin\frac{1}{n} - 2)$. [6]

4. Let $f(x) = \frac{x^2+x-5}{x-1}$.

   (a) Find the asymptotes of f.

   (b) Locate the intervals where the function is increasing and decreasing.

   (c) Locate the intervals where the function is convex and concave.

   (d) Find the points of local maximum and local minimum.

   (e) Sketch the graph of the function.

(After solving Problem 4, at the end its solution, write the explicit answers of (a) - (e) together at one place .) [10]