For $n \geq 1$, let $x_n = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}$. Does the sequence $(x_n)$ converge? Justify your answer. (Do not use statements of the problems appeared in the Assignments or Practice Problems for justifications). [5]

**Solution:**
For any $n \in \mathbb{N}$, $|x_{2n} - x_n| = \left|\frac{1}{2n+1} + \frac{1}{2n+3} + \cdots + \frac{1}{2n-1}\right|$. [2]
Hence $|x_{2n} - x_n| \geq \frac{n}{4n-1} \geq \frac{n}{3n+1} \geq \frac{1}{4} \cdot \frac{n}{n+1} \geq \frac{1}{8}$ for all $n \in \mathbb{N}$. [2]
Therefore $|x_{2n} - x_n| \to 0$ as $n \to \infty$.
Since $(x_n)$ does not satisfy the Cauchy Criterion, it does not converge. [1]

OR

For any $n$, $x_n \geq \frac{1}{2}y_n$ where $y_n = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{n}$.
For every $n \in \mathbb{N}$, $|y_{2n} - y_n| = \left|\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right| \geq \frac{n}{2n} = \frac{1}{2}$. [2]
Therefore $|y_{2n} - y_n| \to 0$ as $n \to \infty$.
Since $(y_n)$ does not satisfy the Cauchy Criterion, it does not converge. [1]
As $(y_n)$ is increasing, it is not bounded. [1]
Therefore, $(x_n)$ is not bounded and hence it does not converge. [1]

(2) Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If $f$ is continuous at $x_0 = 1$, show that $f$ is continuous at $y_0 = 2$. [5]

**Solution:** Note that $f(0) = f(0+0) = 2f(0) \Rightarrow f(0) = 0$ and $f(x+(-x)) = f(0) = 0 \Rightarrow f(-x) = -f(x)$.
Hence $f(x+y) = f(x) - f(y)$. [1]
Suppose $y_n \to y_0 = 2$. [1]
Then $y_n - 1 \to 1$. [1]
Since $f$ is continuous at $x_0 = 1$, $f(y_n - 1) \to f(1)$ [1]
Hence $f(y_n) - f(1) \to f(1)$ and therefore $f(y_n) \to f(1) + f(1) = f(2)$. [1]
Hence $f$ is continuous at $y_0 = 2$. [1]
(3) Let \( f : [0, 1] \rightarrow \mathbb{R} \) be continuous and \((x_n)\) be a sequence in \([0, 1]\). Suppose
\[
\lim_{n \to \infty} \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} = \alpha
\]
for some \( \alpha \in \mathbb{R} \). Show that there exists \( x_0 \in [0, 1] \) such that \( f(x_0) = \alpha \). [5]

**Solution:** Let \( m = \inf \{ f(x) : x \in [0, 1] \} \) and \( M = \sup \{ f(x) : x \in [0, 1] \} \).

Then \( m \leq \frac{1}{n} (f(x_1) + f(x_2) + \cdots + f(x_n)) \leq M \) for every \( n \). [2]

Hence \( m \leq \alpha \leq M \). [2]

By IVP, there exists \( x_0 \in [0, 1] \) such that \( f(x_0) = \alpha \). [1]