Department of Mathematics and Statistics, I.I.T. Kanpur

MTH101A -Mid-Semester Examination - 12.09.2011

Maximum Marks: 80 Time: 7:30 - 09:30 AM

Instructions: 1. Please write your Name and Roll Number and **Section Number** correctly on the answer booklet. If any of these is missing, marks will be deducted.
2. Attempt each question on a new page and attempt all the parts of a question at the same place.

1. (a) Investigate the convergence of the sequence $a_n = \dfrac{n^2}{3^n}$, $n = 1, 2, \ldots$. [5]

Ans:

- Note $a_n \geq 0$.
  Consider $\dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)^2}{3^{(n+1)}}\dfrac{3^n}{n^2} = \dfrac{1}{3}\left(1+\dfrac{1}{n}\right)^2 \to \dfrac{1}{3}$. [3]
- $\dfrac{a_{n+1}}{a_n} \to \dfrac{1}{3} < 1$. Therefore, $a_n \to 0$ as $n \to \infty$. [2]

(b) Determine if the series $\displaystyle\sum_{n=1}^{\infty}(n+2)\left(1-\cos\left(\frac{1}{n}\right)\right)$ is convergent or divergent. [5]

Ans:

- Let $a_n = (n+2)\left(1-\cos\left(\frac{1}{n}\right)\right)$. and let $b_n = n\left(1-\cos\left(\frac{1}{n}\right)\right)$. Note $a_n, b_n \geq 0$. Then as $\frac{a_n}{b_n} \to 1$, the series $\sum_{n=1}^{\infty} a_n$ and the series $\sum_{n=1}^{\infty} b_n$ converge or diverge together. [1]
- Let $c_n = \frac{1}{n}$. Then $\lim_{n\to\infty}\frac{b_n}{c_n} = \lim_{n\to\infty}\frac{n(1-\cos(\frac{1}{n})}{1/n} = \lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{\sin x}{2x} = 1/2$. [2]
- Therefore, the series $\sum_{n=1}^{\infty} b_n$ and the series $\sum_{n=1}^{\infty} c_n$ converge or diverge together. But since $\sum \frac{1}{n}$ is divergent, the series $\sum a_n$ is also divergent. [2]

(c) Determine all values of x for which the series $\displaystyle\sum_{n=2}^{\infty}\frac{x^n}{n\,(\ln n)^2}$ converges. Give reasons for your answer. [10]

Ans:

- Consider $\dfrac{a_{n+1}}{a_n} = \dfrac{n(\ln n)^2}{(n+1)(\ln(1+n))^2} \to 1$ as $n \to \infty$. [3]
- Therefore, the given series is convergent for all $x : |x| < 1$. [2]
- For the case $x = 1$ note that $a_n \geq 0, a_n \downarrow$. Therefore by Cauchy's condensation test, the series $\sum \frac{1}{n\,(\ln n)^2}$ converges iff the series $\sum 2^k a_{2^k} =$

$\sum 2^k \frac{1}{2^k (\ln 2^k)^2}$ converges. But then this is same as the series $\sum \frac{1}{k^2 (\ln 2)^2}$, which is convergent. [3]

- For $x = -1$ the series converges since it converges absolutely by previous case.

  Alternately, the Leibniz test also can be used here to conclude that the series $\sum \frac{(-1)^n}{n (\ln n)^2}$ converges as $\frac{1}{n (\ln n)^2} \geq 0$ and $\frac{1}{n (\ln n)^2} \downarrow 0$. [2]

  Hence the given series converges for all $x : |x| \leq 1$.

2. (a) Compute the limit $\lim_{x \to \infty} \left(x^2 - x^3 \sin\left(\frac{1}{x}\right)\right)$. [10]

   Ans:

   - $x^2 - x^3 \sin\left(\frac{1}{x}\right) = \frac{1 - x\sin(1/x)}{1/x^2}$ [3]
   - Therefore the given limit is same as $\lim_{x \to \infty} \dfrac{1 - x\sin(1/x)}{1/x^2} = \lim_{y \to 0} \dfrac{1 - \frac{\sin y}{y}}{y^2}$ [4]
   - The above is equal to $\lim_{y \to 0} \frac{y - \sin y}{y^3} = \lim_{y \to 0} \frac{\sin y}{6y} = 1/6$. [3]

   (b) Let $f : [1, 3] \mapsto \mathbb{R}$ be a continuous function that is differentiable on $(1, 3)$ with derivative $f'(x) = (f(x))^2 + 4$ for all $x \in (1, 3)$. Determine whether it is true or false that $f(3) - f(1) = 5$. Justify your answer. [5]

   Ans:

   - By Mean Value theorem, $5 = f(3) - f(1) = f'(c)(3-1)$, for some $c \in (1, 3)$. [3]
   - But we are given that $f'(c) = (f(c))^2 + 4$, therefore $\frac{5}{2} = (f(c))^2 + 4 \Rightarrow (f(c))^2 = \frac{-3}{2}$,, which is not possible. Hence the statement is false. [2]

