Department of Mathematics, Indian Institute of Technology, Kanpur
MTH101A: Mid Semester Exam 1- 17-9-2012

Maximum Marks- 70 **7:30-9:30 a.m.**

---

1. Answer all questions.
2. Please number the pages and indicate on a tabular column on the first page, the pages in which the respective questions have been answered.

---

(1) Determine whether the following sequences (a_n) are convergent or divergent (Please provide proper justifications for your answers):

1. $a_1 = 0,\ a_2 = 3,\ a_n = \dfrac{2a_{n-1} + a_{n-2}}{3},\ \forall\ \ n > 2.$

   Solution: $|a_{n+1} - a_n| = \dfrac{1}{3}|a_n - a_{n-1}|$ and $0 < \dfrac{1}{3} < 1$. The sequence is Cauchy. [3]
   By the Cauchy criterion the sequence (a_n) is convergent. [2]

2. $a_n = \sqrt{\ln(n+1)}$.

   Solution: We show that the sequence (a_n) is not bounded.
   Fix $n_0 \in \mathbb{N}$ and let $N = e^{n_0^2}$
   For any $n > N$, $a_n > a_N = \sqrt{\ln(e^{n_0^2} + 1)} > n_0$. [3]
   The sequence is divergent since an unbounded sequence cannot be convergent. [1]

3. Let b_k denote the number of prime numbers less than or equal to k. (For example, $b_4 = 2$ since the prime numbers less than or equal to 4 are 2 and 3.)
   Let $a_1 = 2,\ a_2 = 3$ and $a_n = \displaystyle\sum_{k=3}^{n} \frac{1}{b_k}\ \forall\ n \geq 3.$ [5+4+5]

   Solution: Since the number of primes less than or equal to k is atmost k, $b_k < k$, $\forall k > 2$. [1]
   Hence $0 < \dfrac{1}{k} < \dfrac{1}{b_k},\ \forall\ \ k \geq 3.$ [1]
   By the comparison test since $\sum \dfrac{1}{k}$ is divergent, so is $\sum \dfrac{1}{b_k}$. [2]
   Hence, (a_n) is a divergent sequence. [1]

(2) 1. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions such that $f(a) \neq g(a)$ for some $a \in \mathbb{R}$. Show that there exists a $\delta > 0$, such that $f(x) \neq g(x)$, $\forall x$ such that $|x - a| < \delta$. [5]

   Solution: Let $h(x) = f(x) - g(x)$. h is continuous on $\mathbb{R}$ and $h(a) \neq 0$. [1]

Choose $\epsilon = |\frac{h(a)}{2}| > 0$. [1]

Since, h is continuous at a, there exists a $\delta > 0$ such that

$|h(x) - h(a)| < \epsilon$, whenever $|x - a| < \delta$.

$\Rightarrow |h(x) - h(a)| < |\frac{h(a)}{2}|$, whenever $|x - a| < \delta$.

$\Rightarrow |h(x)| > |\frac{h(a)}{2}| > 0$, whenever $|x - a| < \delta$. [2]

Hence $f(x) \neq g(x)$, whenever $|x - a| < \delta$. [1]

2. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous with $f(0) = -1$ and $f(1) = 3$. Let $S = \{x \in [0, 1] | f(x) = 0\}$.

   (a) Show that S is non empty.

   Solution: Since f is continuous on $[0, 1]$, by the Intermediate Value property for f, there exists a $c \in (0, 1)$, such that $f(c) = 0$.
   $c \in S$ and hence $S \neq \emptyset$. [2]

   (b) Let α be the supremum of the set S. Show that $\alpha \in (0, 1]$.

   Solution: Since 1 is an upper bound for S, $\alpha \leq 1$. [1]

   Since $c \in S$ and $c > 0$, $\alpha \geq c > 0$. Hence $\alpha \in (0, 1]$. [2]

   (c) Show that $f(\alpha) = 0$. [2+3+4]

   Solution: Assume, if possible that $f(\alpha) \neq 0$. Since f is continuous at α, (by problem 2(1)) there exists a $\delta > 0$ such that $f(x) \neq 0$ for all $|x - \alpha| < \delta$. [2]

   In other words, no element in $(\alpha - \delta, \alpha + \delta)$ belongs to S which contradicts that fact that α is the supremum of S. Hence, $f(\alpha) = 0$. [2]

(3) 1. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function which has a local maximum at $x = 0$. Show that $f''(0) \leq 0$.

