Indian Institute of Technology Kanpur

MTH-101AA
QUIZ I, 11-12-2020
TENTATIVE MARKING SCHEME

(1) For $n \geq 1$, let $x_n = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}$. Does the sequence (x_n) converge? Justify your answer. (Do not use statements of the problems appeared in the Assignments or Practice Problems for justifications). [5]

Solution:
For any $n \in \mathbb{N}$, $|x_{2n} - x_n| = |\frac{1}{2n+1} + \frac{1}{2n+3} + \cdots + \frac{1}{4n-1}|$. [2]
Hence $|x_{2n} - x_n| \geq \frac{n}{4n-1} \geq \frac{n}{4n+4} \geq \frac{1}{4}\frac{n}{n+1} \geq \frac{1}{8}$ for all $n \in \mathbb{N}$. [2]
Therefore $|x_{2n} - x_n| \nrightarrow 0$ as $n \to \infty$.
Since (x_n) does not satisfy the Cauchy Criterion, it does not converge. [1]
OR

For any n, $x_n \geq \frac{1}{2}y_n$ where $y_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$
For every $n \in \mathbb{N}$, $|y_{2n} - y_n| = |\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}| \geq \frac{n}{2n} = \frac{1}{2}$. [2]
Therefore $|y_{2n} - y_n| \nrightarrow 0$ as $n \to \infty$
Since (y_n) does not satisfy the Cauchy Criterion, it does not converge. [1]
As (y_n) is increasing, it is not bounded. [1]
Therefore, (x_n) is not bounded and hence it does not converge. [1]

(2) Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If f is continuous at $x_0 = 1$, show that f is continuous at $y_0 = 2$. [5]

Solution: Note that $f(0) = f(0 + 0) = 2f(0) \Rightarrow f(0) = 0$
and $f(x + (-x)) = f(0) = 0 \Rightarrow f(-x) = -f(x)$.
Hence $f(x - y) = f(x) - f(y)$. [1]
Suppose $y_n \to y_0 = 2$. [1]
Then $y_n - 1 \to 1$. [1]
Since f is continuous at $x_0 = 1$, $f(y_n - 1) \to f(1)$ [1]
Hence $f(y_n) - f(1) \to f(1)$
and therefore $f(y_n) \to f(1) + f(1) = f(2)$. [1]
Hence f is continuous at $y_0 = 2$.

(3) Let $f : [0,1] \to \mathbb{R}$ be continuous and (x_n) be a sequence in $[0,1]$. Suppose

$$\lim_{n\to\infty} \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} = \alpha$$

for some $\alpha \in \mathbb{R}$. Show that there exists $x_0 \in [0,1]$ such that $f(x_0) = \alpha$. [5]

Solution: Let $m = inf\{f(x) : x \in [0,1]\}$
and $M = sup\{f(x) : x \in [0,1]\}$.
Then $m \leq \frac{1}{n}(f(x_1) + f(x_2) + \cdots + f(x_n)) \leq M$ for every n. [2]
Hence $m \leq \alpha \leq M$. [2]
By IVP, there exists $x_0 \in [0,1]$ such that $f(x_0) = \alpha$. [1]