   (c) Are there any value(s) of k for which the equation $x^4 - 4x + k = 0$ has two distinct roots in the interval $[0, 1]$? Give reasons. [5]

   Ans:

   - Consider $f(x) = x^4 - 4x + k$. Note that $f'(x) = 4x^3 - 4 = 4(x^3 - 1) < 0$ in $(0, 1)$. [2]
   - Therefore the function is strictly decreasing in the interval $[0, 1]$, and hence there is no value of k for which the given equation has two distinct roots in $[0, 1]$. [3]

3. (a) Trace the curve $f(x) = \dfrac{2x^2 - 3}{x + 1}$ marking the local maxima/minima, intervals where f is increasing or decreasing, points of inflection and asymptotes if any. [10]

   Ans:

   - $f(x) = \dfrac{2x^2 - 3}{x + 1} = 2x - 2 - \dfrac{1}{x + 1}$. $f(x) = 0$ when $x = \pm\sqrt{\dfrac{3}{2}}$. [1]
   - $f^{'}(x) = 2 + \dfrac{1}{(x + 1)^2} > 0, \ \forall\, x \neq -1$. [1]

- f is increasing in both $(-\infty, -1)$ and $(-1, \infty)$. There are no local maxima/local minima. [2]
- $y = 2x - 2$ is an asymptote as $f(x) - y \to 0$ as $x \to \pm\infty$. $x = -1$ is a horizontal asymptote. [2]
- As $x \to \infty$, $f(x) \to \infty$ and as $x \to -\infty$, $f(x) \to -\infty$.
  As $f(x) = 2(-1+h) - 2 - \frac{1}{-1+h+1}$, $f(x) \to -\infty$ when $x \to (-1)^+$ and $f(x) \to \infty$ as $x \to (-1)^-$. [1]
- $f''(x) = \dfrac{-2}{(x+1)^3} > 0$ when $x \in (-\infty, -1)$ and $f''(x) < 0, if x \in (-1, \infty)$. Thus function is concave up in $(-\infty, -1)$ and is concave down in $(-1, \infty)$. [2]
- $x = -1$ is a point of inflection. [1]

(b) Prove that the equation $x^3 + 3x + 1 = 0$ has exactly one real root. Take $x_0 = 0$ in the Newton's method and find x_2 to estimate this root. [10]

Ans:

- Let $f(x) = x^3 + 3x + 1$. Then f is a continuous function. As $x \to \infty$, $f(x) \to \infty$ and as $x \to -\infty$, $f(x) \to -\infty$. [2]
- Therefore, by IVP, $f(x) = 0$ for some $x \in \mathbb{R}$. [2]
- Since $f'(x) = 3x^2 + 3 > 0$, f has exactly one real root. [1]
- In Newton's method, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, n = 1, 2, \ldots$. Here $f'(x) = 3x^2 + 3$. [3]
- Therefore, $x_0 = 0$, gives $x_1 = 0 - 1/3 = -1/3$. $x_2 = -1/3 - \frac{-1/27}{10/3} = -29/90$. [2]

4. (a) For what values of x, can we replace $\sin x$ by $x - \dfrac{x^3}{6}$ with an error of magnitude less than or equal to 5×10^{-4}. Give reasons for your answer. [10]

Ans:

- By Taylor's theorem, $\sin x = x - \frac{x^3}{3!} + \cos c \frac{x^5}{5!}$ where $c \in (0, x)$. [5]
  Therefore, $|\sin x - (x - \frac{x^3}{6})| = |\cos c \frac{x^5}{5!}| \leq |\frac{x^5}{5!}|$ [3]
- This is less than or equal to $\frac{5}{(10)^4}$ if $|x|^5 < \frac{5 \times 5!}{(10)^4}$. [2]

(b) For $x > -1, x \neq 0$ prove that $(1+x)^\alpha > 1 + \alpha x$, whenever $\alpha < 0$ or $\alpha > 1$. [10]

Ans:

- Consider the function $g(x) = (1+x)^\alpha - 1 - \alpha x$.
  We need to show that $g(x) > 0$ under the given conditions. $g'(x) = \alpha(1+x)^{\alpha-1} - \alpha, x \neq -1$ [1]
- Consider $g''(x) = \alpha(\alpha - 1)(1+x)^{\alpha-2}$, which is always positive $\forall x > -1$, for the given $\alpha's$. [3]

- Therefore, g' is strictly increasing in $(-1, \infty)$. [2]
- Therefore, if $x > 0$, then $g'(x) > g'(0) = 0$, and so $g(x) > g(0) = 0$. [2]
- For $-1 < x < 0$, $g'(x) < g'(0) = 0$, and so g is strictly decreasing in $(-1, 0)$, which in turn gives $g(x) > g(0) = 0$. [2]