Solution: Since 0 is a point of local maxima for f, $f'(0) = 0$. [1]

Further there exists an $\epsilon > 0$, such that $f(x) \leq f(0)$, $\forall |x| < \epsilon$. [1]

For each $x \in (0, \epsilon)$, by the Mean Value Theorem, there exists $c_x \in (0, x)$, such that $f(x) - f(0) = f'(c_x)(x - 0) \leq 0$. [2]

$\frac{f'(c_x) - f'(0)}{c_x - 0} = \frac{f'(c_x)}{c_x} \leq 0$. [2]

$f''(0) = \lim_{x \to 0} \frac{f'(c_x)}{c_x} \leq 0$. [2]

Aliter:

0 is a local maximum $\Rightarrow f'(0) = 0$. [1]

Hence there exists a $\delta > 0$ such that
f is increasing in $(-\delta, 0)$ and f is decreasing in $(0, \delta)$. [2]
Hence, $f'(0) < 0$ if $x > 0$ and $f'(0) > 0$ if $x < 0$. [1]
$f''(0^+) = \lim_{h\to 0^+} \dfrac{f'(h) - f'(0)}{h} = \lim_{h\to 0^+} \dfrac{f'(h)}{h} \leq 0.$
$f''(0^-) = \lim_{h\to 0^-} \dfrac{f'(h) - f'(0)}{h} = \lim_{h\to 0^-} \dfrac{f'(h)}{h} \leq 0.$
Since $f''(0)$ exists, $f''(0) = f''(0^+) = f''(0^-)$. [4]

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a thrice differentiable function on $[-1, 1]$ with $f(-1) = 0$, $f(1) = 1$ and $f^{'}(0) = 0$. Using Taylor's Theorem prove that
$f'''(c) \geq 3$ for some $c \in (-1, 1)$. [8+6]

Solution: By Taylor's Theorem
$f(1) = f(0) + f^{'}(0) + \frac{1}{2}f^{''}(0) + \frac{1}{3!}f'''(c_1)$, for some $c_1 \in (0, 1)$. [2]
$f(-1) = f(0) - f^{'}(0) + \frac{1}{2}f^{''}(0) - \frac{1}{3!}f'''(c_2)$, for some $c_2 \in (-1, 0)$. [2]
On subtracting, we get $f'''(c_1) + f'''(c_2) = 6$, which implies atleast one of $f'''(c_1)$ or $f'''(c_2) \geq 3$. [2]

(4) 1. Show that the equation $x^{13} + 7x^3 - 5 = 0$ has exactly one real root.

Solution: Let $f(x) = x^{13} + 7x^3 - 5$. Here, $f(x) < 0\ \forall\ x \leq 0$, $f(0) = -5$ and $f(1) = 3$. By the intermediate value property, there exists $c \in (0, 1)$, such that $f(c) = 0$. So, f has atleast one real root. [1]
If f has more than one real roots, (from above) they must all be positive. But, $f'(x) = x^2(13x^{10} + 21) \neq 0$ unless $x = 0$. Since $f'(x)$ has no positive root, f has atmost one real root. [2]

2. Use the Cauchy Condensation Test to determine the behaviour of the p-series $\sum_n \dfrac{1}{n^p}$ for all p.

Solution: Let $p \geq 0$. Then $(\dfrac{1}{n^p})$ is a decreasing sequence of nonnegative terms. By the Cauchy Condensation test, $\sum_n \dfrac{1}{n^p}$ converges iff $\sum_k 2^k \dfrac{1}{2^{kp}}$ converges. [1]
$\sum_k 2^k \dfrac{1}{2^{kp}} = \sum_k \dfrac{1}{2^{(p-1)k}}$. If $p > 1$, $0 < \frac{1}{2^{(p-1)}} = r < 1$. [1]
Since $\sum_k r^p$ is a geometric series, it is convergent. Hence $\sum_n \dfrac{1}{n^p}$ converges whenever $p > 1$ [2]
When $p \leq 1$, the gemoetric series diverges. [2]

If $p \leq 0$, the Cauchy Condensation test fails and the nth term test tells us that the series $\sum_n \frac{1}{n^p}$ is divergent.

