Department of Mathematics and Statistics, I.I.T. Kanpur
MTH101A - Quiz 2B Examination - 20.10.2011
Maximum Marks: 20  Time: 17:30-18:00 hrs

NAME: ___________________________  Roll No.: ______  Section: ______

1. Sketch the curves \( r = -\sin(2\theta) \) and \( r = 1/2 \). Further, find the area of the region that is inside the curve \( r = -\sin(2\theta) \) and also inside the circle \( r = 1/2 \). [10]

2. Let \( C \) be the (infinite) cylinder generated by revolving the line \( y = x + \sqrt{6} \) about the line \( y = x \). Let \( S \) be the solid sphere \( x^2 + y^2 + z^2 \leq 4 \). Find the volume of the portion of the sphere which lies inside the cylinder \( C \). [10]

\[ \text{Area} = 8 \left[ \frac{1}{2} \int_{0}^{\pi/12} (\sin 2\theta)^2 + \frac{1}{2} \int_{\pi/12}^{\pi/6} \frac{1}{4} \, d\theta \right] \]

\[ = 8 \left[ \frac{1}{2} \int_{0}^{\pi/12} \frac{1 - \cos 4\theta}{2} + \frac{1}{2} \int_{\pi/12}^{\pi/6} \frac{1}{4} \, d\theta \right] \]

\[ = 8 \left[ \frac{1}{2} \int_{0}^{\pi/12} \frac{1 - \cos 4\theta}{2} + \frac{1}{2} \cdot \frac{1}{4} (\pi/4 - \pi/12) \right] \]

\[ = \frac{\pi}{6} + \frac{\pi}{6} - \frac{8}{2} \cdot \frac{1}{2} \cdot 4 \left[ \sin 4 \cdot \frac{\pi}{12} - \sin 0 \right] \]

\[ = \frac{\pi}{3} - \frac{1}{2} \cdot \frac{\pi}{3} \]

\[ = \frac{\pi}{3} - \frac{\pi}{6} \cdot \sqrt{3}/2, \quad ---(2) \]
1. Sketch the curves $r = -\cos(2\theta)$ and $r = \frac{1}{2}$. Further, find the area of the region that is inside the curve $r = -\cos(2\theta)$ and also inside the circle $r = \frac{1}{2}$. [10]

2. Let $C$ be the (infinite) cylinder generated by revolving the line $y = -x + \sqrt{6}$ about the line $y = -x$. Let $S$ be the solid sphere $x^2 + y^2 + z^2 \leq 4$. Find the volume of the portion of the sphere which lies inside the cylinder $C$. [10]

\[
\text{Area} = 8 \left[ \frac{1}{2} \int_{0}^{\pi/6} \left( \frac{1}{2} \right)^2 d\theta + \frac{1}{2} \int_{\pi/6}^{\pi/4} (\cos 2\theta)^2 d\theta \right] \quad (3)
\]

\[
= 8 \left[ \frac{1}{8} \left( \frac{\pi}{6} \right) + \frac{1}{4} \left( \frac{\pi}{4} - \frac{\pi}{6} \right) + \frac{1}{4} \int_{\pi/6}^{\pi/4} \cos 4\theta \, d\theta \right] \quad (4)
\]

\[
= \frac{\pi}{6} + \frac{\pi}{6} + \frac{3}{4} \int_{\pi/6}^{\pi/4} \cos 4\theta \, d\theta \quad (5)
\]

\[
= \frac{\pi}{6} + \frac{\pi}{6} - \frac{1}{2} \cos 30 \quad (6)
\]

\[
= \frac{\pi}{6} + \frac{\pi}{6} - \frac{1}{2} \cdot \frac{\sqrt{3}}{2} \quad (7)
\]

\[
= \frac{\pi}{3} - \frac{\sqrt{3}}{4} \quad (2)
\]
The distance between \((0,0)\) and the line \(y = -x + \sqrt{6}\) is:

\[
\frac{\sqrt{6}}{\sqrt{2}} = \sqrt{3}. \quad \text{-------(2)}.
\]

The axis of the cylinder can be considered as \(125\) \(y\)-axis.

Volume of the whole sphere is:

\[
\frac{4}{3} \pi r^3 = \frac{32}{3} \pi. \quad \text{-------(2)}
\]

We find the volume of the region generated by revolving \(D\) around \(y\)-axis by washer method.

The volume is:

\[
\int_1^4 \pi \left[(4-y^2)^2 - 3^2\right] dy.
\]

\[
= \int_1^4 \pi \left[16 - 8y^2 + y^4 - 9\right] dy
\]

\[
= \pi \left[16y - \frac{8y^3}{3} + \frac{y^5}{5}\right]_1^4
\]

\[
= 4\pi - \frac{128}{3} + \frac{1024}{5}
\]

\[
= \frac{5}{3} \pi
\]

The required volume is:

\[
6\pi + \frac{16\pi}{3} = \frac{28\pi}{3} \quad \text{-------(2)}
\]

**Alternate Set:**

The required volume is:

\[
\text{Vol(cylinder)} + 2\text{ volume(c)}.
\]

\[
= 3\pi \cdot 2 + 2\text{ volume(c)} \quad \text{-------(2)}
\]

Volume \(c\) is:

\[
\int_1^2 \pi (y^2 + 4) dy = \int_1^2 \pi (4 - y^2) dy
\]

\[
= \pi \left[4y - \frac{y^3}{3}\right]_1^2
\]

\[
= 4\pi - \frac{8\pi}{3} + \frac{2\pi}{3}
\]

\[
= \frac{5}{3} \pi
\]

The required volume is:

\[
6\pi + \frac{16\pi}{3} = \frac{28\pi}{3} \quad \text{-------(2)}
\]