3. Using the mean value theorem determine

$$\lim_{x \to 0} \frac{(1+x)^n - 1}{x}.$$

[3+6+5]

Solution: Choose the function $f : \mathbb{R} \to \mathbb{R}$, $f(y) = (1+y)^n$. [1]
For $x > 0$, f is continuous on $[0, x]$ and differentiable on $(0, x)$.
By the Mean Value theorem, there exists a $c_x \in (0, x)$ such that
$\frac{f(x) - f(0)}{x - 0} = \frac{(1+x)^n - 1}{x} = f'(c_x).$ [2]
$f'(c_x) = n(1 + c_x)^{n-1}$.
Since $0 < c_x < x$, as $x \to 0$, $c_x \to 0$.
Hence $\lim_{x \to 0} \frac{(1+x)^n - 1}{x} = \lim_{c_x \to 0} n(1 + c_x)^{n-1} = n$ [2]

(5) 1. Determine whether the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{2^2} + \frac{1}{5} - \frac{1}{2^3} + \frac{1}{7} - \frac{1}{2^4} \dots$$

is convergent or divergent.

Solution:
Consider the above series as $\sum (-1)^n a_n$. Since $(|a_n|)$ is not a decreasing sequence, we cannot use the Leibniz test.
If $\sum a_n$, is convergent then the sequence of partial sums (s_n) must be convergent and hence the subsequence (s_{2n}) must also be convergent. [1]
$s_n = \sum_{m=0}^{n/2-1} \frac{1}{2m+1} - \sum_{m=1}^{n/2} \frac{1}{2^m} = x_n - y_n$, where n is even. Since $\frac{1}{2i+1} < \frac{1}{2i}$, by comparison test (x_n) diverges and hence (x_n) is an unbounded sequence. [2]
$\sum_{i=0} \frac{1}{2^i}$ is a geometric series and converges to 2. (y_n) is an increasing sequence and $y_n < 2$ for all n. [1]
For any $K \in \mathbb{R}$, there exists an $n_0 \in \mathbb{N}$ such that $\forall\ n > n_0$, $x_n > K + 2$.
This implies that $s_n = x_n - y_n > K + 2 - 2 = K$. Hence the sequence of partial sums is unbounded and so the series $\sum a_n$ diverges. [2]

2. Sketch the graph of the function $f(x) = \frac{x^2}{x^2 - 1}$.
Indicate clearly with proper justifications

(a) domain of definition of f,
(b) the x and y intercepts if any,
(c) behaviour of f at $\pm\infty$ and asymptotes if any,
(d) intervals where f is increasing, decreasing and local extrema if any,
(e) regions where f is concave/convex and points of inflection.

[6+8]

Solution: As $f(-x) = f(x)$, f is even and it is sufficient to concentrate on $(0, \infty)$. f is defined on $\mathbb{R} \setminus \{-1, 1\}$ and the only x and y intercepts are when $x = 0$ and $y = 0$. [1]

$\dfrac{x^2}{x^2-1} = 1 + \dfrac{1}{x^2-1}$. Hence $\lim\limits_{x\to\pm\infty} f(x) = 1$. So, $y = 1$ is a horizontal asymptote. [1]

$\lim\limits_{x\to 1^+} f(x) = \infty$ and $\lim\limits_{x\to 1^-} f(x) = -\infty$, $x = 1$ and $x = -1$ are vertical asymptotes. [1]

$f'(x) = \dfrac{-2x}{(x^2-1)^2}$. $f'(x) < 0$ when $x > 0$. Hence the function if decreasing on $(0, 1)$ and $(1, \infty)$ [1]

The function is increasing on $(-\infty, -1)$ and $(-1, 0)$.

f has a local maximum at $x = 0$ and $f(0) = 0$. [1]

$f''(x) = \dfrac{2+6x^2}{(1-x^2)^3}$. $f'' > 0$ on $(1, \infty)$ and $f'' < 0$ on $(-1, 1)$. Hence f is convex on $(1, \infty)$ and $(-\infty, -1)$ and concave on $(-1, 1)$. [1]

Graph [2]